1 Statistical Physics and Cosmology

The original aim of these notes was to provide a brief summary of the essential points of each of the sixteen lectures. In previous years a handwritten version was circulated after each lecture and this is reflected in the rather informal structure of the current typscript. The notes are not intended as a systematic account of statistical physics, nor as a treatise on cosmology. For that the student is directed to the recommended textbooks.

It should be borne in mind that these notes do not contain all the explanations and motivational remarks, nor all the steps which were spelt in the lectures. The required algebraic manipulations are almost always rather straightforward and should cause no great difficulty. What is more difficult to acquire is the physical intuition behind the calculations. A major aim of the lectures as delivered is to attempt to impart that intuition.

1.1 Pre-requisites

The study of statistical physics and cosmology is not one which can be embarked upon with absolutely no pre-requistites. It calls upon quite a wide range of facts and ideas drawn from various sources. However as far as the present course is concerned, the small number of astrophysical facts needed are contained herein. One also needs some familiarity with elementary special relativity and quantum mechanics. The course has been deliberately designed so that no knowledge of general relativity is required.

1.2 Web Pages

Students wishing to find out more about modern cosmology including pretty pictures of galaxies etc might like to consult the DAMTP Relativity Group web pages. The URL is www.damtp.cam.ac.uk/user/gr/public/

1.3 Suitable textbooks

The following is a list of suitable textbooks at the level of the course. Unfortunately Pointon is out of print but it should be available in college libraries. Students report that they have found Guenault most helpful. For cosmology, For comsology I think that Roos or the newcomer, Linder, are probably the books closest to the present level. They do however contain some general relativity Weinberg's book is outstanding and usually found very helpful. It succeeds in conveying all the main ideas in an an entirely non-technical way and yet contains many of the main formulae in a technical appendix. The older books by Sciama and by Bondi are very good on the basic facts. In particular that by Bondi discusses the Newtonian approach to cosmology adopted in this course. Those by Kolb and Turner and by Börner will take you well into the frontiers of the subject but are probaly to advanced for a first look at the subject.

Statistical Physics

1 ⋆ A J Pointon: *Introduction to Statistical Physics for Students*

2 T Guenault: *Statistical Physics*

3 M G Bowler: *Lectures on Statistical Mechanics*

Cosmology

1 M Berry: *Principles of Cosmology and Gravitation*

2 ⋆ S Weinberg *The First Three Minutes*

3 J N Islam *An Introduction to Mathematical Relativity*

4 M Roos *Introduction to Cosmology*

5 E W Kolb and M S Turner *The Early Universe*

6 D W Sciama *Modern Cosmology*

7 G Börner *The Early Universe*

8 H Bondi *Cosmology*

9 E V Linder *First Principles of Cosmology*

Background Reading on Modern Physics

1 F J Blatt *Modern Physics*

Contents

2 Introduction

Modern cosmology is based on two basic observational facts.

The Hubble expansion The observed luminous universe is made up of systems of stars like our own Milky Way called galaxies . Each galaxy contains up to 10^{12} stars.

In 1929 Edwin Hubble discovered that the light coming towards us from these galaxies is systematically "redshifted", the observed wavelengths λ_o are increased by a factor $(1+z)$ over their emitted values λ_e. The (positive) quantity z is called the "redshift". He interpreted this as being due to the Doppler effect. The galaxies he claimed are systematically moving awy from us. Special relativity gives the recession speed (assuming they move along the line of sight) from the formula

$$1 + z = \frac{\lambda_o}{\lambda_e} = \sqrt{\frac{c+v}{c-v}}.$$

For small z and hence small $\frac{v}{c}$ we have

$$z \approx \frac{v}{c}.$$

Hubble obtained an estimate of the distances r of the galaxies by selecting those he thought had the same intrinsic total luminosity L , that is, the same total energy emitted per unit time in their rest frame. Nowadays such objects are referred to as "standard candles". He actually measured their observed brightnesses S (the energy we receive per unit are per unit time). If the usual laws of geometry and optics hold and we ignore the redshifting, we have

$$S = \frac{L}{4\pi r^2}.$$

Hubble plotted $logz$ against $-logS$ and obtained a straight line with slope $\frac{1}{2}$. He deduced the *Hubble Law*

$$v = Hr,$$

where the constant of proportionality H is called "Hubble's constant".

The Cosmic Microwave Background In 1965 two engineers at the Bell Telephone Labs in New Jersey, (Penzias and Wilson) were investigating interference in the propagation of microwaves (i.e. electromagnetic radiation of about a centimetre wavelength). They realized that the earth is bathed in an isotropic bath of microwaves with wavelengths peaked at about 3 cm wavelength. The spectrum is "Planckian", that is it is characteristic of a "black body" with temperature $T \approx 3K$. Another way to say this is that the earth is imerssed in a gas of about 400 photons per cm^3 , with a thermal distribution.

These two facts lead, almost immediately to the

Hot Big Bang Theory which states that the universe we see today has expanded from a previous state of high density and high temperature.

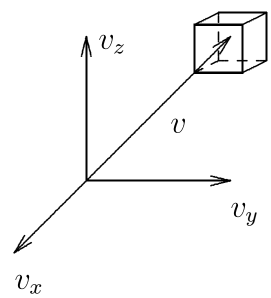
To understand the universe therefore we must understand matter at high density and high pressure. In order to do so we need first to develop the kinetic theory of gases and the elements of statistical mechanics. Having done that we shall return to apply our knowledge to cosmology.

3 Statistical Physics

3.1 Kinetic Theory of Gases

The aim of kinetic theory of gases is to describe their macroscopic properties , such as pressure P, temperature T, and volume V and the relationships bewteen them in terms of microscopic dynamical properties of the molecules, atoms, or other particles making up the gas. The theory was developed originally in the eighteenth century to treat such familar gases as oxygen and hydrogen at ordinary pressures and temperatures. Later it was extended to cover more exotic gases in which the particles move relativistically including photons or neutrinos. It is these cases which are important for cosmology.

Initially we focus on $\bar{n}(v)d^3v =$, the average number of particles per unit volume with velocity vectors $\vec{v}$ inside a cell d^3v in velocity space. Later, when we consider photons or neutrinos which always move at the speed of light, it will turn out to be more convenient to work in terms of the distribution of their momenta $\vec{p}$.



Let's introduce polar coordinates in velocity space:

$$d^3v = v^2 dv\, d\Omega = v^2 dv \sin\theta\, d\theta\, d\phi, \tag{1}$$

where $d\Omega$ is the element of solid angle.

We now assume an *isotropic distribution*:

$$\bar{n}(\vec{v}) = \bar{n}(|\vec{v}|). \tag{2}$$

The number of particles with speed $v \to v + dv$ $(v = |\vec{v}|)$ per unit volume is

$$f(v)dv = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \bar{n}(v)v^2 \sin\theta\, d\theta\, d\phi = 4\pi\bar{n}(v)v^2 dv. \tag{3}$$

If we assume that the only energy is kinetic energy (i.e. we have a gas of "free particles"), then the *energy density* is

$$U = \int\int\int \varepsilon(v)\bar{n}(\vec{v})d^3v, \tag{4}$$

where

$$\begin{aligned} \varepsilon(v) &= \frac{1}{2}mv^2 && \text{non-relativistic case} \\ \varepsilon(v) &= \frac{mc^2}{\sqrt{1-v^2/c^2}} && \text{relativistic case.} \end{aligned} \tag{5}$$

Thus

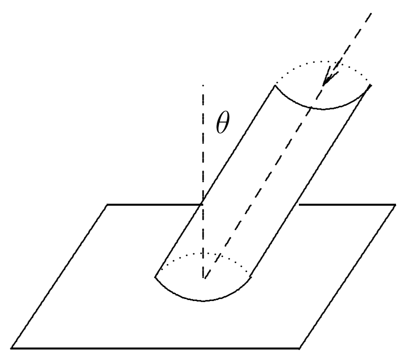
$$U = \int_0^\infty f(v)\frac{mv^2}{2}dv \qquad \text{non-relativistic} \tag{6}$$

$$U = \int_0^c f(v)\frac{mc^2}{\sqrt{1-v^2/c^2}} \tag{7}$$

relativistic (n.b. includes rest mass energy).

Number flux The number of particles striking the surface per unit area per unit time in the direction (θ, ϕ) is

$$f(v)dv\, v\cos\theta\,\frac{\sin\theta\, d\theta\, d\phi}{4\pi} \tag{8}$$



since

$$\frac{\sin\theta\, d\theta\, d\phi}{4\pi} f(v)dv \tag{9}$$

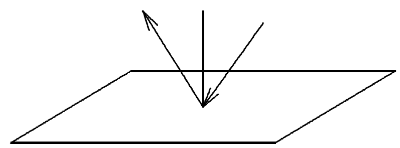
is the number/unit volume in solid angle $\sin\theta\, d\theta\, d\phi$, and all particles striking the surface in unit time arrived from inside the cylinder whose volume is $v\cos\theta$.

The total numbers striking the surface per unit area, per unit time is

$$\int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi\, \frac{f(v)}{4\pi} v\, dv\, \cos\theta\, sin\theta = \frac{f(v)v\, dv}{4}. \tag{10}$$

Pressure: This is the momentum reversed per unit area per unit time in direction (θ, ϕ),

$$f(v)dv\, 2pv\cos\theta\, \cos\theta\, \frac{\sin\theta\, d\theta\, d\phi}{4\pi}. \tag{11}$$



The total pressure is

$$
\begin{aligned}
P &= \int dv \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi\, f(v) \frac{2pv}{4\pi} \cos^2\theta \sin\theta \\
P &= \frac{1}{3} \int p f(v) v dv.
\end{aligned}
$$

In the non-relativistic case $p = mv$, so $P = \frac{1}{3}\int_0^\infty mv^2 f(v) dv$,

$$
\boxed{P = \frac{2}{3} U} \tag{12}
$$

This does not include the rest mass energy.

In the relativistic case $p = mv/\sqrt{1 - v^2/c^2}$, so in the extreme relativistic case as $v \to c$, $\varepsilon \to pc$ and

$$
\boxed{P \to \frac{1}{3} U} \tag{13}
$$

This includes the rest mass energy. However if the particles are ultra-relativistic the contribution from the rest mass will be negligible.

The previous results followed without knowing the detailed form of $\bar{n}(\vec{v})$. Maxwell *guessed* that

i) $\bar{n}(\vec{v}) = \bar{n}\left(v_x^2 + v_y^2 + v_z^2\right)$

ii) $\bar{n}(\vec{v}) = g(v_x) g(v_y) g(v_z)$

(i.e. v_x, v_y, v_z are independently distributed) and deduced that $\bar{n}(\vec{v})$ is a Gaussian or Normal distribution:

$$
\bar{n}(\vec{v}) = \text{constant} \times \exp\left[-\text{constant}' \left(v_x^2 + v_y^2 + v_z^2\right)\right]. \tag{14}
$$

We shall derive this (in the non-relativistic case) using *statistical mechanics*.

3.2 Statistical Mechanics

The idea is that for given values of the macroscopic variables there are many possible microscopic configurations. In order to calculate the macroscopic variables we need to know how the total energy is distributed among the many possible microstates of the system. We specify a *distribution* by giving the *occupation number* n_i telling us how many particles are in state i with energy E_i. Each distribution occurs with a certain *weight* $W(n_1, n_2, \ldots)$ equal to the number of different microscopic configurations giving rise to the same distribution of energies.

3.2.1 The Most Probable Distribtion

The idea is that we get a good estimate of the behaviour of the system by confining attention to the *most probable distribution* which is obtained by maximising $\log W$ subject to the *constraints*:

$$\sum_i n_i = N, \qquad \text{total number of particles} \tag{15}$$

$$\sum_i E_i n_i = E, \qquad \text{total energy of particles.} \tag{16}$$

Because N is very large (of the order of 10^{23} per cm^3 for a gas at room temperature and atmospheric pressure) we treat the n_i's as continuous variables and extremize

$$\log W - \beta\left(\sum_i E_i n_i - E\right) - \alpha\left(\sum_i n_i - N\right) \tag{17}$$

with respect to the n_i where α and β are Lagrange multipliers.

$$\boxed{\left.\frac{1}{W}\frac{\partial W}{\partial n_i}\right|_{n_i=\bar{n}_i} = \beta E_i + \alpha} \tag{18}$$

Example: *Maxwell-Boltzmann* statistics for N *distinguishable* particles. We can imagine picking out the particles from a bag containing N particles in $N!$ ways and assigning them to the k possible energy states. The order in which we assign the n_i particles to the i'th state does not matter and therefore:

$$W = \frac{N!}{n_1! n_2! \cdots n_k!} \tag{19}$$

$$\frac{1}{W}\frac{\partial W}{\partial n_i} = -\log n_i. \tag{20}$$

Using the Stirling approximation $\log x \simeq x \log x - x$,

$$\bar{n}_i = \exp -(\beta E_i + \alpha). \tag{21}$$

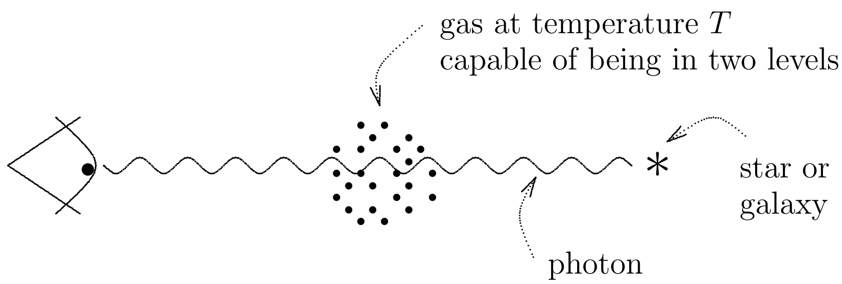
In fact (see later) $\beta = 1/(kT)$ where k is Boltzmann's constant, T = temperature, and $\alpha = -\beta\mu$, where μ is the chemical potential.

If we are given N and E we can calculate T and μ by substituting the Boltzmann distribution into the constraints (15, 16).

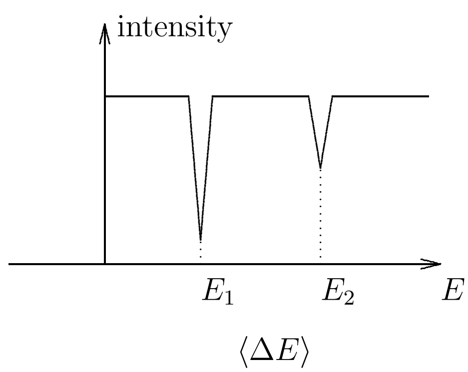
Thus for a two-level system, $k = 2$,

$$\frac{\bar{n}_1}{\bar{n}_2} = \exp -\frac{\Delta E}{kT}, \qquad \Delta E = E_1 - E_2. \tag{22}$$

An application to Cosmology



The observed light intensity shows absorption lines.



Assume the rate of absorption $\propto \bar{n}_1/\bar{n}_2$, then knowing ΔE and $\bar{n}_1/\bar{n}_2$ we estimate T, e.g.

i) Cyanogen molecules (CN) in our galaxy were observed *before* Penzias & Wilson in 1965 to have $T \sim 3K$.

ii) In 1994, observations of a cloud of neutral carbon atoms seen in front of the quasar Q1331+170 which has a redshift $z \simeq 1.776$ were observed to have $T \simeq 7.4 \pm 0.8K$. At redshift z the microwave background had

$$T = (1 + z)\, 2.726 \pm 0.010 \tag{23}$$

(we shall prove this later).

3.2.2 Consistency and Interpretation of Lagrange Multipliers

To complete our calculation we need:

i) to check that $W(n_1, n_2, \ldots, n_k)$ is indeed highly peaked at its most probable value.

ii) to give an interpretation of α and β.

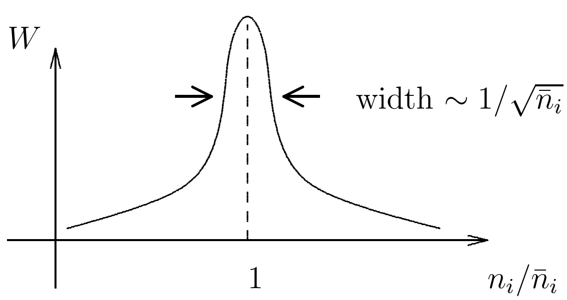
Sharpness of Maximum Use Taylor expansion for $\log W$:

$$\frac{\partial^2 \log W}{\partial {n_i}^2} = -\frac{1}{n_i}. \tag{24}$$

This implies

$$W = \overline{W} \exp - \sum_i \frac{(n_i - \bar{n}_i)^2}{2\bar{n}_i} \qquad \text{(Gaussian)}. \tag{25}$$

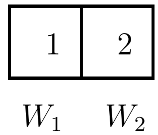
The standard deviation of $(n_i - \bar{n}_i)/n_i$ is $1/\sqrt{\bar{n}_i}$, which is very small for large $\bar{n}_i$.



3.2.3 Interpretation of α and β

There are 3 steps.

I: Thermal Equilibrium of Independent Systems



Assume $W = W_1 W_2$, $\log W = \log W_1 + \log W_2$.

i) Allow the exchange of energy, but *not* particles. There are three constraints and therefore three Lagrange multipliers (not four):

Extremize:

$$\log W_1 + \log W_2 - \alpha_1 \left(\sum_i n_i^{(1)} - N^{(1)} \right) - \alpha_2 \left(\sum_i n_i^{(2)} - N^{(2)} \right)$$
$$-\beta \left(\sum_i n_i^{(1)} E_1^{(1)} + \sum_i n_i^{(2)} E_1^{(2)} \right). \tag{26}$$

Then

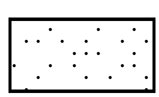
$$\begin{aligned} \frac{1}{W_1} \frac{\partial W_1}{\partial n_i^{(1)}} &= \beta \left(E_i^{(1)} - \mu_1 \right), \qquad & \alpha_1 &= -\beta \mu_1 \\ \frac{1}{W_2} \frac{\partial W_2}{\partial n_i^{(2)}} &= \beta \left(E_i^{(2)} - \mu_2 \right), \qquad & \alpha_2 &= -\beta \mu_2 \end{aligned} \tag{27}$$

e.g. Maxwell-Boltzmann case

$$\bar{n}_i^{(1)} = \exp -\beta \left(E_i^{(1)} - \mu_1 \right) \tag{28}$$
$$\bar{n}_i^{(2)} = \exp -\beta \left(E_i^{(2)} - \mu_2 \right). \tag{29}$$

Thus, *two systems in thermal contact at equilibrium have the same value of* β (cf. temperature).

ii) Remove the partition and allow free exchange of both energy *and* particles, then there are two constraints and two Lagrange multipliers.



Extremize:

$$\log W_1 + \log W_2 - \alpha\left(\sum_i n_i^{(1)} + \sum_i n_i^{(2)} - N\right)$$
$$-\beta\left(\sum_i n_i^{(1)} E_i^{(1)} + \sum_i n_i^{(2)} E_i^{(2)}\right). \tag{30}$$

This implies

$$\frac{1}{W_1}\frac{\partial W_1}{\partial n_i^{(1)}} = \beta\left(E_i^{(1)} - \mu\right), \qquad \alpha = -\beta\mu \tag{31}$$
$$\frac{1}{W_2}\frac{\partial W_2}{\partial n_i^{(2)}} = \beta\left(E_i^{(2)} - \mu\right). \tag{32}$$

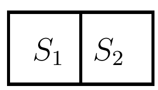
Therefore, *two systems in thermal contact that are allowed to exchange particles have the save values of* μ (cf. chemical potential).

II: Definition of Entropy

$$S = k \log \overline{W} = k \max \log W \tag{33}$$

S is a measure of the likelihood or probability of the equilibrium configurations.

Consider two independent systems allowed to come into equilibrium.



$$S_1 = \max_{n_i^{(1)}} \log W_1, \qquad S_2 = \max_{n_i^{(2)}} \log W_2. \tag{34}$$

This system is subject to four constraints.

The combined entropy S_3,

$$S_3 = \max_{n_i^{(1)}, n_i^{(2)}} \left(\log W_1 + \log W_2\right) \tag{35}$$

is subject to fewer constraints (i.e. 3 or 2).

$$S_3 \geq S_1 + S_2. \tag{36}$$

Entropy cannot decrease when systems come into equilibrium (2nd Law of Thermodynamics).

III: Quasi-Static Variation of Equilibria We have seen that μ and β have properties resembling chemical potential and $1/(kT)$, but to make the correspondence quantitative we consider changing (very slowly) N, E, and V through a sequence of equilibrium configurations. We know (or. lookup) from thermodynamics that we have the *Thermodynamic Relation*,

$$\underset{\substack{\text{change in}\\ \text{internal}\\ \text{energy}}}{dE} = \underset{\substack{\text{work}\\ \text{done}}}{-PdV} + \underset{\substack{\text{gain in}\\ \text{potential}\\ \text{energy}}}{\mu dN} + \underset{\substack{\text{increase}\\ \text{in heat}\\ \text{energy}}}{TdS}. \tag{37}$$

[It is not obvious that the increase in heat energy can be written as TdS. This is established in thermodynamics books. However we shall be able to prove this using statistical mechanics.]

Regarding $S = k \log \overline{W}$ as a function of E, N, and V.

$$\frac{1}{k} dS = \sum_i \frac{1}{\overline{\overline{W}}} \left(\frac{\partial \overline{W}}{\partial n_i} \right)_{n=\bar{n}_i} d\bar{n}_i = \sum_i (\beta E_i + \alpha) d\bar{n}_i, \tag{38}$$

where

$$E = \sum_i \bar{n}_i E_i, \qquad N = \sum_i \bar{n}_i. \tag{39}$$

Then

$$\frac{1}{k} dS = \beta dE + \alpha dN - \sum_i \beta_i \bar{n}_i dE_i. \tag{40}$$

We interpret $\sum \bar{n}_i dE_i = dW = -PdV$ as the work done on the system assuming that no particles or energy are added (i.e. "*adiabatically*"). We find that if $\beta = 1/(kT)$, $\alpha = -\beta\mu$, equation (40) becomes

$$\boxed{dE = TdS + \mu dN - PdV} \tag{41}$$

Conclusion: We have shown that:

1) The quantities T and μ are equal for two systems in equilibrium.

2) For quasi-static variations of equilibrium configurations

$$dE = TdS + \mu dN - PdV.$$

3) For two systems allowed to come into equilibrium

$$S_3 \geq S_1 + S_2.$$

These are the "*zero'th, first, and second laws of thermodynamics*" respectively. They justify calling $\beta = 1/(kT)$ the temperature and $\mu = -\alpha/\beta$ the chemical potential.

3.2.4 Boltzmann Distribution with Degeneracy

If there are g_i distinct energy states with energy E_i then

$$W = N! \prod_i \frac{g_i^{n_i}}{n_i!}. \tag{42}$$

Going through the same steps as before, we find that

$$\boxed{\bar{n}_i = g_i \exp\left[-\beta(E_i - \mu)\right]} \tag{43}$$

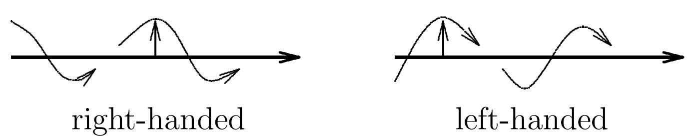
e.g.

i) spin degeneracy of an *electron* at rest: $g = 2$, $\uparrow$ or $\downarrow$.

   In absence of magnetic fields there exist two states with angular momentum projection $\pm\hbar/2$.

ii) a *photon* with fixed momentum has two polarisation states (plane polarised or circularly polarised). For circular polarisation angular momentum along the direction of motion is $\pm\hbar$:

   "two helicity states"



iii) any particle with fixed energy E is degenerate w.r.t. the direction of the momentum $\vec{p}$.

3.3 Non-Relativistic Maxwell-Boltzmann Gas

To find the states we solve the non-relativistic Schrödinger equation for a particle of mass M in a box of sides (a, b, c) subject to periodic boundary conditions:

$$-\frac{\hbar^2}{2M}\nabla^2\Psi_i = E_i\Psi_i. \tag{44}$$

The solutions are

$$\Psi_i = \exp 2\pi i\left(\frac{xl}{a} + \frac{ym}{b} + \frac{zn}{c}\right) = \exp i\vec{k}\cdot\vec{x}, \tag{45}$$

with $i \leftrightarrow (l, m, n) \in \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$, and

$$E_i = \frac{\hbar^2}{2M}(2\pi)^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) = \frac{h^2}{2M}\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right), \tag{46}$$

where $h = 2\pi\hbar$.

Allowed momenta These are given by

$$\vec{p} = \hbar\vec{k} = h\left(\frac{l}{a}, \frac{m}{b}, \frac{n}{c}\right), \tag{47}$$

and are uniformly distributed on a lattice in momentum space with density

$$\frac{abc}{h^3} = \frac{V}{h^3}, \tag{48}$$

where $V = abc$ is the volume of the box.

Now if V is large the levels are very closely spaced and we can pass from a summation over l, m, n to an integral by the rule:

$$\sum_i \leftrightarrow \sum_{l,m,n} \leftrightarrow \int_{\mathbb{R}^3} \frac{V d^3p}{h^3}. \tag{49}$$

Thus

$$\bar{n}(\vec{p})d^3\vec{p} = \frac{d^3p}{h^3}\exp\left(-\frac{\frac{p^2}{2M} - \mu}{kT}\right), \tag{50}$$

which implies

$$\bar{n}(\vec{v}) \propto \exp\left(-\frac{M\vec{v}^{\,2}}{2kT}\right), \tag{51}$$

which is indeed a Gaussian distribution as Maxwell predicted.

The total number of particles in volume V is given by

$$\begin{aligned} N &= V\int \bar{n}(p)d^3p = \frac{V}{h^3}\int\int\int d^3p \exp\left(-\frac{p^2}{2MkT}\right)\exp\frac{\mu}{kT}, \\ &= \frac{V}{h^3}\exp\frac{\mu}{kT}\left[\int_{-\infty}^{+\infty} dp_x \exp\left(-\frac{p_x^2}{2MkT}\right)\right]^3, \end{aligned}$$

$$\boxed{N = \frac{V}{h^3}(2\pi MkT)^{3/2}\exp\frac{\mu}{kT}} \tag{52}$$

To calculate the total energy E one can differentiate N w.r.t. M:

$$\frac{\partial N}{\partial M} = \frac{V}{MkT}\int \frac{p^2}{2M}\bar{n}(p)d^3p \tag{53}$$

Then

$$E = MkT\frac{\partial N}{\partial M}, \tag{54}$$

which gives

$$\boxed{E = \frac{3}{2}kT\frac{V}{h^3}(2\pi MkT)^{3/2}\exp\mu/(kT)} \tag{55}$$

or using the expression for N above,

$$\boxed{E = \frac{3}{2}NkT} \tag{56}$$

A note on boundary conditions Students encountering the density of states formula for the first time are often worried that it depends upon the choice of periodic boundary conditions. For example if one imposes Dirichlet boundary conditions, $\Psi = 0$ on the sides of the box, as one often does in quantum mechanics courses one would get (un-normalized)-solutions of the form

$$\Psi = \sin(\frac{l\pi x}{a})\sin(\frac{m\pi x}{b})\sin(\frac{n\pi x}{c}),$$

with (l, m, n) positive integers. These are of course not momentum eigen-states but they are energy eigen-states. One may label these energy eigens states by a lattice of points lying in the *positive octant* of momentum space but now with $2\times 2\times 2 = 8$ times the density we found before. The net result is that the number of states having energies between E and $E + dE$ is unchanged. In practice it is usually most convenient to use periodic boundary conditions. The number of states with energies between E and $E + dE$ (in the limit of large volume V) are found not to depend on the precise boundary conditions. To check this one may repeat the calculation when one imposes the Neumann bouundary condition that the normal derivative of Ψ vanishes on the sides of the box. This amounts to replacing the sine function with the cosine function. If you are still sceptical try various combinations of periodic, Neumann and Dirichlet boundary conditions.

Pressure

$$P = \frac{2}{3}U = \frac{2}{3}\frac{E}{V} = \frac{NkT}{V}. \tag{57}$$

This is *Boyle-Charles Law*

$$\boxed{PV = NkT} \tag{58}$$

Entropy

$$\frac{S}{k} = \log \overline{W} = N \log N - N + \sum \bar{n}_i \log g_i - \sum (\bar{n}_i \log \bar{n}_i - \bar{n}_i), \tag{59}$$

but $\log g_i - \log \bar{n}_i = \beta(E_i - \mu)$, therefore,

$$S = Nk \log N + k\beta(E - \mu N). \tag{60}$$

Substitution from the expressions for E gives

$$\boxed{S = Nk \log \left[\frac{V}{h^3} (2\pi MkT)^{3/2} \right] + \frac{3}{2} Nk} \tag{61}$$

Application For an *adiabatic expansion* the added heta energy $\delta Q = 0$ which implies that the change in entropy $dS = 0$. Thus an adiabatic process is one for which $S =$ constant at constant N. This implies

$$\boxed{VT^{3/2} = \text{constant}} \tag{62}$$

Using Boyle-Charles Law

$$\boxed{PV^{5/3} = \text{constant}} \tag{63}$$

Alternative Derivation of Pressure Using the Thermodynamic Relation

$$S = S(E, T, N), \tag{64}$$

$$TdS = dE - \mu dN + PdV. \tag{65}$$

Therefore

$$P = T \left(\frac{\partial S}{\partial V} \right)_{E,N}. \tag{66}$$

Now use the expression $E = \frac{2}{3} NkT$ to eliminate T from (61) and get

$$S = Nk \log \left[\frac{V}{h^2} \left(2\pi M \frac{2}{3} E \right)^{3/2} \right] + \frac{3}{2} Nk, \tag{67}$$

which implies

$$P = \frac{NkT}{V}. \tag{68}$$

3.3.1 Gibbs Paradox

We used earlier that for independent systems at some T and μ:

1	2

i) $W_3 = W_1 W_2$.

ii) $S_3 = S_1 + S_2$, entropy is extensive.

Both are false in our case!. This is esaily seen using the formula for the weight

$$W = N! \prod_i \frac{g_i^{n_i}}{n_i!}. \tag{69}$$

As a consequence one arrives at certain paradoxes, e.g. the entropy of a gas can change by removing a fictitious partition. Sackur & Tetrode suggested (before quantum mechanics) the replacement:

$$W \to \prod_i \frac{g_i^{n_i}}{n_i!} \tag{70}$$

(removing $N!$) since $(N_1 + N_2) \neq N_1! N_2!$.

This alters the entropy to

$$\boxed{S = \frac{5}{2} Nk + Nk \log \left[\frac{V}{Nh^3} (2\pi MkT)^{3/2} \right]} \tag{71}$$

Now S is "extensive", i.e. sending $N \to 2N$ and $V \to 2V$ sends $S \to 2S$. The old formula has the effect that if $N \to 2N$ and $V \to 2V$ then $S \to 2S + Nk \log 2$!

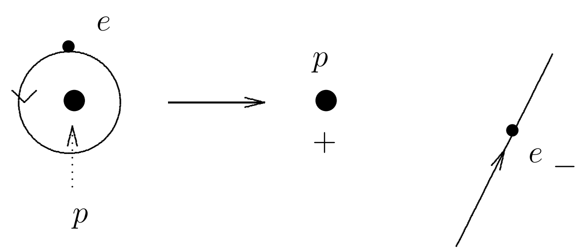
Note however that both formulae give the *same* answer for the pressure P and other physically measurable quantities (as long as N is held fixed). The origin of this problem is that we have assumed our particles are *distinguishable*. This can only be corrected by using *quantum statistics*. This we will do shortly.

3.3.2 Application to Cosmology: Ionisation

The following calculation is representative of how we use the ideas we have developed above are used to study gases with more than one species of particle. We think of each species of particle as defining a system. They are allowed to exchange energy and if chemical reactions are allowed they are also allowed to change their number, subject to the *constraints* imposed by conservation laws.

A neutral gas of hydrogen atoms will heat up on adiabatic compression ($T \propto V^{-2/3}$) and eventually become ionised:

$$\begin{array}{ccccccc} H & \rightarrow & e & + & p & - & E_{\text{ion}} \\ \text{hydrogen} & & \text{electron} & & \text{proton} & & \\ \text{atom} & & g=\frac{1}{2} & & g=\frac{1}{2} & & \\ & & -e & & +e & & \end{array} \tag{72}$$



where E_{ion} is the ionisation energy $\simeq 13.6\, eV$.

At low temperature H is neutral $\Rightarrow$ transparent.
At high temperature H is ionised $\Rightarrow$ opaque, because it is an electrically conducting "plasma".

Calculation of the Amount of Ionisation: Saha's Equation We maximise

$$\log W = \log W_{+} + \log W_{-} + \log W_{0}, \tag{73}$$

subject to three constraints, where

$$\begin{array}{ccl} + & \leftrightarrow & \text{proton} \\ - & \leftrightarrow & \text{electron} \\ 0 & \leftrightarrow & H \text{ atom} \end{array} .$$

During ionisation,

$$\begin{array}{rcl} N_{+} & \rightarrow & N_{+}+1 \\ N_{-} & \rightarrow & N_{-}+1 \\ N_{0} & \rightarrow & N_{0}-1 \end{array} \Rightarrow \begin{array}{rcl} N_{+}+N_{0} & = & \text{constant} \\ N_{-}+N_{0} & = & \text{constant} \end{array}$$

Let

$$E_+ = \frac{p_+^2}{2M_+}, \quad E_- = \frac{p_-^2}{2M_-}, \quad E_0 = \frac{p_0^2}{2M_0}, \tag{74}$$

be kinetic energies. The hydrogen atom's energy is potential plus kinetic,

$$E_H = \frac{p_0^2}{2M_0} - E_{\rm ion}, \qquad (E_{\rm ion} > 0). \tag{75}$$

Energy conservation implies that

$$n_+E_+ + n_-E_- + n_0(E_0 - E_{\rm ion}) = \text{constant}, \tag{76}$$

so we extremize

$$\begin{aligned} &\log W - \beta(n_+E_+ + n_-E_- + n_0(E_0 - E_{\rm ion})) \\ &\quad -\alpha_+(n_+ + n_0) - \alpha_-(n_- + n_0). \end{aligned} \tag{77}$$

This is as if we had three independent systems, but with all three temperatures equal,

$$\beta_+ = \beta_- = \beta_0, \tag{78}$$

and

$$\alpha_0 = \alpha_+ + \alpha_- - \beta E_{\rm ion}, \tag{79}$$

or chemical potentials,

$$\begin{aligned} \mu_+ &= -\alpha_+/\beta, \\ \mu_- &= -\alpha_-/\beta, \\ \mu_0 &= \mu_+ + \mu_- + E_{\rm ion}. \end{aligned} \tag{80}$$

Therefore

$$\frac{N_\pm}{V} = \frac{g_+}{h^3}(2\pi M_\pm kT)^{3/2} \exp\beta\mu_\pm, \tag{81}$$

$$\frac{N_0}{V} = \frac{g_0}{h^3}(2\pi M_0 kT)^{3/2} \exp\beta(\mu_+ + \mu_- + E_{\rm ion}), \tag{82}$$

$$\boxed{\frac{n_+n_-}{n_0} = \frac{g_+g_-}{g_0}\left(\frac{2\pi M_e kT}{h^2}\right)^{3/2} \exp\left(-\beta E_{\rm ion}\right)} \tag{83}$$

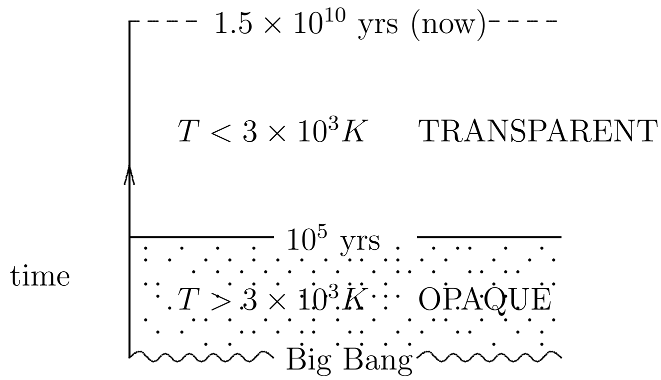
where we have used the fact that $M_+ \simeq M_0 = M_p$, the proton mass, and M_e is the electron mass.

Note that

$$\begin{aligned} \frac{n_+n_-}{n_0} &\ll 1, \qquad \text{unless} \\ kT &\geq E_{\rm ion} \end{aligned}$$

i.e. $T \gtrsim 3 \times 10^3 K$.

Application to Cosmology At around $t \simeq 10^5$ yrs after the Big Bang the redshifted temperature of the universe, $T = 3K(1+z)$ was around $3 \times 10^3 K$ ($z \sim 10^3$) and all hydrogen would have been ionised and hence opaque. We will study this in detail later.



3.4 Relativistic Maxwell-Boltzmann Gas

To find the allowed energy levels we need the analogue of Shrödinger for a relativistic particle. Strictly speaking, this takes us a little beyond elementary quantum mechanics. However we can follow the pattern which should be familar. We make the replacements:

$$E \to -\frac{\hbar}{i}\frac{\partial}{\partial t}, \qquad \vec{p} \to \frac{\hbar}{i}\vec{\nabla}, \tag{84}$$

in the special relativity relation between energy and momentum

$$E^2 = \vec{p}^{\,2} c^2 + M^2 c^4, \tag{85}$$

to get what is called the *Klein-Gordon Equation*,

$$\boxed{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = \vec{\nabla}^{\,2}\phi - \frac{M^2 c^2}{\hbar^2}\phi} \tag{86}$$

n.b. if $M = 0$ we recover the *wave equation*.

We solve this by separation of variables in a box of sides (a, b, c) subject to periodic boundary conditions:

$$\phi = e^{-iEt/\hbar} e^{i\vec{k}\cdot\vec{x}}, \qquad \vec{k} = 2\pi\left(\frac{l}{a}, \frac{m}{b}, \frac{n}{c}\right), \tag{87}$$

as before, and we conclude that the allowed values of the momentum $\vec{p}$ are *the same as in the non-relativistic case*. This is true both for massive and for massless particles.

Thus we obtain the *Juttner distribution*:

$$\bar{n}(\vec{p})d^3p = \frac{d^3p}{h^3}\exp\left[-\frac{\sqrt{\vec{p}^2c^2+M^2c^4}-\mu}{kT}\right], \tag{88}$$

$$N = \frac{V}{h^3}\int d^3p\,\exp\left(-\sqrt{\vec{p}^2c^2+M^2c^4}\right)\exp\frac{\mu}{kT}. \tag{89}$$

This is difficult to evaluate exactly unless $M = 0$ (zero rest mass case). We shall proceed indirectly in order to illustrate the use of the thermodynamic relation. Define the *Helmholtz Free Energy* $F = F(T, V, N)$ by

$$F \equiv E - TS \quad \Rightarrow \quad dF = -PdV - SdT + \mu dN, \tag{90}$$

$$\boxed{P = -\left.\frac{\partial F}{\partial V}\right|_{T,N}} \tag{91}$$

Now

$$N = \frac{V}{h^3}\exp\frac{\mu}{kT}f(T, M, c), \tag{92}$$

for some function $f(T, M, c)$. Moreover we showed earlier(60) that in general, for the Maxwell Boltzmann distribution,

$$S = Nk\log N + \frac{1}{T}(E - \mu N). \tag{93}$$

Therefore,

$$F = N(\mu - kT\log N), \tag{94}$$

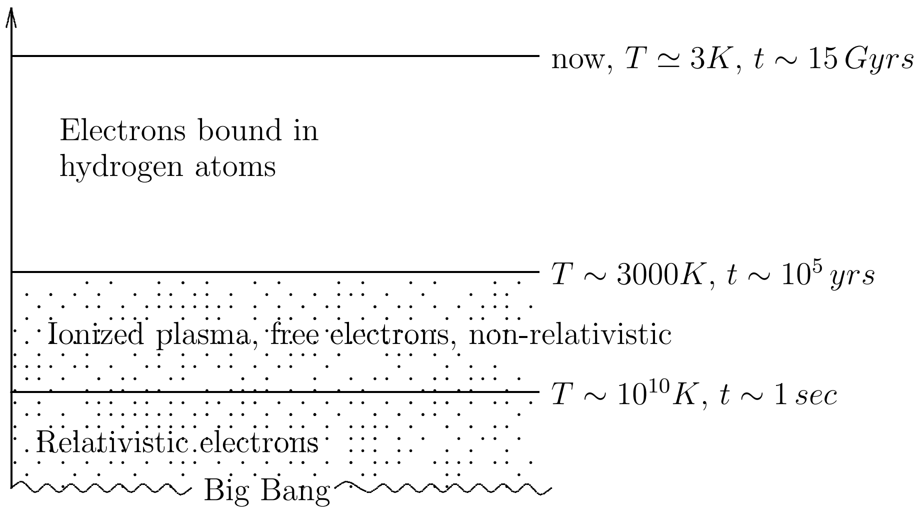
$$= -kTN\log\left(\frac{V}{h^3}f(T, M, c)\right). \tag{95}$$

Using (91), $P = NkT/V$ or

$$\boxed{PV = NkT} \tag{96}$$

i.e. the Juttner gas obeys the familiar Boyle-Charles Law.

Order of Magnitudes The Juttner distribution differs significantly from the Maxwell distribution when $Mc^2 \simeq kT$. In the early universe the lowest temperature when this happens is $T = m_e c^2/k \simeq 10^{10} K$ since the electron with mass m_e is the lightest known massive particle:



3.5 Bosons and Fermions

Swapping the position of two particles which are *indistinguishable* should make no physical difference,

$$|\Psi(\vec{x}_1, \vec{x}_2)| = |\Psi(\vec{x}_2, \vec{x}_1)| \,.$$

We may realize this in two ways:

$$\begin{array}{lrcl} \text{Bosons:} & \Psi(\vec{x}_1, \vec{x}_2) & = & \Psi(\vec{x}_2, \vec{x}_1) \qquad \text{or} \\ \text{Fermions:} & \Psi(\vec{x}_1, \vec{x}_2) & = & -\Psi(\vec{x}_2, \vec{x}_1) \end{array}$$

The spin statistics theorem states that:

Bosons	have	integer	spins	$\frac{s}{\hbar} \in \mathbb{Z}$
Fermions	have	$\frac{1}{2}$ integer	spins	$\frac{s}{\hbar} \in \mathbb{Z} + \frac{1}{2}$

Examples

electron	$s = \frac{1}{2}\hbar$	fermion
photon	$s = \hbar$	boson
pion	$s = 0$	boson
proton	$s = \frac{1}{2}\hbar$	fermion
neutron	$s = \frac{1}{2}\hbar$	fermion
quark	$s = \frac{1}{2}\hbar$	fermion
neutrino	$s = \frac{1}{2}\hbar$	fermion

3.6 Bose-Einstein distribution

For Bosons the counting problem amounts to putting n_i indistingushable objects in g_i boxes. We lay out the n_i objects in a line, separated by $g_i - 1$ partitions. We have $(n_i + g_i - 1)!$ ways of laying them out in order. However the order in which we select the objects and the order in which we select the partitions does not matter. Therefore:

$$W = W_{\mathrm{BE}} = \prod_i \frac{(g_i + n_i - 1)!}{(g_i - 1)! n_i!}, \tag{97}$$

assume $n_i \gg 1$

$$\frac{\partial \log W_{\mathrm{BE}}}{\partial n_i} = \log(g_i + n_i - 1) - \log n_i \tag{98}$$

Then

$$\bar{n}_i = \frac{g_i - 1}{e^{\beta(E_i - \mu)} - 1},$$

assume $g_i \gg 1$. This is the *Bose-Einstein distribution*

$$\boxed{\bar{n}_i = \frac{g_i}{e^{\beta(E_i - \mu)} - 1}} \tag{99}$$

To get the classical, Maxwell-Boltzmann limit assume $g_i \gg n_i \Leftrightarrow \beta(E_i - \mu) \gg 1$, then

$$\bar{n}_i \simeq g_i \exp\left[-\beta(E_i - \mu)\right] \tag{100}$$

3.6.1 The photon gas

$E = pc = \hbar\omega$, $\mu = 0$

$$\boxed{\bar{n}(p) d^3p = 2 \times \frac{V}{h^3} \frac{d^3p}{\exp(\beta pc) - 1}} \tag{101}$$

(n.b. the factor of 2 comes from two polarisation states). We set $\mu = 0$ because photon number is not conserved.

Planck spectrum: The number of photons with energy $\omega \to \omega + d\omega$ per unit volume is

$$\frac{1}{c^3} \frac{1}{\pi^2} \frac{\omega^2 d\omega}{\exp \beta\hbar\omega - 1} \tag{102}$$

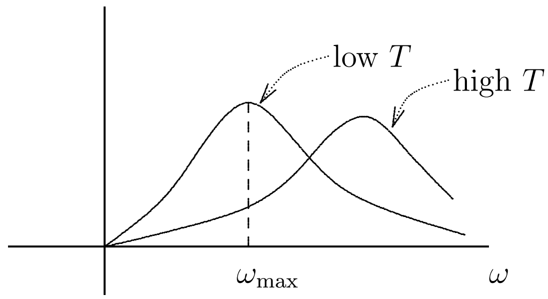
Wien Displacement law: This expression is of the form

$$\omega^2 d\omega f\left(\frac{\hbar\omega}{kT}\right)$$

for some function f.

Consequences of the Wien Displacement Law

i) the maximum scales with T as $\omega_{\max} \propto T$, i.e. $\lambda_{\max} T = \text{constant}$.



ii) if we slowly (i.e. adiabatically) expand a box of thermal photons such that

$$\begin{aligned} V &\rightarrow V(1+z)^3, \\ \omega &\rightarrow \omega/(1+z), \\ T &\rightarrow T/(1+z), \end{aligned}$$

the shape of the distribution will not change and the number of photons in the interval $\omega \rightarrow \omega d\omega$ will equal the number in the interval

$$\frac{\omega}{1+z} \rightarrow \frac{\omega + d\omega}{1+z}$$

This happens to the *microwave background* as the universe expands.

Note that from the observed energy flux we can estimate the temperature ($T \sim 3K$) and the number density ($\sim 400\, cm^{-3}$), and the energy density ($\sim 10^{-35}\, g/cm^3$). Since there is less than one proton and electron per m^3, *photons are by far the most plentiful particles in the universe.*

$$\begin{aligned} N &= \int_0^\infty n(\omega) d\omega = \int_0^\infty \frac{V}{c^3} \frac{1}{\pi^2} \frac{\omega^2 d\omega}{\exp(\beta\hbar\omega) - 1}, && (103) \\ &= 8\pi \left(\frac{kT}{hc}\right)^3 V \int_0^\infty \frac{x^2 dx}{e^x - 1}. && (104) \end{aligned}$$

Using

$$\int \frac{x^n dx}{e^x - 1} = n!\zeta(n+1), \quad \text{where} \quad \zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} \qquad (105)$$

we obtain

$$\boxed{N = 16\pi \left(\frac{kT}{hc}\right)^3 V\zeta(3)} \tag{106}$$

$$E = \int_0^\infty \hbar\omega n(\omega)\, d\omega, \tag{107}$$

$$= \frac{8\pi h}{c^3} \left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1}. \tag{108}$$

Using $\zeta(4) = \sum_1^\infty 1/n^4 = \pi^4/90$, we find

$$\boxed{E = \frac{8\pi^5}{15h^3} \frac{k^4}{c^3} T^4 V} \tag{109}$$

Entropy

$$E = aT^4V, \qquad a = \frac{8\pi^5 k^4}{1h^3c^3}. \tag{110}$$

$\mu = 0 \quad \Rightarrow \quad dE = TdS = -PdV$, therefore

$$\left.\frac{\partial S}{\partial E}\right|_V = \frac{1}{T} = \left(\frac{aV}{E}\right)^{1/4} \quad \Rightarrow \quad S = \frac{4}{3}(aV)^{1/4}E^{3/4} + f(V). \tag{111}$$

$S = 0$ as $T \downarrow 0$ implies

$$\boxed{S = \frac{4a}{3} T^3 V} \tag{112}$$

Pressure

$$F = E - TS = -\frac{a}{3}T^4V, \qquad P = -\left.\frac{\partial F}{\partial V}\right|_T. \tag{113}$$

$$\boxed{P = \frac{1}{3}\frac{E}{V} = \frac{1}{3} \text{ energy density}} \tag{114}$$

Adiabatic Expansion

$$\begin{aligned} S = \text{ constant } \Rightarrow \quad VT^3 &= \text{constant} \\ \Rightarrow \quad PV^{4/3} &= \text{constant} \end{aligned}$$

3.7 Fermi-Dirac distribution

If no more than one of g_i states may be occupied, then given n_i particles, we must choose n_i of the states to fill

$$W_{\rm FD} = \prod_i \binom{g_i}{n_i} = \prod_i \frac{g_i!}{n!(g_i - n_i)!}, \tag{115}$$

therefore

$$\frac{\partial \log W_{\rm FD}}{\partial n_i} = -\log n_i + \log(g_i - n_i) \tag{116}$$

and

$$\boxed{\bar{n}_i = \frac{g_i}{\exp\beta(E_i - \mu) + 1}} \tag{117}$$

3.7.1 Massless Neutrinos

$$\bar{n}(\vec{p})d^3p \; = 2 \times \frac{V}{h^3} \frac{d^3p}{\exp\beta pc + 1} \tag{118}$$

(n.b. the factor of 2 comes from the two helicity states).

$$N \;\; = \;\; 8\pi \left(\frac{kT}{hc}\right)^3 V \int_0^\infty \frac{x^2 dx}{e^x + 1}, \tag{119}$$

$$E \;\; = \;\; \frac{8\pi h}{c^3} \left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x + 1}, \tag{120}$$

but

$$\int_0^\infty \frac{x^n dx}{e^x + 1} = \left(1 - \frac{1}{2^n}\right) \int_0^\infty \frac{x^n dx}{e^x - 1}, \tag{121}$$

therefore

$$\boxed{\begin{aligned} N^{\rm neutrinos} &= \frac{3}{4} N^{\rm photons} \\ E^{\rm neutrinos} &= \frac{7}{8} E^{\rm photons} \end{aligned}} \tag{122}$$

3.7.2 Degenerate Fermi Gas

Let $\mu = E_F$, "Fermi energy".

$$\begin{aligned} \lim_{\beta\to\infty} \frac{1}{\exp\beta(E-E_F)+1} &= 1 \qquad E < E_F \\ &= 0 \qquad E > E_F \end{aligned} \tag{123}$$

Therefore, if $\beta E_F \gg 1$ all states are uniformly populated inside a sphere of radius

$$P_F = \sqrt{\frac{E_F^2 - M^2c^4}{c^2}}, \tag{124}$$

in momentum space.

$$\frac{N}{V} \equiv n = \frac{4\pi}{3h^3} g_s P_F^3 \tag{125}$$

where g_s is spin degeneracy.

$$\frac{E}{V} = \frac{4\pi g_s}{h^3}\int_0^{P_F} P^2\sqrt{P^2c^2+M^2c^4}dP \tag{126}$$

Calculation of pressure as a function of density $dS = 0$ implies $P = -\frac{\partial E}{\partial V}|_S$. The answer depends on $n(\frac{\hbar}{Mc})^3$.

For $n \ll (\frac{Mc}{\hbar})^3$, the particles are *non-relativistic*, and

$$\frac{E}{V} = g_sMc^2 + \frac{4\pi g_s}{h^3}\frac{P_F^5}{10M} \tag{127}$$

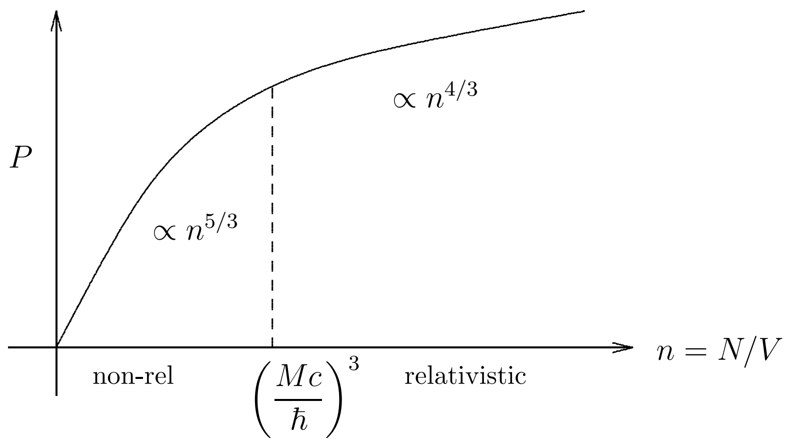
$$= g_sMc^2 + \left(\frac{4\pi g_s}{h^3}\right)^{2/3}\left(\frac{3N}{V}\right)^{5/3}\frac{1}{10M}, \tag{128}$$

$$P = \frac{2}{3}\left(\frac{E}{V} - g_sMc^2\right). \tag{129}$$

For $n \gg (\frac{Mc}{\hbar})^3$, the particles are *relativistic* and

$$\frac{E}{V} = \frac{4\pi g_s}{h^3}\frac{P_F^4 c}{4} = \frac{3c}{4}\left(\frac{4\pi g_s}{3h^3}\right)^{-1/3}\left(\frac{N}{V}\right)^{4/3} \tag{130}$$

$$P = \frac{1}{3}\left(\frac{E}{V}\right) \tag{131}$$



Neutrinos Neutrinos were postulated by Fermi in the 1930's to explain missing energy in β-decay in nuclei. The basic reaction is

$$n \to p + e^- + \bar{\nu}_e \qquad \sim 13 \text{ minutes} \tag{132}$$

(where $\bar{\nu}_e$ is an (anti)-neutrino). Neutrinos:

i) are electrically neutral,

ii) move at the velocity of light,

iii) have helicity or spin $= \hbar/2$.

Experiments carried out in 1957 by Wu at the suggestion of Lee & Yang showed that neutrinos emitted in β-decay spin in a right-handed sense (i.e. parity or left-right symmetry is violated).

Particles & Anti-Particles According to Dirac, to every particle there is an antiparticle, e.g.

i) *electron* (2 spin states) $\longleftrightarrow$ *positron* (2 spin states) — massive, $M \neq 0$

ii) a *photon* is its own antiparticle and it has 2 spin states — massless, $M = 0$

iii) *neutrino* ν_e, left-handed, 1 state $\longleftrightarrow$ *anti-neutrino* $\bar{\nu}_e$, right-handed, 1 state — massless, $M = 0$

	In addition to the	electron-neutrinos ν_e	associated to the	electron e
Three Families	there are	muon-neutrinos ν_μ	associated to the	muon μ
	and	tau-neutrinos ν_τ	associated to the	tau particle τ

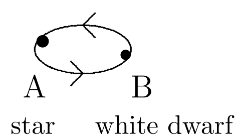
3.8 White Dwarfs, Neutron Stars and Black Holes

Once a star has exhausted it's nuclear fuel it can only be supported against the inward forces due to its own gravitional field by "degeneracy pressure" resulting from the Pauli Exclusion Principle. Depending upon which particle is degenerate we get a white dwarf or a neutron star. If degeneracy pressure is insufficient then nothing else can hold the star up and it must collapse to form a black hole. The discussion given below is mainly qualitative and concentrates on orders of magnitude.

3.8.1 White Dwarfs

In this case we have degenerate assembly of electrons, with non-degenerate protons to give overall charge neutrality. A typical radius is comparable with that of the earth, but the mass is comparable with that of the sun. The density is $\sim 10^5\, g/cm^3$.

Example: Sirius Bessel observed elliptical orbit in 1844. Alvan Clark observed a faint companion in 1862. Adams observed it to be white hot in 1914.



Using the theory of Newtonian orbits one may deduce that the faint companion has mass $\sim \frac{4}{5} M_\odot$. The observed luminosity tohether with the black body emission allows one to estimate that radius $< R_\odot/19$ (where $M_\odot$ and $R_\odot$ are solar mass and radius). This gives a density of $60,000\, g/cm^3$.

In 1924 R.H. Fowler suggested it was made of a superperfect gas (i.e. whta we now call degenerate). S. Chandrasehkar showed that no such configuration exists for $M > 1.5 M_\odot$, *Chandrasehkar Limit*.

3.8.2 Neutron Stars

Landau suggested that for more massive stars it is possible that inverse β-decay takes place:

$$e^- + p \to \nu_e + n. \tag{133}$$

The dense degenerate assembly of neutrons is called a *neutron star*. Landau observed that there is also an upper bound for the mass of a neutron star $M/M_\odot < \sim 6$.

3.8.3 Black Holes

If

$$\frac{GM}{R} > \frac{c^2}{2} \quad \Rightarrow \quad R < R_S = \frac{2GM}{c^2} \tag{134}$$

not even light can escape from the collapsing star (where R_S is the Schwarzschild radius).

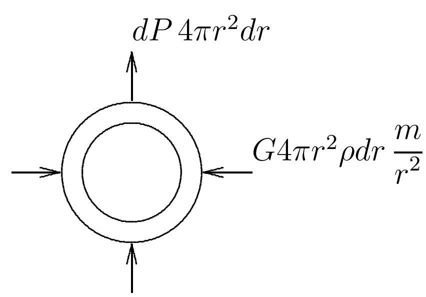
3.8.4 Stellar Structure

Each element in the static star is in equilibrium under two forces, a pressure gradient and gravity,

$$\vec{\nabla} P + \rho \vec{\nabla} \Phi = 0 \tag{135}$$

where ρ is the density, and Φ is the Newtonian potential defines such that $-\rho\vec{\nabla}\Phi$ is the gravitaional force on a unit volume of material of density ρ.

Assume spherical symmetry: $m(r)$ is the mass inside a sphere of radius r



$$\boxed{\begin{aligned} \frac{dP}{dr} &= -G\rho\frac{m(r)}{r^2}, \\ \frac{dm}{dr} &= 4\pi r^2 \rho. \end{aligned}} \tag{136}$$

Boundary conditions

$$\left.\frac{\partial P}{\partial r}\right|_{r=0} = 0, \qquad P|_{r=R} = 0. \tag{137}$$

If we assume a relation $P = P(\rho)$ between pressure P and density ρ , referrred to as an "equation of state" we may integrate these equations.

3.8.5 Virial Theorem

We can get a feel for the solutions by establishing a general relationship , valid for all Newtonian stars, relating their kinetic and potential energy. In fact a result like this hold for any gravitationally bound system.

$$\int_0^R 4\pi r^3 dP = -\int_0^R \frac{Gm(r)}{r} dm(r)\,. \tag{138}$$

Integrating by parts,

$$\boxed{3\int_0^R P 4\pi r^2 dr = \int_0^R \frac{Gm(r)dm(r)}{r}} \tag{139}$$

E.g., $P = (\gamma - 1)\varepsilon$ where $\varepsilon =$ energy density

$$3(\gamma - 1)\int_0^R 4\pi r^2 \varepsilon dr = \int_0^R \frac{Gm(r)dm(r)}{r} \tag{140}$$

Therefore

$$\begin{aligned} 3(\gamma-1)\,\text{Thermal Energy} &= -\,\text{Gravitational Potential Energy} \\ \text{Thermal} + \text{Potential Energy} &= (4-3\gamma)\,\text{Thermal Energy} \end{aligned} \tag{141}$$

Now any bound sytem must have negative total energy and we therefore deduce that

$$\boxed{\text{A star cannot be bound stably if } \gamma \leq \frac{4}{3}}$$

The final conclusion is that if the density increases so that material becomes sufficiently relativistic that $\gamma \leq \frac{4}{3}$, the star becomes unstable.

3.8.6 Order of Magnitude Estimates

(*factors of order unity ignored*)

Pressure support implies,

$$\frac{P}{R} \simeq \frac{GM}{R^2}\rho \quad \Rightarrow \quad \frac{P}{\rho} \simeq \frac{GM}{R} \tag{142}$$

White Dwarfs

i) nucleons dominate the mass, therefore $\rho \simeq n_c m_N$, where m_N is nucleon mass and

$$\begin{aligned} n_c &= \text{critical number density of electrons} \\ &= \text{number density of protons} \end{aligned}$$

ii) electrons provide the pressure,

$$P \simeq \frac{\hbar^2}{m_e} n_c^{5/3}, \qquad n_c \simeq \left(\frac{m_e c}{\hbar}\right)^3, \tag{143}$$

therefore

$$\frac{P}{\rho} \simeq \frac{\hbar^2}{m_e m_N} n_c^{2/3}, \tag{144}$$

and

$$\boxed{\frac{GM}{c^2 R} \simeq \frac{m_e}{m_N}} \tag{145}$$

The Chandrasehkar mass $M \simeq \frac{4\pi}{3} R^3 \rho$ gives

$$\boxed{M = \frac{1}{m_N^2}\left(\frac{\hbar c}{G}\right)^{3/2}} \tag{146}$$

Neutron Stars In free space,

$$n \to p + e^- + \bar{\nu}_e \qquad (\beta\text{-decay}). \tag{147}$$

In the deep potential well of the star, the inverse is favoured,

$$p + e \to n + \nu_e \quad (\text{inverse } \beta\text{-decay}). \tag{148}$$

Pressure is now provided by neutrons. Replace m_e by m_N in the above:

$$\boxed{\frac{GM}{c^2 R} \simeq \mathcal{O}(1)} \tag{149}$$

$$\boxed{M = \frac{1}{m_N^2}\left(\frac{\hbar c}{G}\right)^{3/2}} \tag{150}$$

Therefore,

escape velocity $\simeq$ light velocity.

4 Cosmology

4.1 Isotropy & Homogeneity of the Universe

We begin our study of cosmology by summarising some of the data about its basic construction.

4.1.1 Cosmic Distance Ladder

- distances of stars in our galaxy are determined by trigonometry.
- distances to nearby galaxies are determined by period-luminosity relation of cepheid variable stars.
- distances to more distant galaxies may also be estimated using Tully-Fisher relation between luminosity L and rotation speed v: $L \propto v^4$.
- distances to yet more distant galaxies are estimated by assuming brightest galaxy in a cluster has some intrinsic luminosity for all clusters.

4.1.2 Hubble Expansion

Deduce velocity v from Doppler shift:

$$\frac{\lambda_{\rm obs}}{\lambda_{\rm em}} = 1 + z = \sqrt{\frac{c+v}{c-v}} \tag{151}$$

where z is redshift.

Last year it was true that the most distant galaxy observed had $z = 4.38$
the most distant quasar observed had $z = 4.9$

In March 1998 a paper appeared claiming to have observed a galaxy with a redshift $z = 5.34$.

Hubble's Law: isotropic expansion about us,

$$\boxed{\vec{v} = H_0 \vec{r}} \tag{152}$$

where H_0 is the Hubble constant, which is "probably" $50 < H_0 < 50\, Km/sec/Mpc$. $H_0 = 50$ implies

$$\begin{aligned} \text{Hubble time} &= \frac{1}{H_0} = 19.5\, Gyrs \\ \text{Hubble radius} &= \frac{c}{H_0} = 6000\, Mpc. \end{aligned} \tag{153}$$

Useful Orders of Magnitude Age of earth $4.5\, Gyrs$.
Age of globular clusters $8 - 15\, Gyrs$.
$c = 300\, Mpc/Gyr$.

$$\begin{aligned} 1\,\text{parsec} &\simeq 3.09 \times 10^{18}\, cm \simeq 3.26\,\text{light years}, \\ &= \text{distance at which } \underbrace{\text{earth-sun distance}}_{\text{1 astronomical unit}} \text{ subtends 1 arcsec.} \end{aligned}$$

Distance to Andromeda Galaxy $\simeq 0.5\, Mpc$.

4.1.3 Cosmic microwave background

is isotropic and thermal.

$$T = 2.735 \pm 0.06\,K, \quad \text{as measured by COBE satellite.}$$

There exist *dipole variations* due to Doppler shift of our galaxy towards Virgo cluster and Hydra Centauraus Supercluster $\sim 10^{-3}\,c$. In fact COBE sees Doppler shift due to the earth's motion about the sun ($\sim 30\,Km/sec$).

There exist *quadrupole variations* $\sim 10^{-5}$ first seen by COBE. These correspond to density fluctuations which grew to form galaxies, stars, planet, etc.

Energy Density

$$\begin{aligned} T \sim 2.7K &\equiv 1\,eV/cm^3 \\ &\cong 400\,\text{photons}/cm^3 \end{aligned} \tag{154}$$

Microwave photons have the same total energy as starlight but are much more numerous.

4.2 Kinematics of Homogeneous Isotropic Expansion

- Each galaxy is assumed to satisfy

$$\vec{v}_i = H(t)\vec{r}_i(t), \tag{155}$$

  i.e. isotropy about *us*.

- *Present value of Hubble's constant* $H_0 = H(t_0)$, where t_0 is time now.

$$\vec{v}_i = \frac{d\vec{r}_i}{dt} = H(t)\vec{r}_i \tag{156}$$

  where $\vec{r}_i$ is proper or physical distance.

$$\boxed{\begin{aligned} \vec{r}_i(t) &= a(t)\vec{x}_i \\ H(t) &= \frac{\dot{a}(t)}{a(t)} \end{aligned}} \tag{157}$$

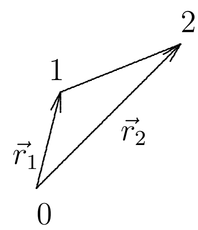
  and $\vec{x}_i$ is independent of time. Therefore,

$$a(t) = a(t_0)\exp\int_{t_0}^{t} H(t')dt' \tag{158}$$

- $a(t)$ is called the *scale factor*.
- $\vec{x}_i$ is called the *comoving coordinate* of the i^{th} galaxy.

4.2.1 Homogeneity and the Cosmological Principle

Choose galaxy 1 as new origin:



$$\begin{aligned}\vec{\tilde{r}}_i &= \vec{r}_i - \vec{r}_1 \\ &= a(t)\vec{x}_i - a(t)\vec{x}_1\end{aligned} \tag{159}$$

$$\boxed{\vec{\tilde{r}}_i = a(t)\vec{\tilde{x}}_i} \tag{160}$$

where $\vec{\tilde{x}}_i = \vec{x}_i - \vec{x}_1$.

Thus, the expansion is isotropic relative to *any* galaxy participating in the Hubble flow.

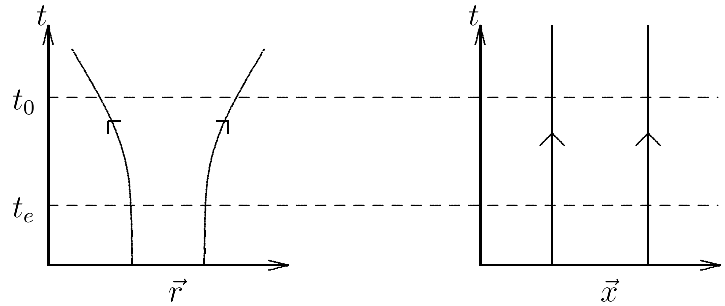
Copernican Principle The earth is not in a specially favoured position in the universe.

Cosmological Principle The universe presents the same aspect from every point except for local irregularities, i.e. the universe seems homogeneous and isotropic when averaged over a suitably large scale.

Therefore, there exists some overall mass density ρ which depends only on time

$$\boxed{\rho = \rho(t)} \tag{161}$$

4.2.2 Behaviour of Distances, Areas, and Volumes



physical distance comoving coordinate

$$\begin{aligned} \vec{r} &= a(t)\vec{x}, \\ d\vec{r} &= a(t)d\vec{x}, \\ dl &= |d\vec{r}| = a(t)\,|d\vec{x}|\,. \end{aligned} \tag{162}$$

Thus,

all	distances	scale	$\propto a(t)$
all	areas	scale	$\propto a^2(t)$
all	volumes	scale	$\propto a^3(t)$.

Example

i) galaxies: number is constant in a comoving volume

$$\begin{aligned} \rho(t)dV &= \text{constant} \\ \rho(t)a^3(t)d^3\vec{x} &= \text{constant} \end{aligned} \tag{163}$$

Therefore,

$$\boxed{\rho a^3 = \text{constant}} \tag{164}$$

ii) pressure free fluid, "dust": same argument $\rho \propto a^3$

iii) fluid with pressure: assume expansion is *adiabatic*,

$$sdV = \text{constant}, \tag{165}$$

where $s = S/V$ is the entropy per unit volume. Then

$$\boxed{sa^3 = \text{constant}} \tag{166}$$

e.g. *microwave photons*, $s \propto T^3$,

$$\boxed{Ta = \text{constant}} \tag{167}$$

Since n_γ = number of photons per unit volume $\propto s \propto T^3$, we also deduce that

$$\boxed{n_\gamma a^3 = \text{constant}} \tag{168}$$

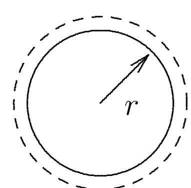
Energy density $\propto T^4 \propto 1/a^4$.

4.3 Dynamics

The equations governing the bahaviour of the scale fcator $a(t)$ cna be derived using simple Newtonian reasoning, subject only to an additional term in the Newtonian force law which enters when the pressure is significant compared with ρc^2. The resulting (consistent) set of equations are identical to those derived from Einstein's theory of General Relativity.

4.3.1 Dynamics of Isotropic Homogeneous Pressure-Free Fluid

Consider a small sphere of radius $r = |\vec{r}| = a(t)\,|\vec{x}|$ and apply Newton's Law to a small shell of thickness dr and mass $\delta m = 4\pi r^2\,dr$



$$\begin{aligned} \delta m\,\ddot{r} &= -\frac{Gm}{r^2}\delta m, && (169) \\ \delta m\,\ddot{a}\,|\vec{x}| &= -\frac{Gm}{a^2}\frac{\delta m}{|\vec{x}|^2}. && (170) \end{aligned}$$

The mass inside radius r is

$$m = \frac{4\pi}{3}\rho r^3 = \frac{4\pi}{3}\rho a^3\,|\vec{x}|^3 = \text{constant}. \tag{171}$$

Then,

$$\delta m\,|\vec{x}|\,\ddot{a} = -\frac{4\pi}{3}\frac{G\rho}{a}\delta m\,|\vec{x}|\,, \tag{172}$$

which gives the *Raychaudhuri equation*,

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G\rho}{3}} \tag{173}$$

n.b.

i) arbitrary radius of sphere cancels

ii) we can ignore GR because

$$\frac{GM}{c^2 r} = \frac{4\pi}{3}\frac{G\rho}{c^2}r^2$$

can be made as small as we like by taking r sufficiently small.

First Integral

$$\ddot{a} = -\frac{GM}{a}, \qquad M = \frac{4\pi}{3}\rho a^3 = \text{constant}. \tag{174}$$

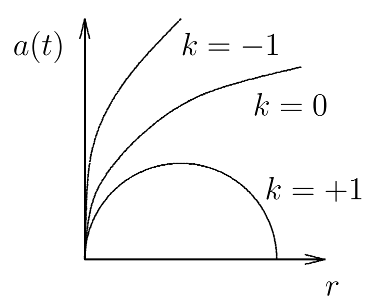
Integrating we obtain

$$\frac{\dot{a}^2}{2} - \frac{GM}{a} = \text{constant} = -\frac{k}{2} \tag{175}$$

which is the *Friedman equation*,

$$\boxed{\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{k}{a^2}} \tag{176}$$

By scaling $a \to \lambda a$ we may take:
$k = +1$ bound $\Rightarrow$ universe recollapses
$k = 0$ marginal
$k = -1$ unbound $\Rightarrow$ universe expands forever



Solutions

$$\dot{a}^2 - \frac{2GM}{a} = -k \tag{177}$$

Einstein-DeSitter: $k = 0$

$$a = a_0 \left(\frac{t}{t_0}\right)^{2/3}, \tag{178}$$

implies,

$$\frac{\dot{a}}{a} = \frac{2}{3}\frac{1}{t_0} = H_0. \tag{179}$$

The age of the universe is

$$\boxed{t_0 = \frac{2}{3}\frac{1}{H_0}} \tag{180}$$

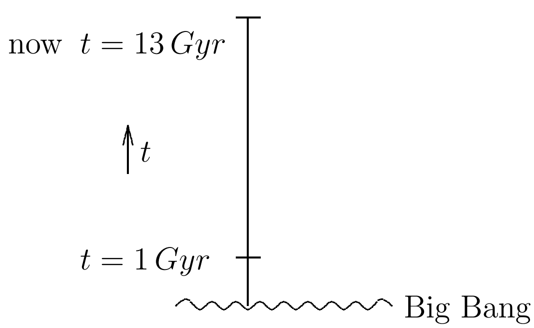
$H_0 \simeq 50\, Km/sec/Mpc$ gives $t_0 \approx 13\, Gyrs$.

Redshift

$$(1+z) = \left(\frac{t_0}{t}\right)^{2/3}, \tag{181}$$

$$\boxed{\frac{t}{t_0} = \frac{1}{(1+z)^{3/2}}} \tag{182}$$

E.g., (typical quasar) $z = 4.5$ gives $t \approx 1\, Gyr$



The density of quasars is bigger by a factor of $(5.5)^3$.

The Age Problem

- Stars in the disc of our galaxy are young and rich in metals ("Population I").
- Stars in the halo, in global clusters are old and poor in metals ("Population II"). Globular clusters are believed to have ages: $12 \leq \text{age} \leq 16\,Gyrs$.

Therefore, if $k = 0$ then Hubble's constant H_0 cannot be too large.

Recollapsing solutions: $k = +1$

$$\dot{a}^2 = \frac{GM}{a} - 1. \tag{183}$$

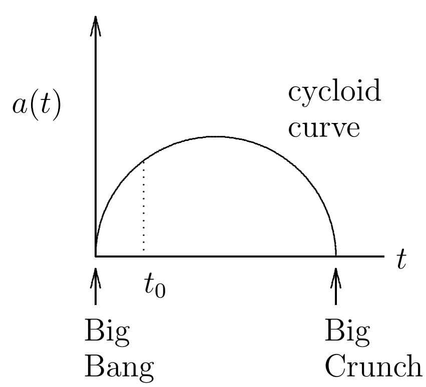
To solve,

$$\frac{a^{\frac{1}{2}}\, da}{\sqrt{2GM - a}} = dt \tag{184}$$

substitute $a = 2GM \sin^2\theta$, then

$$2GM\, 2\sin^2\theta\, d\theta = dt \tag{185}$$

$$2GM\left(\theta - \frac{\sin 2\theta}{2}\right) = t \tag{186}$$

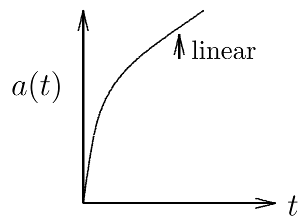


$$\begin{aligned} \text{max scale factor} \quad a_{\max} &= 2GM \\ \text{total age} \quad t_{\text{total}} &= 2GM\pi \end{aligned}$$

Expanding solutions: $k = -1$ Substitute $a = 2GM\sinh^2\theta$,

$$2GM\left(\frac{\sinh 2\theta}{2} - \theta\right) = t, \tag{187}$$

$a \sim t$ as $t \to \infty$.



Limiting forms

- as $t \downarrow 0$, $a(t) \sim t^{2/3}$, $\forall k$.
- as $t \uparrow \infty$, $a(t) \sim t$, $k = -1$ ("Milne" model).
- the $k = 0$ case is an *attractor to the past*, but a *repellor to the future*.

The flatness problem Today we are close to the Einstein-DeSitter model, which is a *repellor*. This must have required very special initial conditions.

4.3.2 Dynamics of Homogeneous Isotropic Fluid with Pressure

We take from GR the result that we must replace ρ by $(\rho + 3P/c^2)$ in the *Raychaudhuri equation*:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)} \tag{188}$$

First Integral is the same as before:

$$\boxed{\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{k}{a^2}} \tag{189}$$

Proof

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\dot{a}^2 - \frac{4\pi G}{3}\rho a^2\right) &= \dot{a}\ddot{a} - \frac{4\pi G}{3}\dot{\rho}a^2 - \frac{8\pi G}{3}\rho\dot{a}a \qquad (190)\\ &= -\frac{4\pi G a^2}{3}\left\{\dot{\rho} + \left(\rho + \frac{P}{c^2}\right)\frac{3\dot{a}}{a}\right\}, \quad \text{on substitution of eq. (188)}\\ &= 0, \quad \text{by Thermodynamic relation.}\end{aligned}$$

Example: Radiation Dominated Universe $\frac{P}{c^2} = \frac{1}{3}\rho$; $\gamma = \frac{4}{3}$. Assume $k = 0$, then

$$\left(\frac{\dot{a}}{a}\right)^2 \propto \frac{1}{a^4} \quad \Rightarrow \quad a \propto t^{1/2}, \tag{191}$$

$$\boxed{\rho = \frac{3}{32\pi G}\frac{1}{t^2}} \tag{192}$$

Age $a \propto t^{1/2}$, therefore

$$\frac{\dot{a}}{a} = \frac{1}{2t} = H, \tag{193}$$

$$\boxed{t_0 = \frac{1}{2H_0}} \tag{194}$$

E.g., if $H_0 = 50\, Km/Mpc/sec$, then $t_0 \approx 9.25\, Gyrs$ which is too short. Thus, probably the universe is *not* dominated by massless particles.

Temperature Time Relation

$$\rho = \left(\frac{8\pi^5 k^4}{30h^3c^5}\right) T^4 g(T) \tag{195}$$

where

$$\begin{aligned} g(T) &= 2 \quad \text{photons} \\ &= \frac{7}{8} \quad \text{neutrinos, etc.} \end{aligned} \tag{196}$$

Therefore

$$T \propto t^{-1/2}. \tag{197}$$

E.g.,

$T \sim 10^9\, K$	$\Rightarrow$	$t \sim 100\, sec$	nucleosynthesis
$T \sim 10^{10}\, sec$	$\Rightarrow$	$t \sim 1\, sec$	electron-pair annihilation

4.3.3 General Upper Bound on the Age of the Universe

The result which follows is most easily seen by noting that if $\ddot{a}$ is negative then the graph of $a(t)$ against t must lie below its tangent. The argument given below however is actually the essential idea behind the famous Hawking-Penrose singularity theorems which show that classical physics implies that the universe had a singularity at the Big bang. Because it is so simple I will give a slightly longer and more formal argument which generalizes.

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) a. \tag{198}$$

Then

$$\left(\frac{\dot{a}}{a}\right)^{\cdot} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) - \left(\frac{\dot{a}}{a}\right)^2, \tag{199}$$

$$\dot{H} = -H^2 - \frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right). \tag{200}$$

Assume $\rho + 3P/c^2$:

$$\dot{H} < -H^2, \tag{201}$$

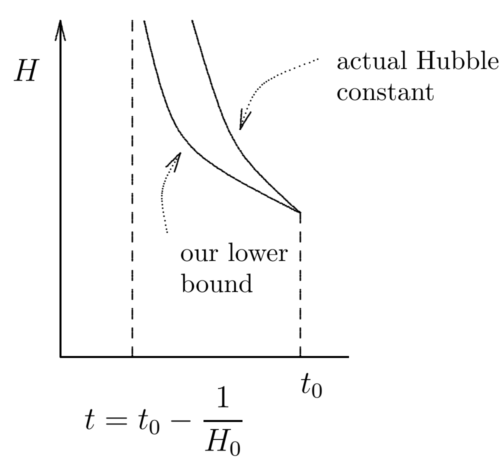
$$\frac{dH}{H^2} < -dt, \tag{202}$$

$$\frac{1}{H} - \frac{1}{H_0} < t - t_0, \tag{203}$$

$$\boxed{H \geq \frac{H_0}{1 + H_0(t - t_0)}} \tag{204}$$

Then

$$\boxed{\text{age} \leq \frac{1}{H_0}}$$



4.4 Redshifting & Motion of Light

In a time dt a photon travels towards us a comoving distance dx and physical distance $a(t)\,dx$. The physical velocity $= a(t)\frac{dx}{dt} = c$, i.e. the motion is given by

$$\boxed{dx = -\frac{c\,dt}{a(t)}} \tag{205}$$

Example

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3}} \qquad \text{(we use units s.t. } a_0 = 1\text{)}, \tag{206}$$

$$-dx = ct_0^{\frac{2}{3}}\frac{dt}{t^{\frac{2}{3}}}, \tag{207}$$

$$x = 3ct_0^{\frac{2}{3}}\left(t_0^{\frac{1}{3}} - t_e^{\frac{1}{3}}\right) \qquad \text{(since } x = 0 \text{ at } t = t_0\text{)}, \tag{208}$$

$$x = 3ct_0\left(1 - \left(\frac{t_e}{t_0}\right)^{\frac{1}{3}}\right), \tag{209}$$

but

$$1 + z = \left(\frac{t_0}{t_e}\right)^{\frac{2}{3}}, \tag{210}$$

therefore,

$$\boxed{x = 3ct_0\left(1 - \frac{1}{(1+z)^{\frac{1}{2}}}\right)} \tag{211}$$

Limiting values:

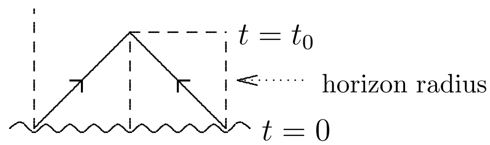
i) z small, implies (Hubble's Law)

$$x \simeq \frac{3}{2}ct_0 z \simeq \frac{c}{H_0}z. \tag{212}$$

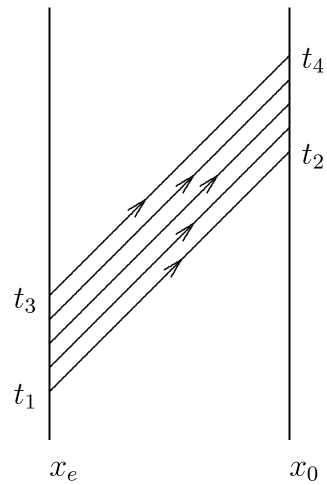
ii) $z \to \infty$,

$$x \to 3ct_0 = \frac{2c}{H_0}. \tag{213}$$

Horizon Light coming from galaxies outside a co-moving radius $2c/H_0$ has not yet had time to reach us.



4.4.1 Proof of Redshift Formula



$$\begin{aligned} \frac{1}{c}(x_e - x_0) &= \frac{1}{c}\int_{x_0}^{x_e} dx\,, & (214)\\ &= \int_{t_1}^{t_2} \frac{dt'}{a(t')}, & (215)\\ &= \int_{t_3}^{t_4} \frac{dt'}{a(t')}. & (216) \end{aligned}$$

Set,

$$\begin{aligned} t_3 &= t_1 + \Delta t_e, \\ t_4 &= t_2 + \Delta t_0. \end{aligned}$$

Then

$$\int_{t_1}^{t_2} \frac{dt'}{a(t')} = \int_{t_1+\Delta t_e}^{t_2+\Delta t_0} \frac{dt'}{a(t')}. \tag{217}$$

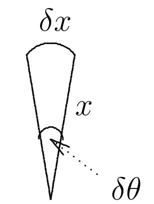
Now in the limit as Δt_0, $\Delta t_e \downarrow 0$,

$$\frac{\Delta t_0}{a(t_0)} = \frac{\Delta t_e}{a(t_e)}, \tag{218}$$

$$\frac{\Delta t_0}{\Delta t_e} = \frac{\lambda_0}{\lambda_e} = \frac{\nu_e}{\nu_0} = 1 + z = \frac{a(t_0)}{a(t_e)}, \tag{219}$$

$$\boxed{1 + z = \frac{a(t_0)}{a(t_e)}} \tag{220}$$

Example: angular sizes A galaxy of size l_0, i.e. physical or proper length l_0, will subtend an angle $\delta\theta$



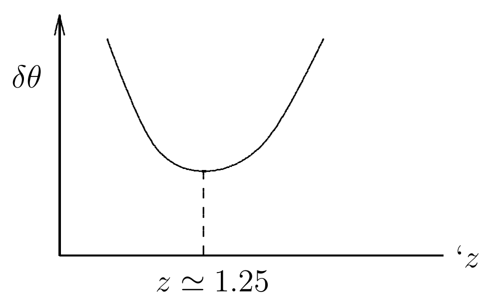
such that,

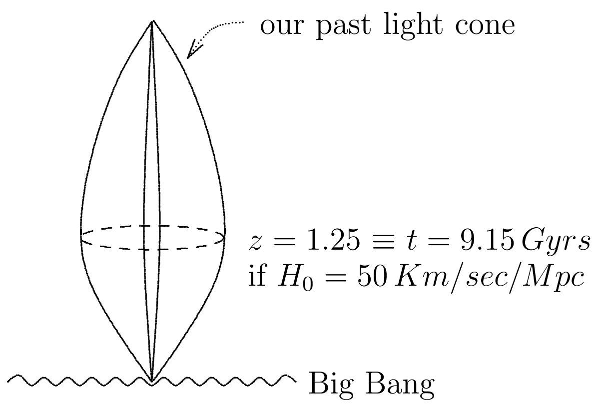
$$\begin{aligned} a(t_e)\delta x &= l_e, \\ \delta\theta &= \frac{\delta x}{x}, \end{aligned} \tag{221}$$

implies,

$$\delta\theta = \frac{l_e}{a(t_e)x}. \tag{222}$$

If we assume $a(t) \propto t^{2/3}$ (Einstein-DeSitter case), we find





4.4.2 Singularity Theorems (Hawking & Penrose)

Since all particles travel no faster than light they must be confined inside the light cone. Following them back into the past we see that if the past light cone reconverges the density will become infinite when the past light cone converges to a point at the big bang. This is the basis of a rigourous mathematical theorem showing that classical general relativity must break down at the big bang singularity.

4.5 Thermal History of the Universe

- The thermal evolution of the the universe depends on the dimensionless rate $\eta = n_B/n_\gamma$, where n_B = number density of baryons, n_γ = the number density of photons.

- $n_B \propto 1/a^3$, $n_\gamma \propto 1/a^3$, implies

$$\boxed{\eta = \text{constant}}$$

- since the photon entropy $\propto n_\gamma$,

$$\frac{1}{\eta} \simeq \frac{\text{entropy}}{\text{baryon}}$$

- $n_\gamma \simeq 400\,/cm^3$

- $n_B \geq 3 \times 10^{-8}\,/cm^3$ in luminous matter

- $n_B < 3 \times 10^{-7}\,/cm^3$ from theory of nucleosynthesis

- $n_B = 3 \times 10^{-6}(H_0/50)^2$

Thus

$$\boxed{10^{-10} < \frac{n_B}{n_\gamma} < 10^{-9}} \tag{223}$$

4.5.1 Decoupling

- At high temperatures hydrogen was ionised to form a "plasma" which is *opaque* to electromagnetic radiation.

- At low temperatures protons and electrons combine to form Hydrogen

  $$p + e^- \to H$$

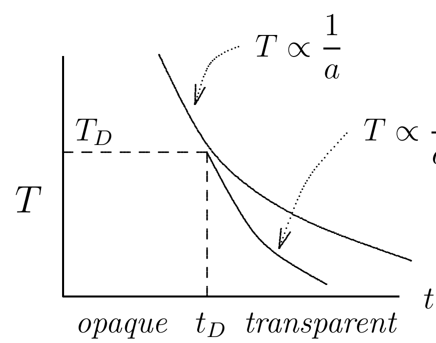
  which is *transparent* to radiation.

- The transition occurs at the ionisation temperature

  $$T_D \simeq \frac{1}{2}\frac{m_e c^2}{k}\alpha^2 \simeq 4,533\,K \simeq 13.6\,eV \tag{224}$$

  where $\alpha = \left(\frac{e^2}{4\pi\hbar c}\right) \simeq 1/137$,

  $$\boxed{T \simeq 4.5 \times 10^3 \quad \Leftrightarrow \quad z \simeq 1.5 \times 10^3} \tag{225}$$



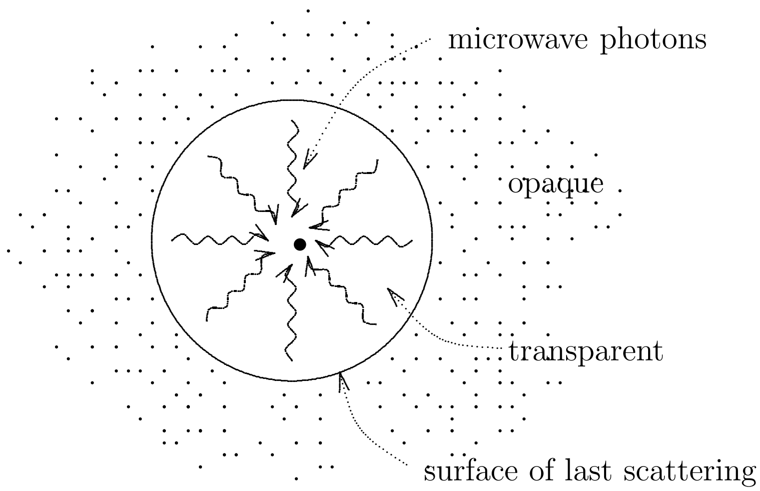
- The collisions between photons and electrons maintain thermal equilibrium before t_D.

- After t_D the protons retain their Planckian spectrum without collisions just by redshifting as explained earlier

- After t_D the hydrogen cools adiabatically as a non-relativistic gas, $PV^{5/3} = \text{constant}$, or $VT^{3/2} = \text{constant}$

  $$\boxed{T \propto \frac{1}{a^2}} \tag{226}$$

  Thus, the hydrogen cools more rapidly than photons.



- Using $z \approx 1.5 \times 10^3$ and $(1+z) = (t_0/t_e)^{2/3}$, if $k = 0$ we deduce that $t \approx 10^5\, yrs$.

4.5.2 Decoupling Temperature & Temperature at End of Radiation Era

- Let f_B be the fraction of total matter density in baryons,

$$\rho_{\text{matter}} = \frac{1}{f_B} m_N n_B. \tag{227}$$

- photon energy density is given by

$$\rho_\gamma = \frac{3!}{2!} \frac{\zeta(4)}{J(3)} \frac{kT}{c^2} n_\gamma \tag{228}$$

- $\rho_{\text{matter}} = \rho_\gamma$ at $T = T_R$, implies

$$\boxed{T_R = \frac{m_N c^2}{f_B k} \frac{2!}{3!} \frac{\zeta(3)}{\zeta(4)} \left(\frac{n_B}{n_\gamma} \right)} \tag{229}$$

$$\boxed{\frac{T_R}{T_D} \simeq \frac{m_N}{m_e} \frac{1}{2\alpha^2} \frac{\zeta(3)}{\zeta(4)} \left(\frac{n_B}{n_\gamma} \right) \frac{1}{f_B}} \tag{230}$$

$$\frac{m_N}{m_e} \simeq 1836; \quad \alpha \simeq \frac{1}{137}; \quad f_B = \mathcal{O}\left(\frac{1}{10} \right); \quad \frac{n_B}{n_\gamma} = \mathcal{O}\left(10^{-8}\right).$$

Therefore, $T_R/T_D = \mathcal{O}(1)$, *the two temperatures roughly coincide.*

4.5.3 Nucleosynthesis

If $T > 10^{10}\,K$, the reactions

$$\begin{aligned} n + e^+ &\rightleftharpoons p + \bar{\nu}_e, \\ p + e^- &\rightleftharpoons n + \nu_e, \\ n &\rightleftharpoons p + e^- + \bar{\nu}_e, \end{aligned}$$

are *rapid* and maintain n and p in equilibrium.

Since $kT \ll m_N c^2$ we use non-relativistic formulae:

$$n_n = \left(\frac{2\pi m_N kT}{h^3}\right)^{3/2} \exp\frac{\mu_n}{kT}, \tag{231}$$

$$n_p = \left(\frac{2\pi m_p kT}{h^3}\right)^{3/2} \exp\frac{\mu_p}{kT}. \tag{232}$$

Now $\mu_n - \mu_p$ is the difference in energy increase due to the addition of one neutron or one proton,

$$\boxed{\mu_n - \mu_p = (m_n - m_p)c^2} \tag{233}$$

$$\frac{n_p}{n_n} = \left(\frac{m_p}{m_N}\right)^{3/2} \exp\left[-\frac{(m_n - m_p)c^2}{kT}\right], \tag{234}$$

i.e.,

$$\boxed{\frac{n_p}{n_n} \simeq \exp\left[-\frac{(m_n - m_p)c^2}{kT}\right]} \tag{235}$$

since $m_p/m_n \simeq 1$.

As T dropped below $\sim 10^{10}\,K$ the electron neutrinos ν_e are no longer in equilibrium (they "decouple") and henceforth their abundance drops without reference to anything else (see later). As T dropped to $\sim 10^9\,K$ the neutrons reacted *slowly* to form deuterium,

$$p + n \rightarrow D + \gamma, \qquad D = {}^2H.$$

Deuterium then reacted rapidly to form Helium-4, ${}^4He$

$$\begin{aligned} D + D &\rightarrow {}^3He + n, \\ D + D &\rightarrow {}^3H + p, \\ {}^3He + n &\rightarrow {}^3H + p, \\ {}^3H + D &\rightarrow {}^4He + n, \end{aligned}$$

(where ^3H is tritium).

As a result almost all the neutrinos reacted to form ^4He.

The abundance by mass Y is defined by:

$$Y = \frac{4n_{^4He}}{n_p + n_n}, \tag{236}$$

but $n_{^4He} \approx \frac{1}{2} n_n$ (because almost all n_n's formed ^4He), therefore

$$Y = \frac{2n_n}{n_p + n_n} = \frac{2}{1 + \frac{n_p}{n_n}} = \frac{2}{\left(1 + \exp\left(-\frac{(m_n - m_p)c^2}{kT_*}\right)\right)} \tag{237}$$

where T_* is called the "*freeze out*" temperature.

The freeze out temperature depends on the rate of expansion of the universe which in turn depends on

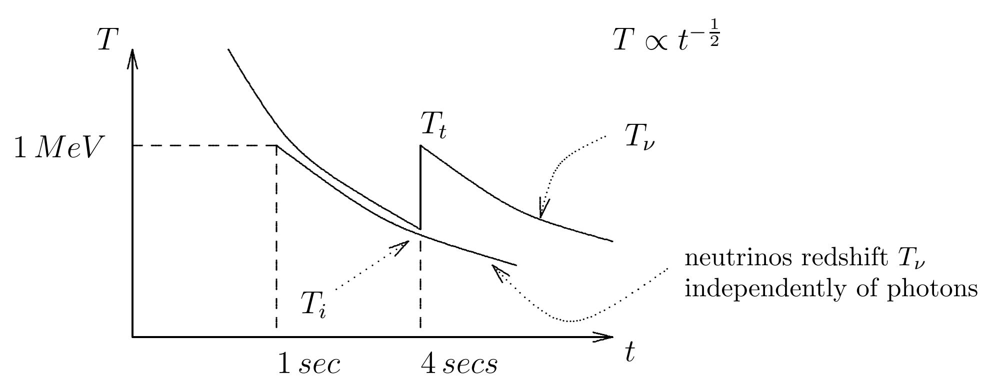
i) $\eta = n_B/n_\gamma$,

ii) the number of neutrino species

$$\begin{aligned} \rho_{\text{rad}} &= \rho_\gamma + \rho_{\nu_e} + \rho_{\nu_\mu} + \rho_{\nu_\tau}, \\ &= \rho_\gamma \left(1 + 3 \times \tfrac{7}{8}\right). \end{aligned} \tag{238}$$

4.5.4 Calculation of Relic Neutrino Number Density

The electron neutrinos, ν_e, went out of equilibrium at $T \simeq 1\,MeV \sim 1\,sec$. Electron-positron pairs annihilated at $T \simeq 0.5\,MeV \sim 4\,secs$. This meant that the temperature rose sharply.



To calculate the reheat temperature we assume that entropy does not decrease. Before:

$$S = (1 + \frac{7}{8} \times 2)\, T_i^3 \times \text{constant} \tag{239}$$

(where 1 comes from the photons and 7/8 is for the electrons and positrons).

After:

$$S = (1 \qquad) \, T_f^3 \times \text{constant} \tag{240}$$

Thus,

$$\frac{T_i}{T_f} = \left(\frac{4}{11}\right)^{\frac{1}{3}}, \tag{241}$$

$$\frac{n_\gamma(T_i)}{n_\gamma(T_f)} = \left(\frac{4}{11}\right). \tag{242}$$

$n_{\nu_e}/n_\gamma = 3/4$ at some temperature, so

$$\frac{n_{\nu_e}}{n_\gamma} = \frac{4}{11} \times \frac{3}{4} = \frac{3}{11}. \tag{243}$$

4.5.5 Comparison with observations

Astronomers have found that

$$Y \simeq 0.23 \pm 0.2, \tag{244}$$

which in turn implies that

i) there are probably no more than three neutrino species,

ii) $3 \times 10^{-10} < \eta < 6 \times 10^{-10}$.

At the same time that the primordial Helium was made (about 25% by weight) ^3He, ^7Li, and ^7Be were also formed. These give good agreement with observations, with $k = 0$ and

$$\frac{\rho_B}{\rho_{\text{matter}}} \simeq 1$$

This indicates that not all the matter in the universe has yet been seen, there must be "*dark matter*" which is probably not baryons but rather some exotic new type of matter.

There is futher evidence for "dark matter" which is based on the observed rotation curves of galaxies and the velocities of galaxies in of clusters of galaxies. In both cases one may use the Virial Theorem to estimate the mass of these systems assuming that they are bound. The estimated mass exceeds the mass of the observed luminous matter, such as stars. the nature of the missing matter is one of the greatest puzzles in astrophysics today.

4.6 Current problems

The cosmological theory developed above is usually thought of as the "standard model". It seems to be well supported by the observations and is accepted at least in outline by most people working in the field. The greatest uncertainty concerns the question of the value of k and whether the cosmological constant is non-zero. In the final lecture of the course I will give a very brief account of some ideas which attempt to take us beyond the standard model and into unknown territory.

4.6.1 Initial Conditions

The hot big bang model is able to account for the present appearance of the universe but it fails to provide an explanation for certain initial conditions. It also leaves unexplained some puzzling questions including:

i) Why is $\eta \simeq 10^{-10}$, i.e. why is there so much entropy per baryon? Why is the total number of baryons non-zero?

ii) Why does the universe have almost zero kinetic energy?

iii) What about "horizons"?

iv) What accounts for the initial fluctuations in density $\delta\rho/\rho \sim 10^{-5}$ on the last scattering surface which grew to make galaxies?

In this lecture we will discuss (ii) (and (iii)).

The Horizon Problem The co-moving radius of the last scattering surface is given by

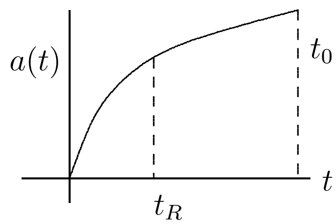
$$\frac{x}{c} = \int_{t_R}^{t_0} \frac{dt}{a(t)} = \int_{t_R}^{t_0} \left(\frac{t_0}{t}\right)^{2/3} dt, \tag{245}$$

$$= 3t_0 \left\{ 1 - \left(\frac{t_R}{t_0}\right)^{1/3} \right\} = 3t_0 \left\{ 1 - \frac{1}{(1+z_R)^{\frac{1}{2}}} \right\} \simeq 3t_0, \tag{246}$$

$$\frac{x}{c} \simeq \frac{2}{H_0}, \qquad x \simeq \frac{2c}{H_0} \tag{247}$$

During the period from the big bang,

$$a(t) = \left(\frac{t}{t_R}\right)^{1/2} \left(\frac{t_R}{t_0}\right)^{2/3}. \tag{248}$$



Thus since the big bang light can have travelled a total comoving distance given by

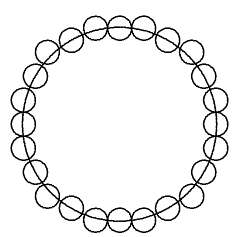
$$\frac{x}{c} = \int_0^{t_R} \left(\frac{t_R}{t}\right)^{1/2} dt \left(\frac{t_R}{t_0}\right)^{-2/3}, \tag{249}$$

$$= 2t_R \left(\frac{t_R}{t_0}\right)^{-2/3} = 2t_0 \left(\frac{t_R}{t_0}\right)^{1/3} = 2t_0 \frac{1}{(1+z_R)^{1/2}}, \tag{250}$$

$$\frac{x}{c} \simeq \frac{2t_0}{z_R^{1/2}} \quad \Rightarrow \quad x \simeq \frac{4}{3}\frac{c}{H_0}\frac{1}{z_R^{1/2}} \tag{251}$$

The ratio

$$\frac{2}{3}\frac{1}{z_R^{1/2}} \simeq \mathcal{O}(10^{-2}). \tag{252}$$



The Kinetic Energy Problem (Also called the horizon problem). Recall that,

$$\Omega = \frac{8\pi G\rho}{3H^2}, \qquad H = \frac{\dot{a}}{a}, \tag{253}$$

tells us how close to the Einstein-DeSitter model we are. The Friedman equation may be re-expressed as

$$\boxed{\frac{1}{\Omega} = 1 - \frac{3k}{8\pi G\rho}\frac{1}{a^2}} \tag{254}$$

$$\begin{aligned} \Omega &> 1 &\Leftrightarrow& \quad k = 1, \\ \Omega &= 1 &\Leftrightarrow& \quad k = 0, \\ \Omega &< 1 &\Leftrightarrow& \quad k = -1. \end{aligned} \tag{255}$$

At present, $\frac{1}{\Omega_0} - 1 = \epsilon_0$, say with $\epsilon_0 = \mathcal{O}(1)$. Let's calculate $\frac{1}{\Omega_R} - 1$, the value at the beginning of the matter dominated era. Since $\rho \propto 1/a^3$,

$$\frac{1}{\Omega_R} - 1 = \epsilon_0 \frac{a(t_R)}{a(t_0)} = \epsilon_0 (1+z)^{-1}, \tag{256}$$

$$\frac{1}{\Omega_R} - 1 = \epsilon_0 \times 10^{-3} = \mathcal{O}(10^{-3}), \tag{257}$$

which requires "*fine tuning*". The problem gets worse the earlier we go, e.g. at $t \sim 1\,sec$, $T \sim 10^{10}\,K$

$$\frac{1}{\Omega_{1\,sec}} - 1 = \epsilon_R \left(\frac{a(1\,sec)}{a(t_R)}\right)^2 = \epsilon_R \left(\frac{T_R}{10^{10}\,K}\right)^2 \tag{258}$$

$$\frac{1}{\Omega_{1\,sec}} - 1 = \epsilon_0 \times 10^{-17} = \mathcal{O}(10^{-17}) \text{ !!!} \tag{259}$$

4.6.2 The solution?

We must replace the assumption $a(t) \propto t^{1/2}$ by a *faster* rate of expansion, eg.

$$\begin{aligned} & a(t) \propto t^n; \quad && n > 1, \\ \text{or} \quad & a(t) \propto e^{Ht}; \quad && H = \text{constant}. \end{aligned}$$

Flatness Problem

$$a(t) \propto t^n \quad \Leftrightarrow \quad \rho \propto \frac{1}{a^{2n}} \tag{260}$$

i.e. $\gamma = 2n/3$. Now,

$$\frac{1}{\Omega} - 1 \propto \frac{1}{\rho a^2} \propto a^{2n-2} \tag{261}$$

which *decreases* with a as long as $n > 1$.

Horizon Problem

$$\int_0^{t_R} \frac{dt}{a(t)} \propto \int_0^{t_R} \frac{dt}{t^n} \to \infty, \qquad \text{if } n \geq 1. \tag{262}$$

Therefore, light may travel arbitrary distances since the big bang.

An equally acceptable solution is to set $a(t) \propto e^{Ht}$ since then $\rho =$ constant. This corresponds to

$$\frac{P}{c^2} = -\rho \tag{263}$$

(i.e. tension and not pressure) and is called a *cosmological term*.m

Index