Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 1

$x = x_0 e^{rt}$ so half life τ must satisfy $\frac{1}{2}x_0 = x_0 e^{r\tau} \Rightarrow \tau = -\frac{1}{r}\log 2$.

reassuringly, this makes sense exactly when $r<0$

Exercise 2

I choose t to be in weeks. $t=0$ now.

So $x(0) = x_0 e^0 = 1500$, $x(-1) = x_0 e^{-r} = 1000$ $\Big\}$ $x_0 = 1500$, $r = -\log\frac{2}{3}$

suppose $t = t_1$ when exactly one case:

$x(t_1) = x_0 e^{rt_1} = 1$, solve for t_1

$$t_1 = \frac{-\log 1500}{-\log 2/3} \simeq -18 \text{ weeks.}$$

so outbreak probably started a few months ago.

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 3

Start from $\frac{dx}{dt} = (B - Dx)x$

notice that $(B-Dx)$ bit gives $x = B/D$ as equilibrium. We want it to be $\hat{x} = 1$, so set $\hat{x} = \frac{D}{B}x$. Sub in $x = \frac{B}{D}\hat{x}$:

$$\frac{d}{dt}\left(\frac{B}{D}\hat{x}\right) = \left(B - D\left(\frac{B}{D}\hat{x}\right)\right)\left(\frac{B}{D}\hat{x}\right)$$

tidy: $$\frac{d\hat{x}}{dt} = \left[B - B\hat{x}\right]\hat{x} = B\hat{x}\left[1-\hat{x}\right]$$

so we have desired form, $\alpha = B$. Drop hats.

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 4

First establish two useful identities:

- $$\frac{1}{2} + \tanh\frac{x}{2} = \frac{1}{2} + \frac{1}{2}\,\frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}} = \frac{e^{x/2}}{e^{x/2} + e^{-x/2}} = \frac{e^{x}}{e^{x}+1}$$

- $$\frac{1}{2} + \coth\frac{x}{2} = \frac{e^{x}}{e^{x}-1}$$ similarly.

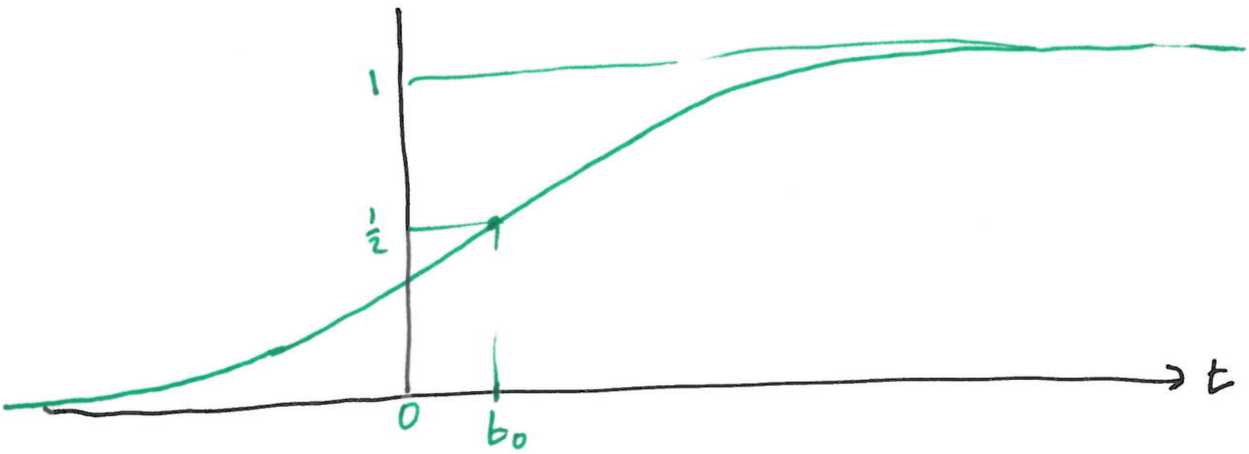
Given solution was $x = \dfrac{x_0 e^{\alpha t}}{(1-x_0) + x_0 e^{\alpha t}}$

If $\underline{\underline{x_0 < 1}}$ $(1-x_0) > 0$, divide top and bottom:

$$x = \frac{\frac{x_0}{1-x_0} e^{\alpha t}}{1 + \frac{x_0}{1-x_0} e^{\alpha t}} = \frac{e^{X}}{1 + e^{X}} \quad \text{with } X = \alpha t + \underbrace{\log\frac{x_0}{1-x_0}}_{\text{real as } x_0 \not< 1}$$

we would like $X = \alpha(t - t_0)$ so set $t_0 = -\frac{1}{\alpha}\log\frac{x_0}{1-x_0}$.

t_0 is when $x = \frac{1}{2}$ (could note $t_0 > 0 \Leftrightarrow x_0 < \frac{1}{2}$!)



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Exercise 4 ctd

If $\underline{\underline{x_0 > 1}}$ avoid taking log of negative by

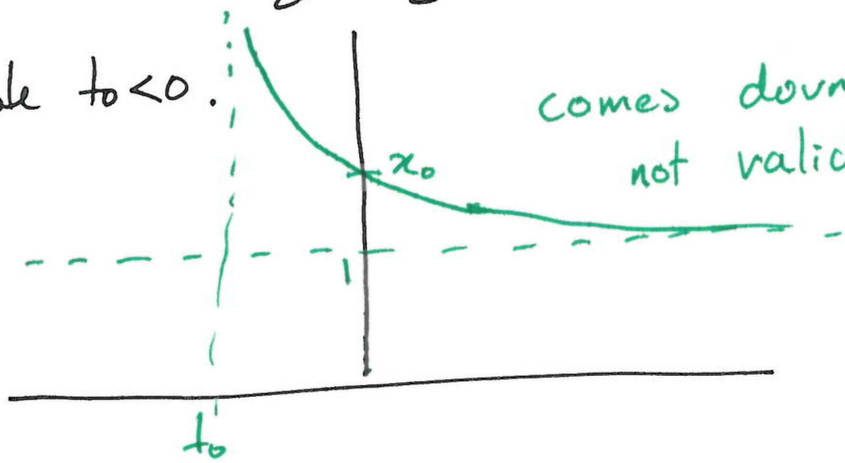
writing $$x = \frac{\left(\frac{-x_0}{1-x_0} e^{\alpha t}\right)}{-1 + \left(\frac{-x_0}{1-x_0} e^{\alpha t}\right)} = \frac{e^X}{e^X - 1}$$

with $$X = \alpha\left(t + \frac{1}{\alpha}\log\left(\frac{-x_0}{1-x_0}\right)\right) = \alpha(t - t_0)$$

so here $$t_0 = -\frac{1}{\alpha}\log\left(\frac{-x_0}{1-x_0}\right) = -\frac{1}{\alpha}\log\frac{x_0-1}{x_0}$$

and $$x = \frac{1}{2} + \frac{1}{2}\coth\frac{\alpha}{2}(t - t_0).$$

note $t_0 < 0$.



comes down from ∞ at t_0.
not valid for $t < t_0$!

note, for $x_0 = 0$ $\quad x = 0 \ \forall t$

$x_0 = 1$ $\quad x = 1 \ \forall t$ ..

so all $x_0 \geq 0$ $\underline{\text{done}}$.

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 5

Logistic : $\frac{dx}{dt} = \underbrace{\alpha x(1-x)}_{f(x)}$

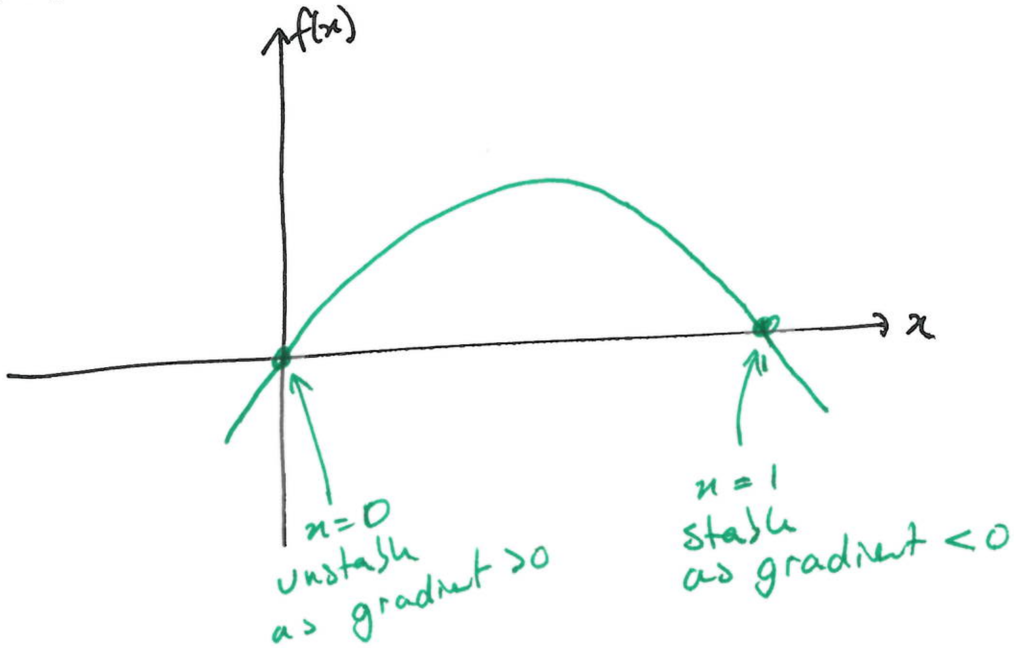
FP $f(x)=0$: $x=0$ or $x=1$

Stability $f'(x) = \alpha(1-2x)$

at $x^*=0$: $f'(x^*) = \alpha$ >0 UNSTABLE

$x^*=1$: $f'(x^*) = -\alpha$ <0 STABLE .

Even lazier :



Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 6.

start from $\dfrac{\partial \eta}{\partial \hat{t}} = -\eta(\hat{t} - \alpha T)$

try $\eta = \eta_0 e^{s\hat{t}}$: $\quad \eta_0 s e^{s\hat{t}} = -\eta_0 e^{s(\hat{t} - \alpha T)}$

cancel: $\quad s = -e^{-s\alpha T}$

rewrite: $\quad \alpha s = -\alpha e^{-\alpha s T}$

which is same as lectures, except αs instead of s. So solutions s will be rescaled by α, but as $\alpha > 0$, this doesn't affect sign of real part. Stability will work out the same.

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 7

As notes, but hope you did better!

Exercise 8

Start at $\mathcal{E}'(t) = (1-x^*)\,\mathcal{E}(t-a) - \mathcal{E}(t)$

- Set $\mathcal{E}(t) = e^{st}$: $s = (1-x^*)e^{-as} - 1$.
- Set $s = \sigma + i\omega$ (σ, ω real) :

$$\begin{cases} \sigma = (1-x^*)\, e^{-a\sigma} \cos a\omega - 1 & (\mathrm{Re}) \\ \omega = (1-x^*)\, e^{-a\sigma}(-\sin a\omega) & (\mathrm{Im}) \end{cases}$$

Exercise 9

Consider (Re) above, trying to exclude possibility $\sigma > 0$.

$$\sigma = \underbrace{\overset{\text{make } |1-x^*|<1 \text{ ?}}{(1-x^*)}\ \overset{<1 \text{ if } \sigma>0}{e^{-a\sigma}}\ \overset{|\cos|\le 1}{\cos a\omega}}_{\text{If we could get this } <1\text{, then RHS} < 0.} - 1$$

so check $1-x^* = 1 - \log\frac{b}{a}$. if $b < e^2 a$, $\log\frac{b}{a} < 2$ $\Leftrightarrow |1-x^*| < 1$. (know $x^* > 0$).

so this is exactly ~~to~~ $\Leftrightarrow |1-x^*| < 1$.

Can finish like this: if $|1-x^*| < 1$, <u>assume $\sigma > 0$</u> :

$$\mathrm{RHS} = \underbrace{(1-x^*)}_{|\cdot|<1}\ \underbrace{e^{-a\sigma}}_{|\cdot|<1}\ \underbrace{\cos a\omega}_{|\cdot|\le 1} - 1 \quad < 0$$

$$\mathrm{LHS} = \sigma \quad > 0$$

$\#$ contradiction.

so can't have $\sigma > 0$.

STABLE.

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 10

Got to $y'(t) = -\underbrace{\alpha f(c^*)}_{=A} y(t) \; -\underbrace{\alpha c^* f'(c^*)}_{=B} y(t-T)$

and also have $\alpha c^* f(c^*) = 1$ (*)

A easy. $A = \alpha f(c^*) = \frac{1}{\alpha}$ by (*).

B trickier as we have $f'(c^*)$. Cannot differentiate (*) as it is constants! Need to go back to $f(c)$ function:

$f(c) = \frac{c^m}{1+c^m}$ so $f'(c) = \frac{mc^{m-1}}{(1+c^m)^2} = \frac{m}{c} \frac{c^m}{1+c^m} \frac{1}{1+c^m}$

$= \frac{m}{c} f(c) \left[1 - f(c) \right]$

now can sub c^* into this:

$f'(c^*) = \frac{m}{c^*} f(c^*) \left(1 - f(c^*)\right)$

so $B = \alpha c^* f'(c^*) = \alpha m f(c^*)(1 - f(c^*))$

and use $f(c^*) = \frac{1}{\alpha c^*}$: $B = \frac{m}{c^*}\left(1 - \frac{1}{\alpha c^*}\right)$.

Finally note $\alpha c^* = \frac{1}{f(c^*)} > 1$

so $1 - \frac{1}{\alpha c^*} > 0$, $B > 0$.

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 11

got $B^2 = T^{-2} g(A\tau) + A^2$, $\quad A = \frac{1}{c^*}$, $B = \frac{m}{c^*}\left(1 - \frac{1}{\alpha c^*}\right)$

so

$$\frac{m^2}{c^{*2}}\left(1 - \frac{1}{\alpha c^*}\right)^2 = T^{-2} g(A\tau)^2 + \frac{1}{c^{*2}}$$

$$m^2\left(1 - \frac{1}{\alpha c^*}\right)^2 = 1 + \frac{c^{*2}}{T^2} g(A\tau)^2.$$

and know $g(A\tau) \in \left(\frac{\pi}{2}, \pi\right)$ i.e $g^2(A\tau) \in \left(\frac{\pi^2}{4}, \pi^2\right)$

and result follows.

> **Solutions to exercises** *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 12

$$\phi(\gamma) = \int_0^\infty b(a)\, e^{-\gamma a}\, e^{-\int_0^a \mu(s)\, ds}\, da.$$ from lecture.

- $b(a) \equiv b$, $\mu(a) \equiv \mu$.

Then $$\phi(\gamma) = \int_0^\infty b\, e^{-\gamma a}\, e^{-\mu a}\, da$$

$$= \frac{-b}{\mu+\gamma}\left[e^{-(\gamma+\mu)a}\right]_0^\infty = \underline{\frac{b}{\mu+\gamma}}.$$

- $\phi(0) = \frac{b}{\mu}$ b is birth rate, $\frac{1}{\mu}$ mean lifetime.

So $b \times \frac{1}{\mu}$ is offspring per lifetime. ✓

- $\phi(\gamma) = 1 \Leftrightarrow \frac{b}{\mu+\gamma} = 1$ i.e. $\gamma = b - \mu$.

- if $b > \mu$ growth. } not surprising.
  if $b < \mu$ decay

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 13

got $x_1, x_2 = \frac{1}{2\alpha}\left((1+\alpha) \pm \sqrt{(1+\alpha)(\alpha-3)}\right)$

x_1 has $+$ say.
x_2 has $-$

$f(x) = \alpha x(1-x)$

so $f(x_1) = \alpha \frac{1}{2\alpha}\left((1+\alpha)+\sqrt{}\right)\left(1 - \frac{1}{2\alpha}(1+\alpha) - \frac{\sqrt{}}{2\alpha}\right)$

$= \frac{1}{4\alpha}\left((1+\alpha)+\sqrt{}\right)\left(\underbrace{2\alpha-\alpha-1}_{\alpha-1} - \sqrt{}\right)$

$= \frac{1}{4\alpha}\left(\alpha+(1+\sqrt{})\right)\left(\alpha-(1+\sqrt{})\right)$

$= \frac{1}{4\alpha}\left(\alpha^2 - 1 - 2\sqrt{} - (1+\alpha)(\alpha-3)\right)$

$= \frac{1}{4\alpha}\left(\cancel{\alpha^2} - \cancel{\alpha^2} + 2\alpha + 2 - 2\sqrt{}\right)$

$= \frac{1}{2\alpha}\left((1+\alpha) - \sqrt{(1+\alpha)(\alpha-3)}\right) = x_2$

just switch all $\sqrt{\cdots} \to -\sqrt{\cdots}$
to show $f(x_2) = x_1$.

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 14

Lectures: x_1, x_2 solve $\alpha^2 x^2 - \alpha(1+\alpha)x + (1+\alpha) = 0$.

$$\frac{d}{dx} f(f(x))\Big|_{x_1} = \frac{d}{dx} f(f(x))\Big|_{x_2} = \alpha^2(1-2x_1)(1-2x_2)$$

quadratic for x_1, x_2 gives $x_1 + x_2 = \frac{1+\alpha}{\alpha}$, $x_1 x_2 = \frac{1+\alpha}{\alpha^2}$

$$\begin{aligned}\frac{d}{dx} f^2 &= \alpha^2(1-2x_1)(1-2x_2)\\ &= \alpha^2\left[1 - 2(x_1+x_2) + 4x_1x_2\right]\\ &= \alpha^2 - 2\alpha(1+\alpha) + 4(1+\alpha) = \underline{-\alpha^2 + 2\alpha + 4}.\end{aligned}$$

modulus $=1$?

$-\alpha^2 + 2\alpha + 4 = 1 \Rightarrow \alpha^2 - 2\alpha - 3 = 0 \quad \alpha = -1$ or 3

$\alpha = 3$ is when p2 appeared

or

$-\alpha^2 + 2\alpha + 4 = -1 \Rightarrow \alpha^2 - 2\alpha - 5 = 0$: $\underline{\alpha = 1 \pm \sqrt{6}}$

$1-\sqrt{6} < 0$, so $\underline{1+\sqrt{6} \text{ is when P2 goes unstable}}$

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 15

Note: x_1, y_1 solve $[\ ]=0$ terms, so good idea to leave them intact. Will also use $x_1+y_1=x_0$

$$J=\begin{pmatrix} [\ \ ]+x\left(\frac{+y}{(x+y)^2}-1\right) & x\left(\frac{-x}{(x+y)^2}-1\right) \\ -y & [\ \ ]+y(-1) \end{pmatrix}$$

sub in x_1, y_1:

$$J\Big|_{x_1,y_1}=\begin{pmatrix} x_1\left(\frac{y_1}{x_0^2}-1\right) & x_1\left(-\frac{x_1}{x_0^2}-1\right) \\ -y_1 & -y_1 \end{pmatrix}$$

Looking for saddle, so check det:

$$\det\left(J\Big|_{FP}\right)=-x_1y_1\left(\left(\frac{y_1}{x_0^2}\cancel{-1}\right)+\left(\frac{x_1}{x_0^2}\cancel{+1}\right)\right)$$

$$=-x_1y_1\,\frac{x_1+y_1}{x_0^2}=-\frac{x_1y_1}{x_0}<0$$

so it is a saddle. (no need to check Tr).

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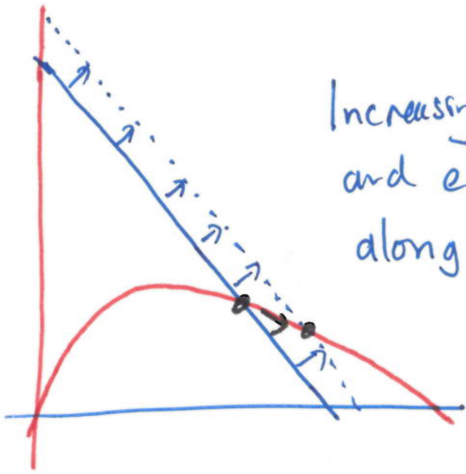
Exercise 16

Hmm... μ: Extra shortening of lifespan due to wolbachia. $\mu > 1$
λ: reduced rate of egg production . . . $\lambda < 1$

make wolbachia less pathogenic to mosquitos to bring λ or μ back towards 1. Which helps invasion more?

$x_0 = 1 - \frac{d}{r}$, $y_0 = \lambda - \mu \frac{d}{r}$ .. so can't change x_0 but can increase y_0 towards x_0.

Null clines



Increasing y_1 moves one nullcline only, and edges the saddle point (x_1, y_1) along and down towards $(x_0, 0)$

GOOD .. smaller introduction of infected mosquitos needed.

$y_0 = \lambda - \mu \frac{d}{r}$ .. well r = egg rate of uninfected
d = death rate of uninfected .. $\frac{1}{d}$ = lifetime

so $\frac{r}{d}$ = mean number of eggs per mosquito which ought to be well above 1.

So $\frac{d}{r} << 1$ so looks like λ might have more impact on y_0, until it is as high as possible, then worth a look at μ.

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Exercise 17

Start from
$$\dot{N} = N(a - bP)$$
$$\dot{P} = P(cN - d)$$

I will do two phases...

(1) move FP to 1, 1. i.e $P = \frac{a}{b}v$, $N = \frac{d}{c}u$

$$\left.\begin{aligned} \frac{d}{c}\dot{u} &= \frac{d}{c}u\left(a - b\frac{a}{b}v\right) \\ \frac{a}{b}\dot{v} &= \frac{a}{b}v\left(c\frac{d}{c}u - d\right) \end{aligned}\right\} \qquad \begin{aligned} \dot{u} &= a\,u(1-v) \\ \dot{v} &= d\,v(u-1) \end{aligned}$$

(2) get rid of a in $\dot{u}$ eqn. $\frac{d}{dt} = a\frac{d}{d\tau}$ $(\tau = at)$

$$\left.\begin{aligned} a\frac{du}{d\tau} &= au(1-v) \\ a\frac{dv}{d\tau} &= dv(u-1) \end{aligned}\right\} \qquad \begin{aligned} u' &= u(1-v) \\ v' &= -\frac{d}{a}v(1-u) \end{aligned}$$

so set $\alpha = \frac{d}{a}$ and we are done.

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 18

FPs $\left.\begin{array}{l}\dot{u}=0\\ \dot{v}=0\end{array}\right\}$ $\quad u=0$ or $[1-v-\varepsilon_u u]=0$

$v=0$ or $[1-u+\varepsilon_v v]=0$.

If $u=0$: $\quad v=0 \quad (0,0)$ ① $\quad$ or $\quad v=\frac{-1}{\varepsilon_v}$ × want $v>0$.

if $1-v-\varepsilon_u u=0$: $\quad v=0 \quad \left(\frac{1}{\varepsilon_u},0\right)$ ② $\quad$ or $\quad 1-u-\varepsilon_v v=0$.

③ $\left.\begin{array}{l}1-v-\varepsilon_u u=0\\ 1-u+\varepsilon_v v=0\end{array}\right\}$ $\quad \begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}\varepsilon_u & 1\\ 1 & -\varepsilon_v\end{pmatrix}^{-1}\begin{pmatrix}1\\1\end{pmatrix}$

$$=\frac{-1}{1+\varepsilon_u\varepsilon_v}\begin{pmatrix}-\varepsilon_v & -1\\ -1 & \varepsilon_u\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$

so $\begin{pmatrix}u^*\\v^*\end{pmatrix}=\frac{1}{1+\varepsilon_u\varepsilon_v}\begin{pmatrix}1+\varepsilon_v\\ 1-\varepsilon_u\end{pmatrix}$ as required.

$\varepsilon_u, \varepsilon_v$ small and positive

so $u^*=\frac{1+\varepsilon_v}{1+\varepsilon_u\varepsilon_v}>\frac{1+\varepsilon_u\varepsilon_v}{1+\varepsilon_u\varepsilon_v}=1$ as $\varepsilon_u<1$

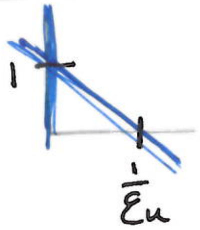
$v^*=\frac{1-\varepsilon_u}{1+\varepsilon_u\varepsilon_v}<\frac{1+\varepsilon_u\varepsilon_v}{1+\varepsilon_u\varepsilon_v}=1$, $v^*>0$ as $\varepsilon_u<1$.

so (u^*,v^*) has both present.

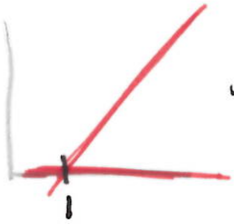
> **Solutions to exercises** *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 19

$\dot{u} = 0$ $\quad u = 0$ or $1 - v - \varepsilon_u u = 0$

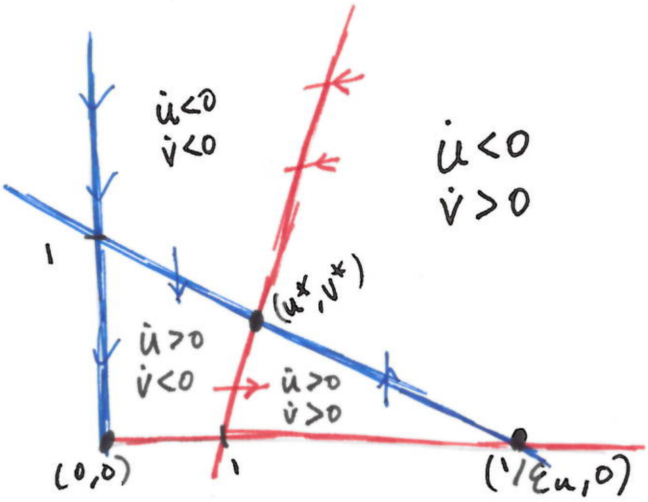


$\dot{v} = 0$ $\quad v = 0$ or $1 - u + \varepsilon_v v = 0$

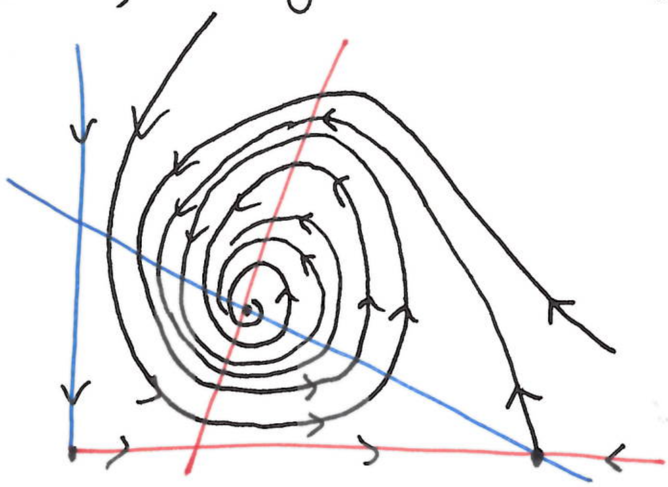


$\varepsilon_u \ll 1$ so $1 < \frac{1}{\varepsilon_u}$

Combine and note signs of $\dot{u}, \dot{v}$:



Draw in trajectories, knowing it is close to cycles but now u^*, v^* is stable:



Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 20 (optional, for dyn sys really)

We have a constant function $H(u,v)$ for $\varepsilon_u = \varepsilon_v = 0$

$$H(u,v) = \alpha(\log u - u) + (\log v - v)$$

so good idea to start there. Flip sign to make it min somewhere, want min at u^*, v^* so modify:

$$F(u,v) = \alpha(u - u^* \log u) + (v - v^* \log v)$$

~~and~~ check $\frac{d}{dt}F$:

$$\frac{d}{dt}F = \alpha\left(1 - \frac{u^*}{u}\right)u' + \left(1 - \frac{v^*}{v}\right)v'$$

$$= \alpha(u-u^*)\left[1 - (v + \varepsilon_u u)\right] - \alpha(v - v^*)\left[1 - u + \varepsilon_v v\right]$$

$1 = u^* - \varepsilon_v v^*$

note $1 = v^* + \varepsilon u^*$

$$= \alpha(u-u^*)\left((v^* + \varepsilon_u u^*) - (v + \varepsilon_u u)\right) - \alpha(v-v^*)\left((u^* - \varepsilon_v v^*) - (u - \varepsilon_v v)\right)$$

tidy fest $= -\varepsilon_u \alpha (u-u^*)^2 - \varepsilon_v \alpha (v-v^*)^2 \leq 0$ yay!

(and actually strict: $\frac{d}{dt}F < 0$, except $=0$ at u^*, v^*.)

Just need to shift by const so $V(u^*, v^*) = 0$

set $V(u,v) = F(u,v) - F(u^*, v^*)$

$$= \alpha\left((u-u^*) - u^* \log \frac{u}{u^*}\right) + \left((v - v^*) - v^* \log \frac{v}{v^*}\right)$$

valid Lyapunov fn (strict!) for all of $u, v > 0$.

> **Solutions to exercises** *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise ~~25~~ 21

Can see want to adsorb $k_1 e_0$ in time rescaling. Could do in two phases, but I'm just going to go for it in one:

$$\frac{d}{dt} = k_1 e_0 \frac{d}{d\tau}, \quad s = s_0 u, \quad c = e_0 v \ldots \qquad \left(\frac{d}{d\tau} = {}'\right)$$

$$\dot{s} = -k_1 e_0 s + (k_1 s + k_2) c$$

$$\dot{c} = +k_1 e_0 s - (k_1 s + k_2 + k_3) c$$

And hold tight...

$$\cancel{k_1} \cancel{e_0} \cancel{s_0} u' = -\cancel{k_1} \cancel{e_0} \cancel{s_0} u + (\cancel{k_1} \cancel{s_0} u + \underset{k_1 s_0}{\underline{k_2}}) \cancel{e_0} v$$

$$\cancel{k_1} \cancel{e_0} \underset{s_0}{\underline{e_0}} v' = + \cancel{k_1} \cancel{e_0} \cancel{s_0} u - (\cancel{k_1} \cancel{s_0} u + \underset{s_0 k_1}{\underline{k_2}} + \underset{s_0 k_1}{\underline{k_3}}) \cancel{e_0} v$$

method of coloured pens!
- cancel in red k_1
- cancel s_0 in blue
- cancel e_0 in green..

What's left?

$$u' = -u + \left(u + \frac{k_2}{k_1 s_0}\right) v \qquad \frac{k_2}{k_1 s_0} = \mu - \lambda \quad \text{so} \quad \lambda = \frac{k_3}{s_0 k_1}$$

$$\underbrace{\frac{e_0}{s_0}}_{\varepsilon} v' = +u - \left(u + \underbrace{\frac{k_2 + k_3}{s_0 k_1}}_{\mu}\right) v$$

Finally

$$u' = -u + (u + \mu - \lambda) v$$

$$\varepsilon v' = +u - (u + \mu) v$$

and $\varepsilon = \frac{e_0}{s_0} \ll 1$ by assumption $e_0 \ll s_0$

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 22

hmm.. $\frac{dp}{dt} = k_3 c = k_3 e_0 v = k_3 e_0 \frac{u}{u+\mu}$. on $v' \simeq 0$.

$s = s_0 u$

so $\left(\frac{dp}{dt}\right)^{-1} = \frac{1}{k_3 e_0} \frac{u+\mu}{u} = \frac{1}{k_3 e_0}\left(1 + \frac{\mu s_0}{s}\right)$ = linear in $\frac{1}{s}$.

[Why do we care? If we had some real system and could measure $\frac{dp}{dt}$ ad s at differt timepoints, we could plot $\left(\frac{dp}{dt}\right)^{-1}$ ad s^{-1} and check to see if points are near! a line. This comes from the world of biochemistry and enzyme kinetics (seach online for Michaelis-Menten..).]

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 23 (2017 version)

Work process by process. Rates are products of things at start of arrows times parameter

$$\dot{x}_0 = -k_1 x_0 c^2 + \mu_1 x_1 + \lambda_1 x_1$$
$$\dot{x}_1 = +k_1 x_0 c^2 - k_2 x_1 c^2 - \mu_1 x_1 + \mu_2 x_2 - \lambda_1 x_1 + \lambda_2 x_2$$
$$\dot{x}_2 = +k_2 x_1 c^2 - \mu_2 x_2 - \lambda_2 x_2$$
$$\dot{c} = -2k_1 x_0 c^2 - 2k_2 x_1 c^2 + 2\mu_1 x_1 + 2\mu_2 x_2$$
$$\dot{p} = +\lambda_1 x_1 + \lambda_2 x_2$$

---

invariant sum.. can see each process adds and removes one x_i, so $x_0 + x_1 + x_2$ should be constant.

[could reduce by two dimensions then.. just eject p as nothing depends on it and eliminate one of x_i using invariant eg $x_0 = \text{const} - x_1 - x_2$]

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 24

Just consider $I(S)$, differentiate wrt S

or go back to $\frac{dI}{dS} = \frac{\dot{I}}{\dot{S}} = -1 + \frac{N}{R_0}\frac{1}{S}$

which is zero at $S = N/R_0$. But why? Is there an intuitive explanation?

Max I is at the point when the epidemic 'turns over', and the number of new cases starts decreasing. This is when the effective R_0 (R_{eff} or r or whatever) hits 1.

$$R_{eff} = \frac{\beta}{\nu} S = \frac{R_0}{N} S \quad .. \quad \text{so } S = \frac{N}{R_0} \text{ is the}$$

critical population size for the disease to transmit to more than one new case, on average. So, it is also when I maxes out, and then decreases.

Exercise 25

Just check notes.

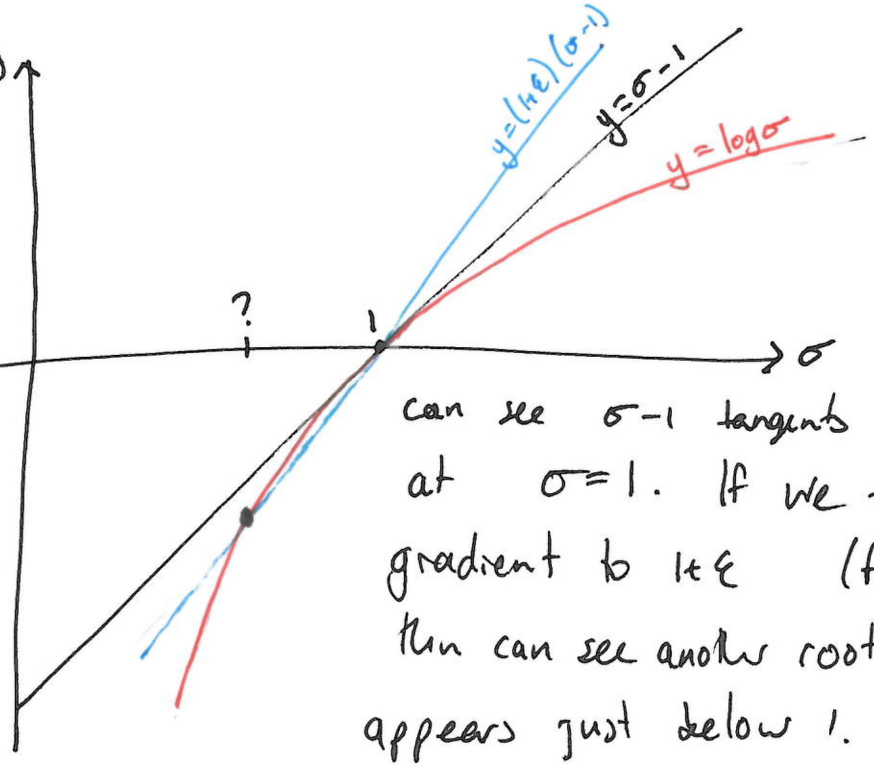
Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 2.6

Set $R_0 = 1+\varepsilon$, use $*_2$: $\quad \sigma - \frac{1}{R_0}\log\sigma = 1$

strategic rearrangement: $\quad \log\sigma = (\sigma-1)(1+\varepsilon) \qquad (*_3)$

not necessary to do this, but hope this makes it clear why we can expect a root just below 1.



can see $\sigma-1$ tangents $\log\sigma$ at $\sigma=1$. If we tilt gradient to $1+\varepsilon$ (for small $\varepsilon>0$) then can see another root for σ appears just below 1.

Set $\sigma = 1-\delta$ and expand $*_3$ in small δ:

$$\log(1-\delta) = -\delta(1+\varepsilon)$$

$$-\delta - \tfrac{1}{2}\delta^2 + O(\delta^3) = -\delta - \delta\varepsilon$$

(correction in pink!)

$\div\,\delta$: $\quad -1 + \frac{1}{2}\delta + O(\delta^2) = -1 + \varepsilon \qquad$ so $\varepsilon = \frac{1}{2}\delta + O(\delta^2)$

i.e. $\quad \underline{\delta = 2\varepsilon} + O(\varepsilon^2)$.

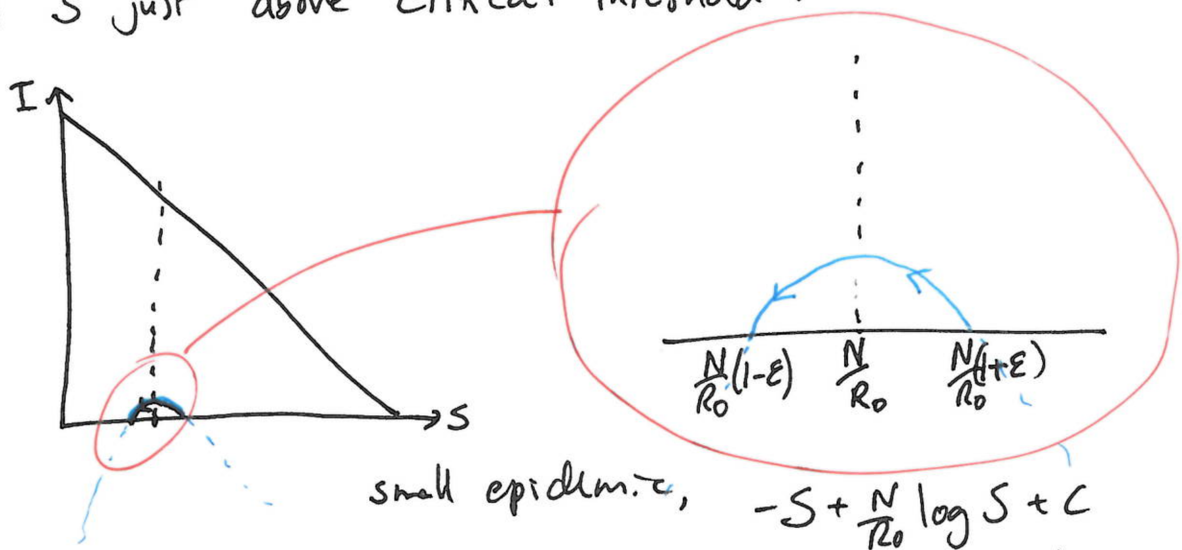
ie if S just over critical population size, epidemic size is twice the excess: it overshoots (see next page)

ctd..

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Ex 26 bonus notes

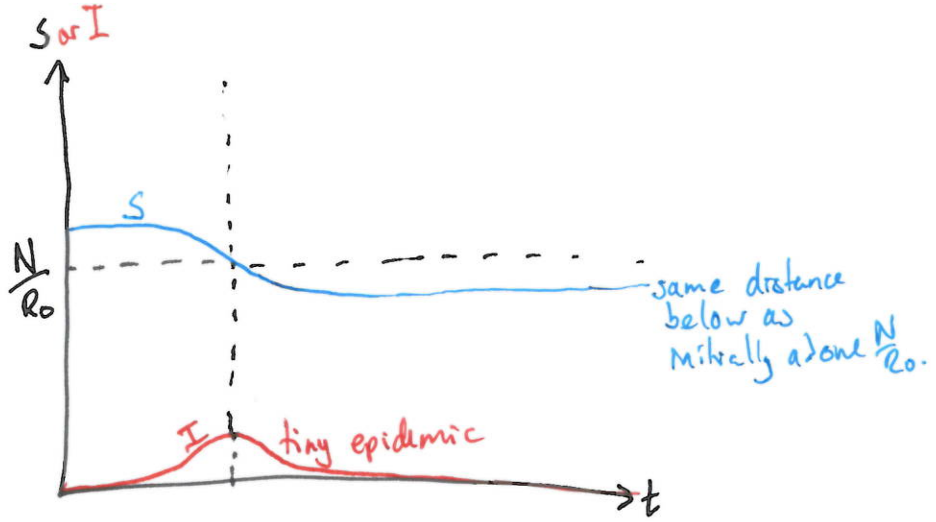
Actually there's a simple way to see why the factor 2: rather than R_0, suppose this is effective R and we have initial S just above critical threshold:



small epidemic, $-S + \frac{N}{R_0}\log S + C$

only just above axis.. locally quadratic

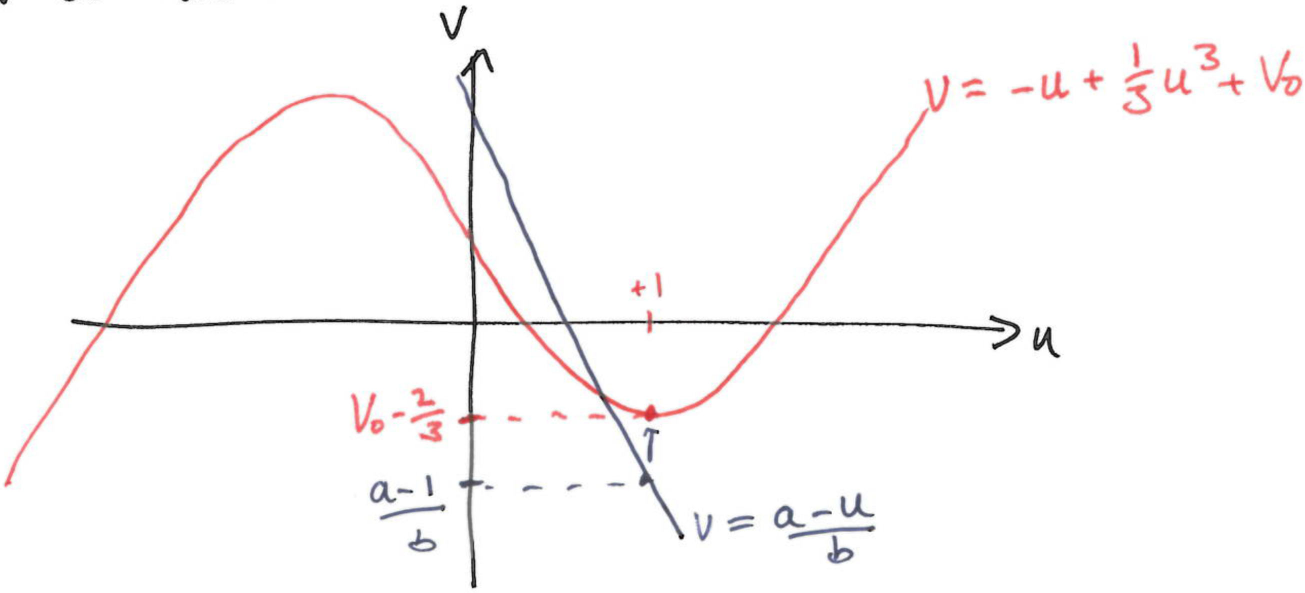
.. quadratic, max halfway between roots.



Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 27

For repeated firing, want min of new $\dot{v}=0$ nullcline to be above $\dot{u}=0$ nullcline. ie



so condition is $v_0 - \frac{2}{3} > \frac{a-\cancel{b}1}{b}$

ie $v_0 > \frac{a-1}{b} + \frac{2}{3}$

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 28

We have $\dot{p}_n = \lambda(p_{n-1} - p_n)$ and $p_{-1}(t) = 0 \ \forall t$
$p_n(0) = 0 \quad n \geq 1$
$p_0(0) = 1$

Base case $\quad \dot{p}_0 = -\lambda p_0, \ p_0(0) = 1 \ \Rightarrow \ p_0(t) = \underline{e^{-\lambda t}}$

Induct for $n \geq 1$ $\quad$ assume $p_{n-1} = \dfrac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}$

then $$\dot{p}_n = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} - \lambda p_n$$

rearrange: $$\underbrace{(\dot{p}_n + \lambda p_n) e^{\lambda t}}_{\frac{d}{dt}\left(e^{\lambda t} p_n\right)} = \frac{\lambda^n t^{n-1}}{(n-1)!}$$

Integrate: $$e^{\lambda t} p_n = \frac{\lambda^n t^n}{n!} + C$$

and $p_n(0) = 0$ for $t \geq 1$ so $C = 0$, hence:

$$p_n = \frac{\lambda^n t^n}{n!} e^{-\lambda t}$$ as required.

This is just Poisson distribution, parameter (λt)

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 29

See lecture notes to check!

Exercise 30

Know $\langle N \rangle = \phi_s \big|_{s=1}$ and here $\phi = e^{\frac{\lambda}{\beta}(1-e^{-\beta t})(s-1)}$

so $\langle N \rangle = \underline{\frac{\lambda}{\beta}(1-e^{-\beta t})}$ ✓

Also know $\mathrm{Var}(N) = \phi_{ss}\big|_{s=1} + \langle N \rangle - \langle N \rangle^2$

$$\text{so } \mathrm{Var}(N) = \cancel{\left(\frac{\lambda}{\beta}(1-e^{-\beta t})\right)^2} + \left(\frac{\lambda}{\beta}(1-e^{-\beta t})\right) - \cancel{\left(\frac{\lambda}{\beta}(1-e^{-\beta t})\right)^2}$$

$$= \underline{\frac{\lambda}{\beta}(1-e^{-\beta t})}$$ ✓

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise ~~32~~ 31

Master equation (lectures): $\dot{p}_n = \lambda\left(p_{n-M} - p_n\right) + \beta\left[(n+1)p_{n+1} - np_n\right]$

which holds for $n \geq 0$ if we say $p_n \equiv 0$ for $n<0$.

$\phi_t = \frac{\partial}{\partial t}\sum s^n p_n = \sum s^n \dot{p}_n$ and sub in for $\dot{p}_n$:

$$\phi_t = \lambda \sum s^n p_{n-M} - \lambda \sum s^n p_n + \beta \sum s^n (n+1) p_{n+1} - \beta \sum s^n n p_n$$

adjust index in each sum to get p_n :

$$\phi_t = \lambda \underbrace{\sum s^{n+M} p_n}_{s^M \phi} - \lambda \underbrace{\sum s^n p_n}_{\phi} + \beta \underbrace{\sum s^{n-1} n p_n}_{\phi_s} - \beta \underbrace{\sum s^n n p_n}_{s\phi_s}$$

$$\phi_t = \lambda\left(s^M - 1\right)\phi - (s-1)\beta\phi$$

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise ~~33~~ 32 (long way)

$\dot{X} = -2\beta X + a + b e^{-\beta t}$ has solution:

$$X = A e^{-2\beta t} + \frac{a}{2\beta} + b e^{-\beta t} \quad \text{(linear DE, just solve)}$$

Equation for $\langle N^2 \rangle^{\cdot}$:

$$\frac{d}{dt}\langle N^2 \rangle = \lambda M^2 + (2\lambda M + \beta)\langle N \rangle - 2\beta \langle N^2 \rangle \quad \text{(lecture)}$$

$$= \lambda M^2 + (2\lambda M + \beta)\underbrace{\left[\frac{\lambda M}{\beta}(1 - e^{-\beta t})\right]}_{\text{subbing in.}} - 2\beta \langle N^2 \rangle$$

so $a = \lambda M^2 + 2\frac{\lambda^2}{\beta}M^2 + \lambda M$, $\quad b = -\left(\frac{2\lambda^2 M^2}{\beta} + \lambda M\right)$

in above form to get solution above.

Let $t \to \infty$, just get $\langle N^2 \rangle = \frac{a}{2\beta} = \underline{\frac{\lambda}{2\beta}\left(1 + \frac{2\lambda}{\beta}\right)M^2 + \frac{\lambda}{2\beta}M}$ ✓

Also as $t \to \infty$, $\langle N \rangle = \frac{\lambda M}{\beta}$.

so $\text{Var}(N) = \langle N^2 \rangle - \langle N \rangle^2 = \left(\frac{\lambda}{2\beta} + \cancel{\frac{\lambda^2}{\beta^2}}\right)M^2 + \frac{\lambda}{2\beta}M - \cancel{\left(\frac{\lambda M}{\beta}\right)^2}$

$$= \frac{\lambda}{2\beta}M(M+1)$$

so if $M \gg 1$, $\underline{\text{Var}(N) \simeq \frac{\lambda}{2\beta}M^2}$

Also OK from start to seek stationary $\langle N \rangle$, $\langle N^2 \rangle$ rather than solve for all t. Could also have thrown out M term in $\langle N^2 \rangle$ early, but kept in here so details can be checked if you want.

(see next page for this)

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 32 fast way

Work at SS. Have $\langle N \rangle = \frac{\lambda M}{\beta}$

'DE' for $\langle N^2 \rangle$ at steady state

$$0 = \lambda M^2 + (2\lambda M + \beta)\langle N \rangle - 2\beta \langle N^2 \rangle$$

sub in $\langle N \rangle$, solve for $\langle N^2 \rangle$:

$$\langle N^2 \rangle = \frac{\lambda}{2\beta} M^2 + \frac{\lambda^2 M^2}{\beta^2} + \frac{\lambda M}{2\beta}$$

could ignore from here: $M \gg 1$

so $\mathrm{Var}(N) = \langle N^2 \rangle - \langle N \rangle^2 = \frac{\lambda}{2\beta} M^2 + \frac{\lambda M}{2\beta}$

for large M it is mainly the M^2 term

$$\mathrm{Var}(N) \simeq \frac{\lambda}{2\beta} M^2$$

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise ~~34~~ 33

Just keep doing same thing:
- use master equation
- shift index to make terms in $p_{m,n}$

$$\dot{\langle M^2\rangle} = \sum m^2 \dot{p}_{m,n}$$

$$= \lambda_1\left(\langle (M+1)^2\rangle - \langle M^2\rangle\right) + \beta_1\left(\langle M(M-1)^2\rangle - \langle M^3\rangle\right)$$

$$+ \lambda_2(0) + \beta_2(0)$$

$$= \lambda_1\left(2\langle M\rangle + 1\right) + \beta_1\left(-2\langle M^2\rangle + \langle M\rangle\right).$$

$$\dot{\langle N^2\rangle} = \sum n^2 \dot{p}_{m,n}$$

$$= \lambda_1(0) + \beta_1(0) + \lambda_2\left(\langle M(N+1)^2\rangle - \langle MN^2\rangle\right)$$

$$+ \beta_2\left(\langle N(N-1)^2\rangle - \langle N^3\rangle\right)$$

$$= \lambda_2\left(2\langle MN\rangle + \langle M\rangle\right) + \beta_2\left(\langle N\rangle - 2\langle N^2\rangle\right)$$

Finally the interesting one:

$$\dot{\langle MN\rangle} = \sum mn\, \dot{p}_{m,n}$$

$$= \lambda_1\left(\langle (M+1)N\rangle - \langle MN\rangle\right) + \beta_1\left(\langle M(M-1)N\rangle - \langle MN\rangle\right)$$

$$+ \lambda_2\left(\langle M^2(N+1)\rangle - \langle M^2N\rangle\right) + \beta_2\left(\langle MN(N-1)\rangle - \langle MN^2\rangle\right)$$

$$= \lambda_1\langle N\rangle - \beta_1\langle MN\rangle + \lambda_2\langle M^2\rangle - \beta_2\langle MN\rangle$$

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Ex 34

Here $W(n,1)=\lambda \ \forall n$, $W=0$ otherwise

So $A=\sum r W = 1.\lambda = \underline{\lambda}$ $\qquad B = 1^2 \times \lambda = \underline{\lambda}.$

FPE: $$\frac{\partial P}{\partial t} = -\lambda \frac{\partial P}{\partial x} + \frac{\lambda}{2}\frac{\partial^2 P}{\partial x^2}$$

but don't need, could use result from lectures that

$$\frac{d}{dt}\langle f(x)\rangle = \int f(x)\frac{\partial P}{\partial t}\,dx = \int \left(f' A + \tfrac{1}{2} f'' B\right) P\,dx$$

$$= \langle A f'(x)\rangle + \tfrac{1}{2}\langle B f''(x)\rangle$$

$$\underline{\frac{d}{dt}\langle f(x)\rangle = \lambda \langle f'(x)\rangle + \tfrac{1}{2}\lambda \langle f''(x)\rangle} \quad \text{for example 1.}$$

- $f(x)=x$: $\quad \langle x\rangle^{\circ} = \lambda \quad \Rightarrow \langle x\rangle = \lambda t + c$

  and initial condition is zero population so $\langle x\rangle = 0$ at $t=0$

  $\Rightarrow c=0 \qquad \underline{\langle x\rangle = \lambda t}$

- $f(x)=x^2$: $\quad \langle x^2\rangle^{\circ} = 2\lambda\langle x\rangle + \lambda$
  $= 2\lambda^2 t + \lambda$

  integrate: $\quad \underline{\langle x^2\rangle = (\lambda t)^2 + (\lambda t)} + \underbrace{\cancel{c}}_{=0}$ by IC again.

- $f(x)=x^3$: $\quad \langle x^3\rangle^{\circ} = 3\lambda\langle x^2\rangle + \frac{6}{2}\lambda\langle x\rangle$
  $= 3\lambda^3 t^2 + 3\lambda^2 t + 3\lambda^2 t$
  $= 3\lambda^3 t^2 + 6\lambda^2 t$

  integrate: $\quad \underline{\langle x^3\rangle = (\lambda t)^3 + 3(\lambda t)^2} \ (\cancel{+c} = 0$ by IC$)$

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Ex 3~~4~~ ctd

• $f(x) = x^4$:

$$\dot{\langle x^4 \rangle} = 4\lambda \langle x^3 \rangle + \frac{4.3}{2} \lambda \langle x^2 \rangle$$

$$= 4\lambda^4 t^3 + 12\lambda^3 t^2 + 6\lambda^3 t^2 + 6\lambda^2 t$$

$$= 4\lambda^4 t^3 + 18\lambda^3 t^2 + 6\lambda^2 t$$

integrate $\langle x^4 \rangle = (\lambda t)^4 + 6(\lambda t)^3 + 3(\lambda t)^2$ $\cancel{+C}$ $=0$ by IC.

Back to master equation (L12): $\dot{p}_n = \lambda(p_{n-1} - p_n)$

could do each directly, or do similar trick of general f :

$$\dot{\langle f(n) \rangle} = \frac{d}{dt} \sum_n f(n) p_n = \sum_n f(n) \dot{p}_n \leftarrow \text{sub in}$$

$$= \sum_n f(n) \lambda (p_{n-1} - p_n) = \lambda \sum_n p_{n-1} f(n) - \lambda \sum_n p_n f(n)$$

$$= \underline{\lambda \langle f(n+1) \rangle - \lambda \langle f(n) \rangle} = \lambda \langle f(n+1) - f(n) \rangle$$

• $f(n) = n$:

$$\dot{\langle n \rangle} = \lambda \langle (n+1) - n \rangle = \lambda$$

integrate $\langle n \rangle = \lambda t + \cancel{C}$ IC $\Rightarrow \langle n \rangle = 0$ at $t=0$ so $C = 0$.

matches $\langle x \rangle$ ✓.

• $f(n) = n^2$:

$$\dot{\langle n^2 \rangle} = \lambda \langle (n+1)^2 - n^2 \rangle$$

$$= \lambda \langle 2n + 1 \rangle = 2\lambda \langle n \rangle + \lambda$$

$$= 2\lambda^2 t + \lambda$$

integrate: $\langle n^2 \rangle = \cancel{2}\lambda^2 t^2 + \lambda t + \underset{=0}{C}$.

✓ matches $\langle x^2 \rangle$

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Ex 3 ctd²

- $f(n) = n^3$:

$$\dot{\langle n^3\rangle} = \lambda \langle (n+1)^3 - n^3 \rangle$$
$$= \lambda\, 3\langle n^2\rangle + \lambda\, 3\langle n\rangle + \lambda$$
$$= 3\lambda(\lambda^2 t^2 + \lambda t) + 3\lambda(\lambda t) + \lambda$$
$$= 3\lambda^3 t^2 + 6\lambda^2 t + \lambda$$

integrate : $\langle n^3 \rangle = \lambda^3 t^3 + 3\lambda^2 t^2 + \lambda t + \cancel{c}_{=0}$

different! not in $\langle x^3 \rangle$.

- $f(n) = n^4$:

$$\dot{\langle n^4\rangle} = \lambda \langle 4n^3 + 6n^2 + 4n + 1 \rangle$$
$$= 4\lambda\left((\lambda t)^3 + 3(\lambda t)^2 + \lambda t\right) + 6\lambda\left((\lambda t)^2 + \lambda t\right) + 4\lambda(\lambda t) + \lambda$$
$$= 4\lambda^4 t^3 + 18\lambda^3 t^2 + 14\lambda^2 t$$

integrate : $\langle n^4 \rangle = (\lambda t)^4 + 6(\lambda t)^2 + 7(\lambda t)^2 + \lambda t$

was coeff 3 in $\langle x^4 \rangle$ missing in $\langle x^4 \rangle$

So first two moments match. Third and fourth are correct to top two orders in λt then differ. Why? note typical $x \sim \langle x \rangle = \lambda t$ so x grows in time. r stays 1, so approximation $x \gg |r|$ gets better in time, so right that highest orders behave.

Approx was second order in TS, so not too surprising that two terms OK.

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 35

Start translating to $\underline{r}$, W. note $\underline{x} = (x_1, x_2)$

$x_1 =$ # wildebeest, $x_n =$ # flies

Four event types:	(W) rate	$\underline{r}$	$r_i r_j$
• Wildebeest added	λ_1	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
• Wildebeest dies	$\beta_1 x_1$	$\begin{pmatrix} -1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
• Fly added	$\lambda_2 x_1$	$\begin{pmatrix} 0 \\ +1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
• Fly dies	$\beta_2 x_2$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

so $$\underline{A} = \sum_{\underline{r}} W(\underline{x}, \underline{r})\, \underline{r} = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta_1 x_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \lambda_2 x_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \beta_2 x_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 - \beta_1 x_1 \\ \lambda_2 x_1 - \beta_2 x_2 \end{pmatrix}$$

$$\underline{B} = \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \beta_1 x_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda_2 x_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \beta_2 x_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \beta_1 x_1 & 0 \\ 0 & \lambda_2 x_1 + \beta_2 x_2 \end{pmatrix}$$

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise ~~37~~ 36

Actually $\underline{A}$ is already liner in $\underline{x}$, so no need to do local approx:

$$\underline{A} = \begin{pmatrix} \lambda_1 - \beta_1 x_1 \\ \lambda_2 x_1 - \beta_2 x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}}_{\underline{\lambda}} + \underbrace{\begin{pmatrix} -\beta_1 & 0 \\ \lambda_2 & -\beta_2 \end{pmatrix}}_{\underline{\underline{a}}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\underline{x}}$$

For $\underline{\underline{B}}$, we are considering steady state ~~at~~ near given x_1, x_2 so

sub in $\underline{\underline{B}} = \begin{pmatrix} \lambda_1 + \beta_1 x_1 & \\ & \lambda_2 x_1 + \beta_2 x_2 \end{pmatrix}$ at $x_1 = \frac{\lambda_1}{\beta_1}$, $x_2 = \frac{\lambda_1 \lambda_2}{\beta_1 \beta_2}$

$$= \underbrace{\begin{pmatrix} 2\lambda_1 & 0 \\ 0 & \frac{2\lambda_1\lambda_2}{\beta_1} \end{pmatrix}}_{= \underline{\underline{b}}} .$$

Lyapunov $\underline{\underline{a}}\,\underline{\underline{C}} + \underline{\underline{C}}\,\underline{\underline{a}}^T + \underline{\underline{b}} = 0$:

$$\begin{pmatrix} -\beta_1 & 0 \\ \lambda_2 & -\beta_2 \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix} + \text{transpose of previous} + \begin{pmatrix} 2\lambda_1 & \\ & \frac{2\lambda_1\lambda_2}{\beta_1} \end{pmatrix} = 0.$$

Pull out components:

$$-2\beta_1 C_{11} + 2\lambda_1 = 0$$

$$\lambda_2 C_{11} - \beta_2 C_{12} - \beta_1 C_{12} = 0$$

$$2\lambda_2 C_{12} - 2\beta_2 C_{22} + \frac{2\lambda_1\lambda_2}{\beta_1} = 0.$$

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 3~~B~~ ctd

Solve for C_{11}, C_{12}, C_{22}:

$$C_{11} = \frac{\lambda_1}{\beta_1} = \mathrm{Var}(x_1)$$

$$C_{12} = \frac{\lambda_1 \lambda_2}{\beta_1(\beta_1+\beta_2)} = \mathrm{cov}(x_1, x_2)$$

$$C_{22} = \frac{\lambda_1 \lambda_2}{\beta_1 \beta_2}\left(1 + \frac{\lambda_2}{\beta_1+\beta_2}\right) = \mathrm{Var}(x_2).$$

Note $C_{12} > 0$ so wildebeest and flies covary positively (not a surprize.. flies attracted by the wildebeest !).

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise ~~38~~ 37

Full problem for completeness, though only gravest mode needed:

$\frac{\partial c}{\partial x} = 0$ at $x = 0$, $c = C_1$ at $x = L$ Steady soln $c^*(x) = C_1$ (const)

basically bacteria end up at C_1 everywhere to balance $x = L$ end.

General solution .. $\frac{G'}{G} = D\frac{F''}{F}$ as before .. $c = c^* + \hat{c}$, try $\hat{c} = F(x)G(t)$.

and now want $F'(0) = 0$, $F(L) = 0$

so must be $\cos kx$ with $kL = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

i.e. $k_n = (2n-1)\frac{\pi}{2L}$ $n = 1, 2, \ldots$

$$G = e^{-\lambda_n t} \qquad \lambda_n = D k_n^2 = \frac{D(2n-1)^2\pi^2}{4L^2}$$

so full soln

$$c(x,t) = C_1 + \underbrace{a_1 \cos\frac{\pi x}{2L}}_{\text{gravest mode now.}} e^{-\frac{D\pi^2}{4L^2}t} + a_2 e^{-\frac{9\pi^2 t D}{4L^2}} \cos\frac{3\pi x}{2L} + \ldots$$

(would have been fine to spot least wiggly cos soln to BCs was $\cos\frac{\pi x}{2L}$ without doing all the detail above)

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 38

got $c = e^{\lambda t}\tilde{c}(x,t)$ and $\frac{\partial \tilde{c}}{\partial t} = D\frac{\partial^2 \tilde{c}}{\partial x^2}$ with $\tilde{c} = 0$ at $x = 0, L$.

so this can be solved exactly as previous:

$$\tilde{c}(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \sin\frac{n\pi x}{L} \qquad \lambda_n = \frac{Dn^2\pi^2}{L^2}$$

note $\lambda_1 < \lambda_2 < \lambda_3 < \dots$

i.e. $c(x,t) = e^{\lambda t}\tilde{c}(x,t) = \sum_{n=1}^{\infty} a_n e^{(\lambda-\lambda_n)t} \sin\frac{n\pi x}{L}$

Ahha! Time exponent of nth term is $\lambda - \lambda_n$ and the greatest of these is $n=1$ $\lambda - \frac{D\pi^2}{L^2}$.

If $(\lambda - \lambda_1) < 0$ then all $(\lambda - \lambda_n) < 0$ so all decay

so $\lambda - \frac{D\pi^2}{L_c^2} = 0$ $\quad \underline{L_c = \sqrt{\frac{D}{\lambda}}\,\pi}$

- If $L < L_c$, $(\lambda - \lambda_n) < 0$ $\forall n$, so $c(x,t) \to 0$ as $t \to \infty$ for <u>any</u> a_n. Bacteria doomed, diffusion leaks them out faster than they can grow.
- If $L > L_c$, at least $(\lambda - \lambda_1) > 0$, so if there is any first term (ie $a_1 \neq 1$) then that will grow and dominate

[if $a_1 = 0$, would need to check $\lambda - \lambda_2$ etc]

key idea: diffusion causes flux out ends, so can beat bacteria growth if tube short, or D large, or λ small.

> **Solutions to exercises** *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise [redacted]39

$$C = t^{\alpha} G(\xi), \quad \xi = \frac{r}{t^{\beta}} \quad \text{so} \quad \frac{\partial \xi}{\partial t} = -\beta \frac{\xi}{t}, \quad \frac{\partial \xi}{\partial r} = \frac{1}{t^{\beta}} .$$

① Get PDE to match in t:

- $$\frac{\partial c}{\partial t} = \alpha t^{\alpha-1} G + t^{\alpha} \frac{\partial \xi}{\partial t} G' = \underbrace{t^{\alpha-1}}_{t,\ \text{no}\ \xi} \underbrace{(\alpha G - \beta \xi G')}_{\xi\ \text{no}\ t.}$$

* see later for derivation if wanted.

- Quote $\nabla^2 f(r) = (n-1)\frac{f'}{r} + f''$ in n-sphericals

$$D\nabla^2 C = D t^{\alpha} \nabla^2 G\left(\frac{r}{t^{\beta}}\right) = D t^{\alpha}\left((n-1)\frac{1}{r} G'(\xi) \frac{\partial \xi}{\partial r} + \left(\frac{\partial \xi}{\partial r}\right)^2 G''(\xi)\right)$$

ok as $\frac{\partial \xi}{\partial r}$ indep r.

$$= D t^{\alpha}\left((n-1) t^{-\beta} \xi^{-1} G' t^{-\beta} + t^{-2\beta} G''\right)$$

$$= \underbrace{t^{\alpha-2\beta}}_{t\ \text{no}\ \xi} \underbrace{D\left(\frac{(n-1)}{\xi} G' + G''\right)}_{\xi\ \text{no}\ t}$$

so $\alpha - 1 = \alpha - 2\beta$

$\beta = \frac{1}{2}$ again.

also would be ok to just say $\nabla^2 \sim r^{-2} = t^{-2\beta} \xi^{-2}$

② Sort integral

$$M = \int C \, dV = \int t^{\alpha} G(\xi)\, dV = \int t^{\alpha} G(r t^{-\beta}) \, S_n(r) \, dr$$

where $S_n(r)$ is surface area of sphere in n-dim $= r^{n-1} S_n(1)$.

$$M = \int t^{\alpha} G(r t^{-\beta}) r^{n-1} S_n(1) \, dr = t^{\alpha} S_n(1) t^{+\beta n} \int G(\xi) \xi^{n-1} d\xi$$

so to be indep of time $\alpha + \beta n = 0$ $\quad \boxed{\alpha = -\frac{n}{2}, \ \beta = \frac{1}{2}}$

again, ok to just say $\int C\, dV \propto t^{\alpha} r^{n} = t^{\alpha + n\beta} \xi^{n}$.

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Ex ~~40~~ 39 pedantry

$*$.. $\nabla^2 f(r) = (n-1)\frac{f'}{r} + f''$ .. where from ??

$$r = \sqrt{x_1^2 + \dots + x_n^2} \quad \text{so} \quad \frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

and $\nabla^2 f(r) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} f(r) = \sum \frac{\partial}{\partial x_i}\left(\frac{x_i}{r} f'(r)\right)$

$$= \sum_{i=1}^{n}\left(\frac{1}{r} f'(r) - \frac{x_i}{r^2}\left(\frac{x_i}{r}\right) f'(r) + \frac{x_i}{r}\cdot\frac{x_i}{r} f''(r)\right)$$ no summation convention !

$$= n \times \frac{f'}{r} - \frac{r^2}{r^3} f' + \frac{r^2}{r^2} f''$$

$$= \underline{\frac{(n-1)}{r} f' + f''} \quad \left(= \frac{1}{r^{n-1}}\left(r^{n-1} f'\right)'\right).$$

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 40

got $$C(x,t) = 1 - \frac{1}{2}\,\mathrm{Erf}\left(\frac{x+w}{\sqrt{4Dt}}\right) + \frac{1}{2}\,\mathrm{Erf}\left(\frac{x-w}{\sqrt{4Dt}}\right)$$

so check C at $x=0$:

$$C(0,t) = 1 - \frac{1}{2}\,\mathrm{Erf}\left(\frac{W}{\sqrt{4Dt}}\right) + \frac{1}{2}\,\mathrm{Erf}\left(\frac{-W}{\sqrt{4Dt}}\right)$$

but Erf is an odd function, so
$$C(0,t) = 1 - \mathrm{Erf}\left(\frac{W}{\sqrt{4Dt}}\right)$$

"Fluoresecence half recovered" – $C = \frac{1}{2}$ at $t = 1$ (work in seconds). Window $W = 1\mu m = 10^{-6}$ (work in metres)

$\mathrm{Erf}(X) = \frac{1}{2}$? for ballpark, use $\mathrm{Erf}(X) \simeq \frac{2}{\sqrt{\pi}} X$

so $X \simeq \frac{\sqrt{\pi}}{4}$ actually not bad.. $\frac{\sqrt{\pi}}{4} \simeq 0.44\ldots$

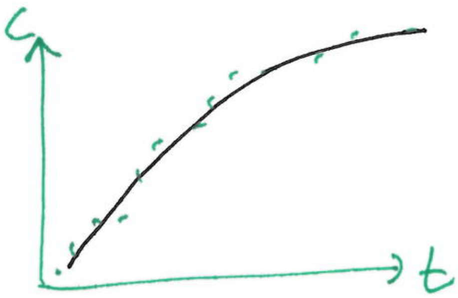
$\mathrm{Erf}^{-1}(\frac{1}{2}) = 0.48\ldots$ (using numerics)

ie $\frac{W}{\sqrt{4Dt}} = \frac{\sqrt{\pi}}{4}$ so $D = \frac{W^2}{t}\frac{4}{\pi}$

$D \simeq \frac{(10^{-6})^2}{1} \cdot \frac{4}{\pi} \simeq 10^{-12}$ ~~units~~ units ? $m^2 s^{-1}$

so $D \sim 1\ (\mu m)^2 s^{-1}$

In practice, usually fit full curve:



> **Solutions to exercises** *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 41

$D = kC^p$ so $[k] = L^2 T^{-2} C^{-p}$ .. see what changes:

$$\left.\begin{aligned} [M] &= CL \\ [k] &= L^2T^{-1}C^{-p} \\ [t] &= T \\ [x] &= L \end{aligned}\right\} \text{elim. } T: \quad \left.\begin{aligned} [M] &= CL \\ [kt] &= L^2C^{-p} \\ [x] &= L \end{aligned}\right\} \text{elim } C \text{ (for } \xi\text{)}: \quad [M^p kt] = L^{p+2}$$

$$\text{so } \xi = \frac{x}{(M^p kt)^{\frac{1}{p+2}}}$$

and for η: elim L without using x:

$$\left[\frac{M^2}{kt}\right] = C^{p+2} \quad \text{so } \eta = \left(\frac{M^2}{kt}\right)^{\frac{1}{p+2}} = \left(\frac{M^{p+2}}{M^p kt}\right)^{\frac{1}{p+2}} = \frac{M}{(M^p kt)^{\frac{1}{p+2}}}$$

so $p=1$ agrees with lecture, $p=0$ and $k=D$ also.

Can continue example from here, but will be on sheet 4 anyway.

Exercise 42

just check all is fine without notes. Common errors (or sub-optimal routes):

- differentiating then reintegrating same thing in PDE
- stray power of 2 or 3
- dividing by F when we actually need $F=0$ soln.
- Forgetting ~~#~~ the integral ($\int C = M$ or $\int F = 1$)

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 43

$$f' = g$$
$$g' = -cg - f(1-f)$$

- Find FPs: $g = 0$, $-cg - f(1-f) = 0$ $\Big\}$ $(0,0)$ or $(1,0)$ ✓
- Find J: general $J = \begin{pmatrix} 0 & 1 \\ -1+2f & -c \end{pmatrix}$ $\quad J\Big|_{0,0} = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix}$, $J\Big|_{1,0} = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}$
- For $(1,0)$, trace $= -c$ det $= -1$ → saddle (didn't need $c>0$!)
- For $(0,0)$, trace $= -c$ det $= 1$ → stable..

  $\lambda^2 + c\lambda + 1 = 0$ so $\lambda = \dfrac{-c \pm \sqrt{c^2-4}}{2}$

real $\Leftrightarrow$ $c \geq 2$ node

complex $\Leftrightarrow$ $c < 2$ focus

(replace with $|c|$ if not using $c>0$).

> **Solutions to exercises** *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 44

$$u = \frac{y}{x+y}, \quad 1-u = \frac{x}{x+y}, \quad \dot u = \frac{\dot y(x+y) - (\dot x + \dot y)y}{(x+y)^2}$$

~~so $\dot u =$~~ so $(x+y)^2 \dot u = x\dot y - y\dot x$ sub $\dot x, \dot y$ now

$$(x+y)^2 \dot u = xy\left(y_0 - \cancel{(x+y)}\right) - yx\left(x_0 - \frac{y}{x+y} - \cancel{(x+y)}\right)$$

logistic terms cancel nicely

$$(x+y)^2 \dot u = xy(y_0 - x_0) + \frac{xy^2}{(x+y)}$$

$\div (x+y)^2$:

$$\dot u = -\underbrace{\frac{x}{x+y}}_{1-u} \cdot \underbrace{\frac{y}{x+y}}_{u} \underbrace{(x_0 - y_0)}_{r} + \underbrace{\frac{x}{x+y}}_{1-u} \cdot \underbrace{\frac{y}{x+y}}_{u} \cdot \underbrace{\frac{y}{x+y}}_{u}$$

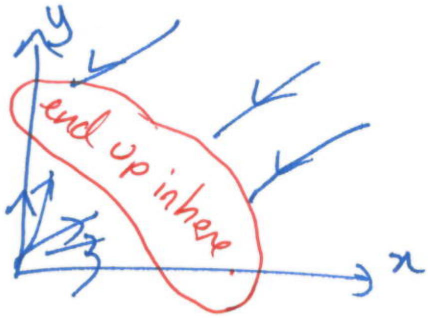
$$= u(1-u)\left[-r + u\right]$$

$$\underline{\dot u = -u(u-r)(u-1)} \quad \text{as req.}$$

Bonus notes:

1) if we try make $v = x+y$, end up with $\dot v$ depends on u and v.

2) ought to be careful that we can only just consider u biologically if total numbers sensible, but can see this is ok from original work



x+y is sensible in long term

Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise 4.5

now add per capita clearane rate from insecticide.

Go back to original eqns:

$$\dot{x} = x\left(x_0 - \frac{y}{x+y} - (x+y)\right) \quad - \lambda x$$

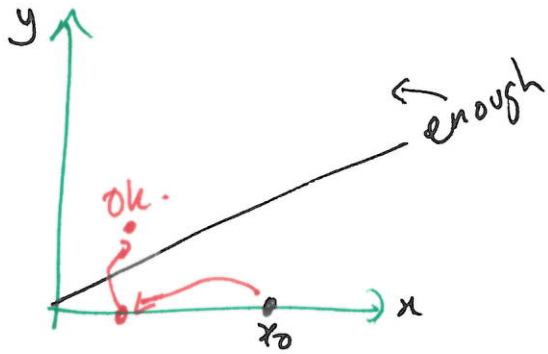
$$\dot{y} = y\left(y_0 \qquad - (x+y)\right) \quad - \lambda y$$

so now we have same but $x_0 - \lambda$, $y_0 - \lambda$

and $r = "x_0 - y_0"$ so this is unchanged, so

width of introduction needed unchanged.

Note: insecticide could still be useful, eg use it before introduction, to get x population down. Now need fewer y to achieve same $\frac{y}{x+y}$.



Solutions to exercises *The exercises are intended to be fairly straightforward and doable after each lecture. These 'solutions' are here in case you want to check what you did, or to see what was intended.*

Exercise ~~45~~ 46

at: (*) $\frac{\alpha\gamma}{\delta}\chi > \frac{(k^2 D_n + \delta)(k^2 D_c + \beta)}{k^2} = k^2 D_n D_c + (\delta D_c + \beta D_n) + \frac{\delta\beta}{k^2}$

minimise RHS wrt k^2:

$\frac{d}{d(k^2)}RHS = D_n D_c - \frac{\delta\beta}{k^4}$ so min at $k_c = \left(\frac{\delta\beta}{D_n D_c}\right)^{\frac{1}{4}}$

at k_c, $RHS = \sqrt{\frac{\delta\beta}{D_n D_c}} \cdot D_n D_c + (\delta D_c + \beta D_n) + \delta\beta\sqrt{\frac{D_n D_c}{\delta\beta}}$

$= 2\sqrt{\delta\beta}\sqrt{D_n D_c} + \delta D_c + \beta D_n = \left(\sqrt{\delta D_c} + \sqrt{\beta D_n}\right)^2$

so if $\chi > \frac{\delta}{\gamma\alpha}\left(\sqrt{\delta D_c} + \sqrt{\beta D_n}\right)^2$, have (*) true for some range of k which includes k_c.

Dimension check for k_c:

k .. in $\cos kx$, so $[k] = L^{-1}$ wavenumber

δ .. in $\dot{n} = -\delta n$ so $[\delta] = T^{-1}$

β $\dot{c} = \ldots - \beta c$ so $[\beta] = T^{-1}$ } rates

D_n, D_c $u_t = D\nabla^2 u$ $[D_n] = [D_c] = L^2 T^{-1}$ – diffusion const.

so $[k_c^4] = \left[\frac{\delta\beta}{D_n D_c}\right] = \frac{T^{-1}T^{-1}}{(L^2 T^{-1})^2} = L^{-4}$

so $[k_c] = L^{-1}$ ✓ good.