AN INTRODUCTION TO BLACK HOLE THERMODYNAMICS

G.W. GIBBONS

Department of Applied Mathematics and Theoretical Physics
Cambridge, U.K.

Notes of Lectures
delivered at the 1980 Summer Institute
of the Laboratoire de Physique Théorique de l'Ecole Normale Supérieure+
August 4-23, 1980

LPTENS 80/28
December 1980

Postal address : 24 rue Lhomond, 75231 Paris Cedex 05 (France)

+Laboratoire Propre du Centre National de la Recherche Scientifique, associé à l'Ecole Normale Supérieure et à l'Université de Paris-Sud.

INTRODUCTION :

These are the notes of 6 lectures given at l'Ecole Normale to an audience comprised mainly of particle physicists on Black Hole thermodynamics. They are meant as a simple introduction to the subject and do not claim to be in any way complete, many topics being left out. The idea is to provide a working knowledge of the subject starting from the minimum prerequisites -i.e. from a first course in General Relativity (G.R.) and Quantum Field Theory (Q.F.T.).

UNITS AND CONVENTIONS :

Throughout we set $\hbar = c = G = k = 1$ (Planck units). In conventional units we have :

mass $\sim 10^{-5}$ g $\sim 10^{19}$ GeV

distance $\sim 10^{-33}$ cm

time $\sim 10^{-44}$ sec

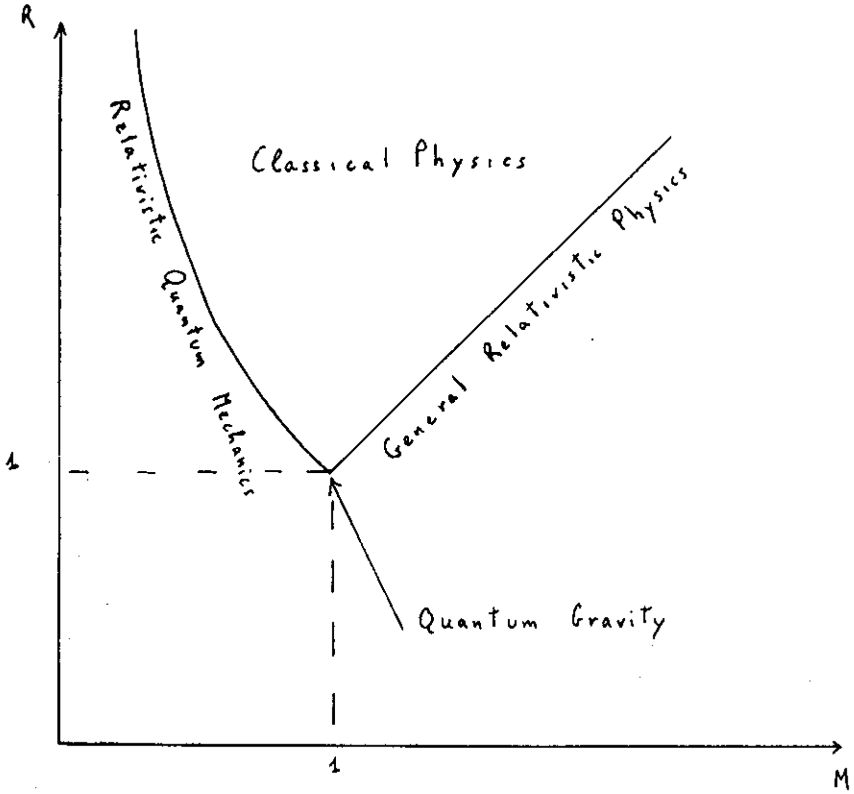
temperature $\sim 10^{32}$ °K.

The Planck scale is that at which Quantum Gravity effects predominate. For a body of mass M and size R if we have

$\frac{GM}{c^2R} \simeq 1$ we must use General Relativity (G.R.)

$\frac{\hbar}{RMc} \simeq 1$ we must use Relativistic Quantum Mechanics (Q.F.T.)

if $\frac{GM}{c^2R} \simeq \frac{\hbar}{RMc} \simeq 1$ we must use Quantum Gravity (Q.G.).



In these units a typical nuclear mass $m \sim 10^{-18}$

metric signature $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

Curvature tensor $(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) V^\gamma = R^\gamma{}_{\sigma\alpha\beta} V^\sigma$

Ricci tensor $R_{\alpha\beta} = R_{\alpha\mu\beta\nu} g^{\mu\nu}$

Einstein Equations $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi T_{\alpha\beta} - \Lambda g_{\alpha\beta}$

LECTURE 1 : CLASSICAL GRAVITATIONAL COLLAPSE

Stars which have exhausted their nuclear fuel are supported by degeneracy pressure. If they are heavy enough the degenerate particles become relativistic (electrons for white dwarfs, neutrons for neutron stars). A stable star is then no longer possible -the star must collapse. Suppose the pressure P and density ρ are related by $P = \alpha \rho^\gamma$

Gravitational Potential Energy $\sim -\frac{M^2}{R}$

Pressure energy $\sim PV \sim PR^3 \sim \alpha \left(\frac{M}{R^3}\right)^\gamma R^3$

Total energy $E \sim \alpha M^\gamma R^{3(1-\gamma)} - M^2 R^{-1}$

if $\gamma > \frac{4}{3}$ a stable minimum exists for all M

if $\gamma < \frac{4}{3}$ no stable minimum exists for all M

if $\gamma = \frac{4}{3}$ $E \sim \frac{M^{4/3}}{R}\left(\alpha - M^{2/3}\right)$

and for $M > \alpha^{3/2}$ E is not bounded below and collapse is inevitable.

This is called the Chandrasekhar-Limit (White dwarfs)

or the Landau-Limit (Neutron stars).

For degenerate neutrons which just fill the Fermi Level p_F we have :

n = no./volume $\sim p_F^3$

Density $\rho \sim m_n p_F^3$

Pressure $\sim n p_F \sim p_F^4$

$P \sim m_n^{-4/3} \rho^{4/3}$

$\therefore$ $M_c \simeq \frac{1}{m_n^2}$

In order of magnitude this is $\sim 1 M_{\odot}$

The collapsing star will eventually disappear inside a sphere of radius R = 2 M (Schwarzschild radius) from which the escape velocity is that of light. Once inside this sphere the star is causally cut off from the outside and is effectively "black". This fact was noticed in Newtonian theory by Michell (1784) and Laplace (1796) and continues to hold in Einstein's theory where the boundary separating those events which can causally affect events far away ("at infinity") from those that cannot is called the Event Horizon. A detailed model of spherically symmetric collapse in the approximation that the star has essentially no pressure support and started collapsing from a homogeneous state was first worked out by Oppenheimer and Snyder in 1939. Penrose (1965) brought out the essential geometrical features by concentrating on the spacetime geometry. We exhibit the evolution by plotting the (time-radius diagram). Each point represents a 2-sphere (orbit of SO(3) isometry group) except for the centre of symmetry (degenerate zero dimensional orbit). As is conventional time is plotted vertically and radius horizontally. The area shaded # represents the star . Also shown are the future light cones of points and the centre of symmetry. After a certain time the light cones start reconverging and refocus at the spacetime singularity which is however hidden inside the limiting light cone which just fails to escape. This light cone or null surface is the event horizon. The spacetime is the union of 3 regions

I) occupied by star (shaded #)

II) vacuum but $r > 2M$ - outside the hole

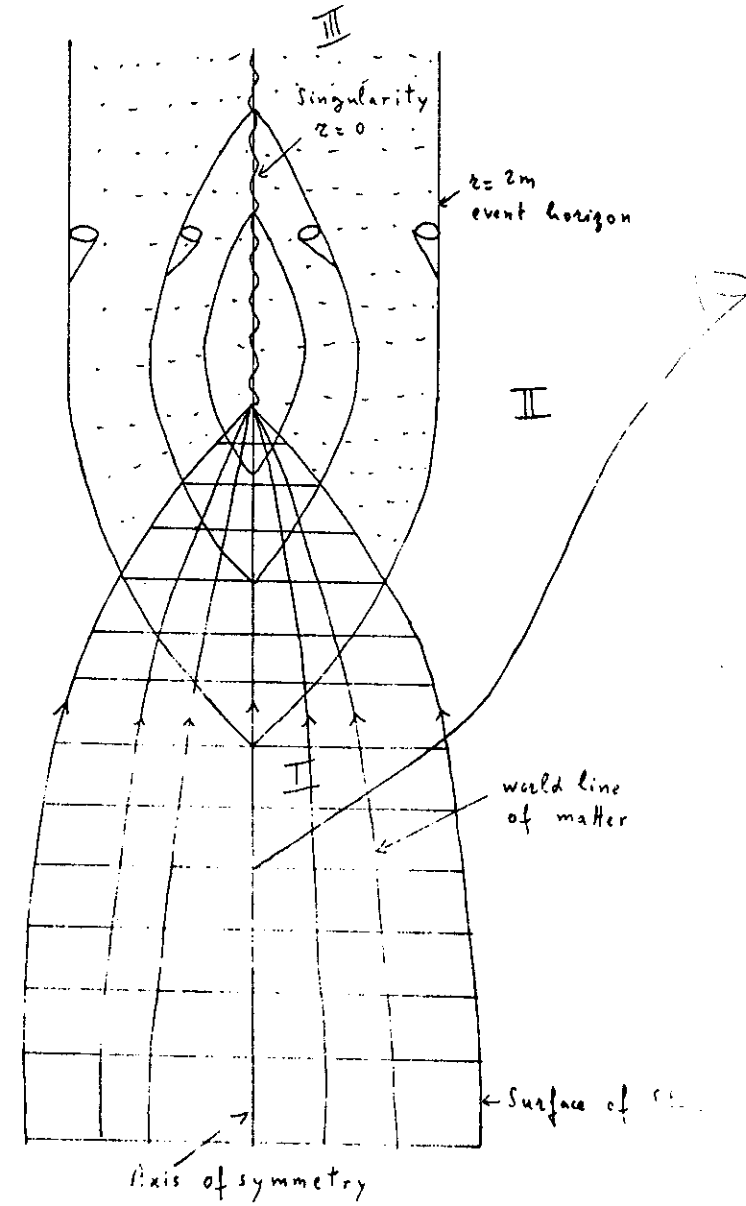
III) vacuum but $r < 2M$ - inside the hole (stippled).

In I) the metric is

$$ds^2 = -dt'^2 + R^2(t')\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)\right] \tag{1}$$

t' is proper time on world lines of matter

χ, θ, φ label the world lines , θ, φ are standard spherical



coordinates and $0 \leq \chi \leq \alpha$. $\chi = 0$ is the centre of symmetry. $\chi = \alpha$ is the surface of the star.

We find that $T_{\mu\nu}\, dx^\mu \otimes dx^\nu = -\rho(t')\, dt' \otimes dt'$

i.e. $$T_{00} = -\rho(t') \tag{2}$$

where $$\rho(t') = \frac{\rho_0}{R(t')} \tag{3}$$

and $$\frac{1}{2}\dot{R}^2 - \frac{4\pi}{3}\frac{\rho_0}{R} = -\frac{1}{2} \tag{4}$$

where ρ_0 is const.

In fact (4) is the "Friedmann equation" and (1) is a portion of a "closed universe".

In regions II) and III) the solution must be a spherically symmetric vacuum solution and thus must be by Birkhoff's Theorem the Schwarzschild solution. This is because there can be no spherically symmetric (transverse) gravitational waves. One can see this topologically ("Hairy Ball Theorem" -there can be no non-singular direction field on S^2) or physically because gravitons couple to the quadrupole moment of the source. In any event we conclude that

In region II) the metric is

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \tag{5}$$

where $r \geq r(t) \geq 2m$

where $r(t)$ is the surface of the star which must be a geodesic in both metric 1 and metric 5 .

The timelike geodesics are obtained by solving the Hamilton-Jacobi equation

$$g^{\alpha\beta}\frac{\partial S}{\partial x^\alpha}\frac{\partial S}{\partial x^\beta} = -1 \tag{6}$$

and setting $$\frac{dx^\alpha}{d\lambda} = g^{\alpha\beta}\frac{\partial S}{\partial x^\beta} \tag{7}$$

where λ is proper time on the geodesic. We find that

(Exercise) $$S = \frac{Et}{m} + W(r) \tag{8}$$

$$\left(\frac{dW}{dr}\right)^2\left(1 - \frac{2m}{r}\right) - \left(\frac{E}{m}\right)^2\left(1 - \frac{2m}{r}\right)^{-1} = -1 \tag{9}$$

$$m\frac{dr}{d\lambda} = \left[E^2 - m^2\left(1 - \frac{2m}{r}\right)\right]^{1/2} \tag{10}$$

$$m\frac{dt}{d\lambda} = \frac{-E}{\left(1 - \frac{2m}{r}\right)} \tag{11}$$

$E^2 \leq m^2$ for a bound orbit.

E is the conserved energy of the particle

$$E = -m\frac{dx^\alpha}{dt}K_\alpha \tag{12}$$

where $$K^\alpha = \delta_0^{\ \alpha} \tag{13}$$

is the timelike Killing vector corresponding to time translations

if $$\delta x^\alpha = \varepsilon K^\alpha \tag{14}$$

$$\delta g_{\alpha\beta} = 0 \tag{15}$$

since $K_{\alpha;\beta} + K_{\beta;\alpha} = 0$ (16) (Killing's equation)

and by Noether's theorem we expect a conserved quantity which is in fact E which satisfies

$$\frac{dE}{d\lambda} = 0 \tag{17}$$

(Exercise: prove (17) using (16) and the geodesic eq.).

Now we apply these equations to the surface of the star.

$$\frac{dR}{dt'} = \sqrt{\frac{8\pi\rho_0}{3R} - 1}$$ (18)(Friedmann equation)

matching the metric we obtain

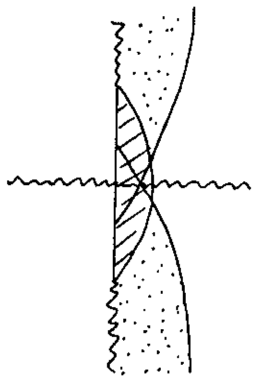
$$g_{00} \text{ const} \Rightarrow 1)\quad t' = \lambda \tag{19}$$

$g_{\theta\theta}$ eq^{n} $\Rightarrow$ 2) $\dot{R}(t')\sin\alpha = r$ (20)

whence $\sin\alpha = \sqrt{1-\frac{E^2}{m^2}}$ (21)

$$m = \frac{4\pi}{3}\sin^3\alpha \qquad (22)$$

N.B. We can choose the origin of time at the moment of time symmetry and reject the solution for earlier times as being unphysical. It would look like this :



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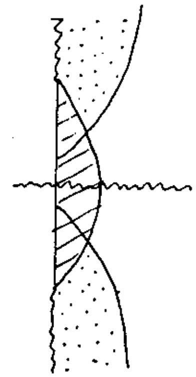


PROPER TIME AND TIME AT INFINITY

Consider any geodesic (including the star surface)

as $r \to 2m$

$$dr \to E\,d\lambda \qquad (23)$$

$$dt \to \frac{-E\,d\lambda}{1-2m/r} \qquad (24)$$

whence

$$dt \to -\frac{dr}{1-\frac{2m}{r}} \qquad (25)$$

$$(\lambda-\lambda_0) \to \lambda_0 e^{-t/2m} \qquad (26)$$

where λ_0 is the finite proper time, the in-falling geodesic passes r = 2m. Clearly according to time t (used by an outside observer) this time is ∞, i.e. $r = 2m \leftrightarrow t = +\infty$. Clearly the coordinates t,r do not cover region III -We need better ones. Before introducing them we digress on the

USE OF NULL COORDINATES IN FLAT SPACE (Light-Cone Gauge)

The metric in spherical polars is

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \qquad (27)$$

set $u = t - r$: retarded time (28)

$v = t + r$: advanced time (29)

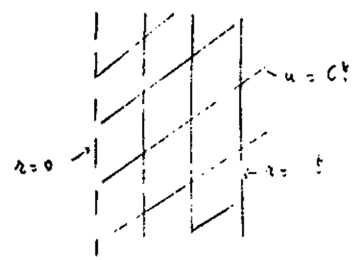
giving the time asymmetric metric forms

$$ds^2 = -du^2 - 2\,dr\,du + r^2\,d\Omega^2 \qquad (30) \text{ retarded}$$

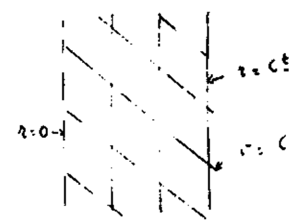
$$ds^2 = -dv^2 + 2\,dr\,dv + r^2\,d\Omega^2 \qquad (31) \text{ advanced}$$

and the time symmetric form

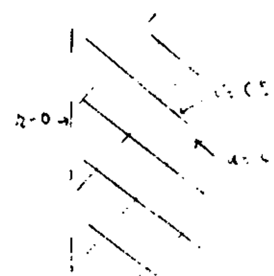
$$ds^2 = -2\,du\,dv + r^2\,d\Omega^2 \qquad (32) \text{ double null}$$



retarded



advanced



double null

A useful device (à la Penrose) is <u>always</u> to plot light cone coordinates at 45°. For <u>scattering</u> of massless particles we introduce future and past null <u>infinity</u> ($\mathcal{I}^-$, $\mathcal{I}^+$)

(script i pronounced SCRI)

$\mathcal{I}^+$ future null is at $v = \infty$

$\mathcal{I}^-$ past null is at $u = -\infty$

setting $x = \frac{1}{r}$ we have

$$ds^2 = \frac{1}{x^2}\left[-x^2\,du^2 + 2\,dx\,du + d\theta^2 + \sin^2\theta\, d\varphi^2\right] \tag{33}$$

$$x = 0 \leftrightarrow \mathcal{I}^+ \qquad (\theta,\varphi) \in S^2 \qquad -\infty < u < +\infty$$

<u>retarded coordinates</u>

$$ds^2 = \frac{1}{x^2}\left[-x^2\,dv^2 - 2\,dx\,dv + d\theta^2 + \sin^2\theta\, d\varphi^2\right]$$

$$x = 0 \leftrightarrow \mathcal{I}^- \qquad (\theta,\varphi) \in S^2 \qquad -\infty < v < +\infty \tag{34}$$

in both cases

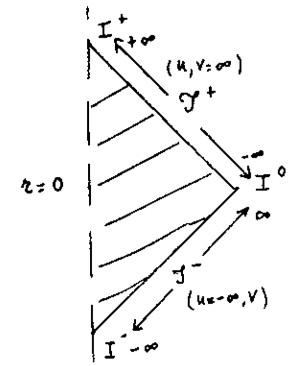
$$\underset{\substack{\uparrow \\ \text{physical metric}}}{ds^2} = \Omega^2 \underset{\substack{\uparrow \\ \text{unphysical metric}}}{(d\tilde{s}^2)} \tag{35}$$

$$\Omega = x \quad , \quad \Omega = 0 \ \text{ on } \ \mathcal{I}^\pm \tag{36}$$

$$\nabla_\mu \Omega \neq 0 \ \text{ on } \ \mathcal{I}^\pm \tag{37}$$

$\mathcal{I}^\pm$ is a <u>null surface</u>.

We thus obtain the <u>Penrose Diagram</u> of flat space



<u>timelike geodesics</u>   start from I^- and go to I^+ (past and future timelike ∞)

<u>spacelike geodesics</u> go to I^0 (spacelike ∞)

<u>null geodesics</u> start from $\mathcal{I}^-$ and end on $\mathcal{I}^+$.

<u>Exercise</u> $U = \tanh\frac{u}{2}$

$V = \tanh\frac{v}{2}$

Show that U and V cover $\mathcal{I}^\pm$ and that $\mathcal{I}^+ \leftrightarrow V = 1$, $\mathcal{I}^- \leftrightarrow U = -1$.

We illustrate the Penrose diagram by considering a scalar S-wave

$$\varphi = \frac{f(t-r)}{r} - \frac{f(t+r)}{r} \tag{38}$$

$$= \frac{f(u) - f(v)}{r} \tag{39}$$

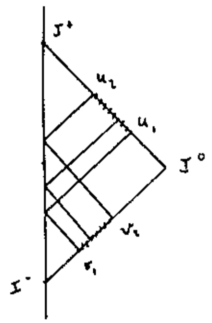
On $\quad \mathcal{I}^- \qquad r\varphi = -f(v) \tag{40}$

$\quad \mathcal{I}^+ \qquad r\varphi = f(u) \tag{41}$

we have a spherical wave which is <u>ingoing</u> on $\mathcal{I}^-$ bounces off the origin

$r = 0$ and is outgoing on $\mathcal{J}^+$.

If $f(x) = 0$ unless $x_1 \leqslant x \leqslant x_2$ we get



Note that in these diagrams the wave fronts are null surfaces which are always at 45°. Also for S-waves geometric optics is exact. (*Exercise* : work out what happens if the waves have angular dependence).

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3. S.W. Hawking, G.F.R. Ellis, "The Large Scale Structure of Spacetime", C.U.P., Cambridge, 1973.
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5. J. Michell, Phil. Trans. Roy. Soc., ser. 74 (1784) 35.

LECTURE 2 : NULL COORDINATES IN SCHWARZSCHILD.

Following the flat space example we set

$$du = dt - \frac{dr}{1-\frac{2m}{r}} \qquad u = t - r^* \qquad \text{retarded null coordinates} \quad (1)$$

$$dv = dt + \frac{dr}{1-\frac{2m}{r}} \qquad v = t + r^* \qquad \text{advanced null coordinates} \quad (2)$$

$$r^* = r + 2m \operatorname{Log}\left(\frac{r-2m}{2m}\right) \quad (3)$$

where r^* is the Regge-Wheeler-Tortoise coordinate.

$r = 2m$ (horizon) is at $r^* = -\infty$

We have

$$ds^2 = -\left(1-\frac{2m}{r}\right)du^2 - 2dr\,du + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \quad (4)$$

retarded Finkelstein null coordinates

valid if $0 < r \leq +\infty$, $-\infty < u < +\infty$

$$ds^2 = -\left(1-\frac{2m}{r}\right)dv^2 + 2dr\,dv + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \quad (5)$$

advanced Finkelstein null coordinates

valid if $0 < r \leq \infty$, $-\infty < v < +\infty$

The symmetric double null form is

$$ds^2 = -\left(1-\frac{2m}{r}\right)du\,dv + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \quad (6)$$

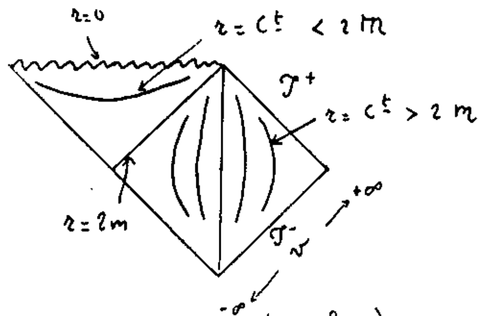
with r defined implicitly by

$$r + 2m \log\left(\frac{r-2m}{2m}\right) = r^* = \frac{v-u}{2} \quad (7)$$

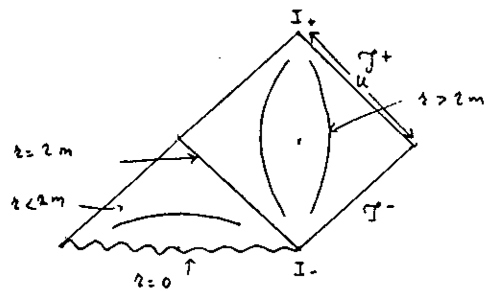
and valid for

$-\infty < u < +\infty$ $\quad -\infty < v < +\infty \quad \Rightarrow \quad r > 2m$

Clearly the advanced set (r, v, θ, φ) penetrate the (future) horizon in the future direction to give the Carter-Penrose Diagram

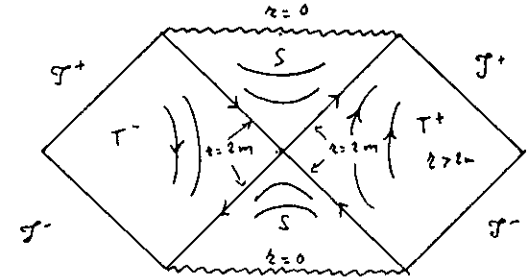


The retarded set (r, u, θ, φ) penetrate the (past) horizon in the past-direction to give the Carter-Penrose diagram :



N.B. r = 0 is a singularity since $R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \sim \left(\frac{M}{r^3}\right)^2$

In our original collapse diagram we could use the coordinates (v, r, θ, φ) to cover the regions II and III provided we are outside the star. Note that the time asymmetric behaviour of the light cones is reflected in the time asymmetric behaviour of the metric in these coordinates - i.e. the metric in (5) has the cross-term $2\,dr\,dv$. By patching the (u, r) and (v, r) coordinates and introducing (u', r) and (v', r) coordinates we get a Carter-Penrose diagram like this



We have two asymptotic regions connected by an Einstein Rosen Bridge. To cover the entire manifold we introduce Kruskal coordinates by

$$\mathcal{U} = -\exp\left(-\frac{u}{4m}\right) \tag{8}$$

$$V = \exp\left(\frac{v}{4m}\right) \tag{9}$$

$$\mathcal{U}V = \left(1 - \frac{r}{2m}\right) e^{r/2m} \tag{10}$$

$$\mathcal{U}/V = -\exp\left(\frac{t}{2m}\right) \tag{11}$$

for $r < 2m$

$$T = \frac{\mathcal{U}+V}{2} = \cosh\frac{t}{4m}\left|1 - \frac{r}{2m}\right|^{\frac{1}{2}} e^{r/4m} \tag{12}$$

$$X = \frac{V-\mathcal{U}}{2} = \sinh\frac{t}{4m}\left|1 - \frac{r}{2m}\right|^{\frac{1}{2}} e^{r/4m} \tag{13}$$

(for $r > 2m$ T', ...)

The metric becomes

$$ds^2 = -\frac{32m^3}{r}\,d\mathcal{U}\,dV + r^2\,d\Omega_2^2 \tag{14}$$

$$d\Omega_2^2 = \sin^2\theta\,d\varphi^2 + d\theta^2$$

or

$$ds^2 = -\frac{32m^3}{r} e^{r/2m}\left(dT^2 - dX^2\right) + r^2\,d\Omega_2^2 \quad (1\ldots)$$

Note that the <u>time translation group</u> is really a group of <u>boosts</u> in the (T,X) plane whose orbits are timelike future directed (T^+) or timelike past directed (T^-) or spacelike (S) depending on which quadrant one is in as shown in the above figure p.15.

Note that this group G_t has the property that

1) it stabilizes the horizon $r = 2m$ as a <u>set</u> but moves points along the null generators,

2) it stabilizes null infinity as a set but moves points along the null generators,

3) it stabilizes the 2-sphere U = 0, V = 0 <u>pointwise</u>.

This 2-sphere (the Boyer-Kruskal crossover axis) has area $16\pi m^2$

The intrinsic 3-metric of a 3-surface of constant time is given by

$$ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \tag{16}$$

Set $$r^2 = \rho^2 \left(1 + \frac{m}{2\rho}\right)^4 \tag{17}$$

(isotropic coordinates)

and we get

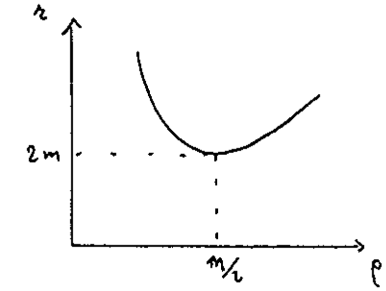
$$ds^2 = \left(1 + \frac{m}{2\rho}\right)^4 \left(d\rho^2 + \rho^2 d\Omega_2^2\right) \tag{18}$$

This is invariant under the discrete involution or inversion given by

$$\rho \to \frac{m^2}{4\rho} \tag{19}$$

which maps one universe into the other and leaves fixed the Boyer-Kruskal axis.

Thus ρ double covers r



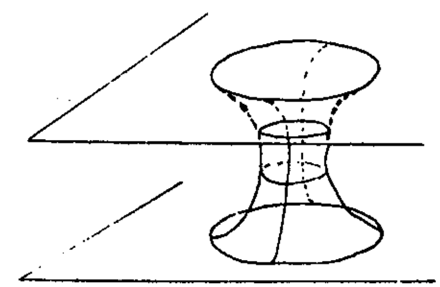
The <u>area</u> of a 2-sphere r = constant is

$$4\pi r^2 = 4\pi\rho^2 \left(1 + \frac{m}{2\rho}\right)^4 \tag{20}$$

which is minimized for $\rho = \frac{m}{2}$ or r = 2m and has the value $16\pi m^2$.

Thus the surfaces t = constant have two "sheets" connected by the Einstein-Rosen Bridge.

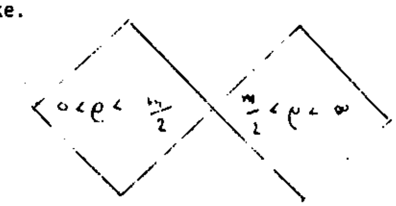
r = 2M



Note that if we use isotropic coordinates

$$ds^2 = -\frac{\left(1 - \frac{m}{2\rho}\right)^2}{\left(1 + \frac{m}{2\rho}\right)^2} dt^2 + \left(1 + \frac{m}{2\rho}\right)^4 \left(d\rho^2 + \rho^2 d\Omega_2^2\right) \tag{21}$$

we still cannot cover all the manifold. We just get the two regions in which $\frac{\partial}{\partial t}$ is timelike.



To complete our description of the Schwarzschild solution we note that it can be isometrically embedded algebraically in R_7 with metric (++++-++) as the real surface

$$X_4^2 - X_5^2 + \frac{4}{3} X_7^2 - 16 m^2 = 0 \tag{22}$$

$$X_7^4 (X_1^2 + X_2^2 + X_3^2) - 576 m^2 = 0 \tag{23}$$

$$\sqrt{3} X_6 X_7 + X_7^2 - 24 m^2 = 0 \tag{24}$$

where

$$X_1 = r \sin\theta \cos\varphi \tag{25}$$

$$X_2 = r \sin\theta \sin\varphi \tag{26}$$

$$X_3 = r \cos\theta \tag{27}$$

$$X_4 = 4m \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} \cosh \frac{t}{4m} \tag{28}$$

$$X_5 = 4m \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} \sinh \frac{t}{4m} \tag{29}$$

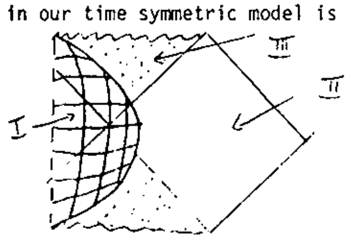
$$X_6 = -2m \left(\frac{2m}{r}\right)^{\frac{1}{2}} + 4m \left(\frac{r}{2m}\right)^{\frac{1}{2}} \tag{30}$$

$$X_7 = 2m\sqrt{3} \left(\frac{2m}{r}\right)^{\frac{1}{2}} \tag{31}$$

Again "time translations" are boosts in the (X_4, X_5) directions. (This embedding was constructed by Ferrari and Francaviglia G.R.G. 10 (1979) 283).

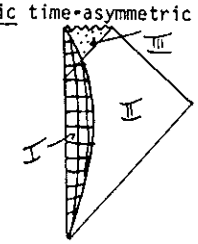
Remember that realistic collapse occupies only a portion of the diagram.

The shaded region in our time symmetric model is absent and occupied by the star to give :



In a realistic time-asymmetric collapse starting from an initially static sta

we get :



GENERIC CHARACTER OF SPHERICAL COLLAPSE :

The final end point of spherical collapse is a static black hole whic hides the inevitable spacetime singularities (predicted by the Penrose-Hawkir Theorems) inside the event horizon. The Hypothesis of Cosmic Censorship essen tially supposes that this hiding of the singularities always takes place. The end point of a collapse with no net angular momentum must then be static. The Israel Theorem states that the only such black holes are the Schwarzschil black holes parametrized by just one parameter the total mass M or equivalent the total surface area A of the horizon given by $A = 16 \pi M^2$

Hawking has defined the event horizon in this situation to be the non trivial boundary to the causal past of $\mathcal{J}^+$ written as $\dot{J}^-(\mathcal{J}^+)$. This is, i simple terms, the boundary between those events which can ever causally affec infinity and those that cannot.

Hawking proved that the event horizon was a null surface -"the last wave front to escape"- and that its area A could never decrease. i.e.

$$dA \geqslant 0 \tag{32}$$

We can't prove this result here. However we note that if the future directed null generator of the horizon -call it φ = constant- is ℓ_α such that

$$\ell_\alpha = \varphi_{,\alpha} \tag{33}$$

$$\ell_\alpha \ell^\alpha = 0 \tag{34}$$

then a particle dropping in with momentum $m \frac{dx^\alpha}{d\lambda} = p^\alpha$ satisfies

$$- m \frac{dx^\alpha}{d\lambda} \ell_\alpha \geq 0 \qquad (35)$$

since ℓ^α is null and p^α is timelike and both are future directed. On the static horizon ℓ^α coincides with the Killing vector K^α which becomes null at r = 2M

So that
$$- K^\alpha \frac{m\, dx^\alpha}{d\lambda} \geq 0 \qquad (36)$$

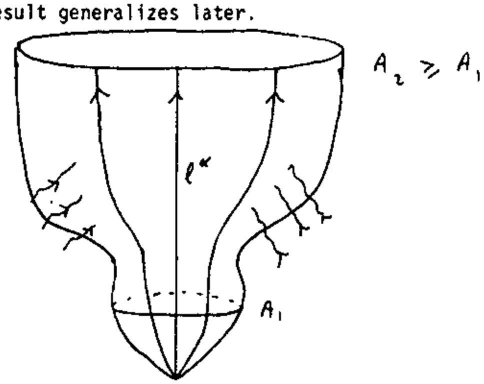
whence
$$E \geq 0 \qquad (37)$$

Now if the hole changes in mass so that $M \to M + dM$ we have

$$dM = \frac{\kappa\, dA}{8\pi} = E \geq 0 \qquad (38)$$

whence
$$dA \geq 0 \qquad (39)$$

where $K = \frac{1}{4M}$ is called the surface gravity of the hole. We shall see that this simple result generalizes later.



Behaviour of event horizon.

LECTURE 3 : CHARGED BLACK HOLES.

Israel also proved that the only static black hole in Einstein-Maxwell theory was the Reissner-Nordstrøm Solution with just 3 parameters

mass M or area A

electric charge Q or potential Φ_H

magnetic charge P.

The metric is

$$ds^2 = - \frac{\Delta}{r^2} dt^2 + \frac{r^2}{\Delta} dr^2 + r^2 d\Omega_2^2 \qquad (1)$$

The vector potential is

$$A_\mu dx^\mu = \frac{Q}{r} dt + P \cos\theta\, d\varphi \qquad (2)$$

$$\Delta = (r - r_+)(r - r_-) = r^2 - 2Mr + Q^2 + P^2 \qquad (3)$$

$$r_\pm = M \pm \sqrt{M^2 - Q^2 - P^2} \qquad (4)$$

where the roots are real and hence a horizon exists

if
$$M^2 \geq Q^2 + P^2 \qquad (5)$$

If
$$M^2 < Q^2 + P^2 \qquad (6)$$

there is no horizon - the singularity at $r = 0$ is naked.

N.B. All elementary particles with charge badly violate this inequality - however as stated before classical G.R. doesn't apply to these objects. So we obtain no contradiction. Classically we don't expect to violate the bound $M^2 \geq Q^2 + P^2$ because electrostatic repulsion would prevent collapse. This is indeed supported by many detailed examples. In fact we would not even expect to achieve the limiting case $M^2 = Q^2 + P^2$.

To describe these metrics we introduce the Regge-Wheeler r^* coordinate by

$$r^* = r + \frac{1}{2K_+} \operatorname{Log}\left|\frac{r-r_+}{r_+}\right| + \frac{1}{2K_-} \operatorname{Log}\left|\frac{r-r_-}{r_-}\right| \qquad (7)$$

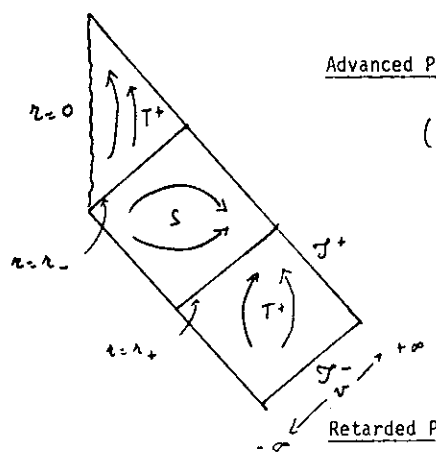
where $$K_\pm = \frac{|r_+ - r_-|}{2r_\pm^2} \qquad (8)$$

where $$dr^* = \frac{dr}{-g_{00}} = \frac{r^2\,dr}{(r-r_+)(r-r_-)} \qquad (9)$$

Let $u = t - r^*$ retarded Finkelstein (10)

$v = t + r^*$ advanced Finkelstein

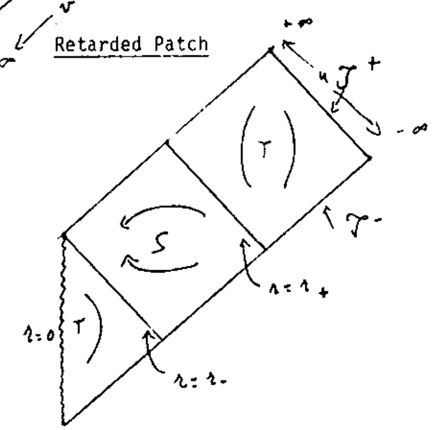
We proceed as before and get :



Advanced Patch

$(0<r<\infty\ ;-\infty<v<\infty\ ;\theta,\varphi)$

Retarded Patch



$(0<r<\infty\ ,\ -\infty<u<\infty\ ;\ \theta,\varphi)$

We get an <u>outer horizon</u> at $r = r_+$

and an <u>inner horizon</u> at $r = r_-$

Each horizon needs its own <u>Kruskal patch</u>

$$U^\pm = \exp - K_\pm u$$

$$V^\pm = \exp K_\pm v$$

<u>N.B.</u> $K_+ \neq K_-$ unless $M^2 = P^2 + Q^2$ when $r_+ = r_-$ and $K_+ = K_- = 0$

In addition one should perform the following <u>Gauge transformation</u>

$$A \rightarrow \tilde{A} = A - d\Lambda$$

$$\Lambda = \frac{Q}{r_+}t = \Phi_H t$$

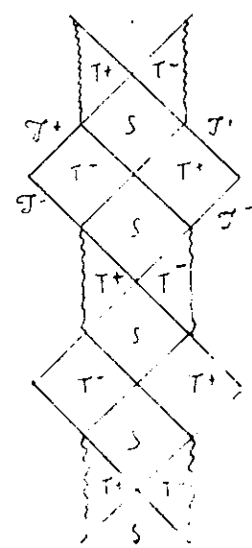
$$\tilde{A}_\mu dx^\mu = \left(\frac{Q}{r} - \frac{Q}{r_+}\right)dt + P\cos\theta\,d\varphi$$

$$\Phi_H = \frac{Q}{r_+}$$

is the <u>electrostatic potential</u> of the hole. Note that $\tilde{A}$ is not a good ga

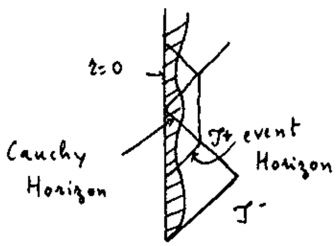
at $\mathcal{J}^\pm$ because $\tilde{A} \not\to 0$ as $r \to \infty$.

<u>THE COMPLETE PENROSE-CARTER DIAGRAM IS</u> :

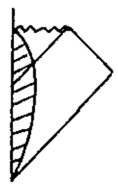


Exercise: (construct this explicitly)

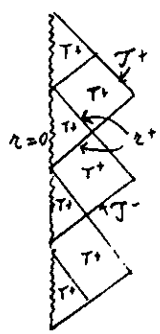
The surfaces $r = r_-$ are called Cauchy Horizons. One can cross them in a finite proper time and experience the entire future history of our world. This suggests they are unstable (they are ∞ blue shift surfaces) and indeed detailed calculations indicate this. One can construct models in which charged collapsing balls enter these regions, e.g.



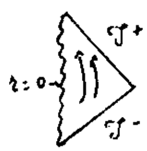
But these are believed to be unstable and generically one expects :



just as in the neutral case. The limiting extreme case looks like this



And the over extreme Naked Singularity like this :

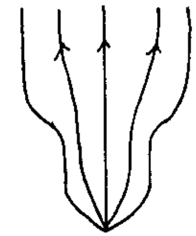


Chinese Proverb "One Picture tells a thousand words".

English Proverb "One Carter-Penrose diagram describes an infinity of worlds".

Some familiarity with spacetime diagrams is as indispensible as a knowledge of Feynman diagrams -both serve the same purpose they describe succintly a complicated spatio-temporal sequence of events in a readily grasped form.

ENERGETICS OF REISSNER-NORDSTRØM :



The canonical momentum $\quad p^\alpha = m\dfrac{dx^\alpha}{d\lambda} - eA^\alpha \qquad (17)$

The conserved energy E for a charged particle is

$$E = -p_\alpha K^\alpha \qquad (18)$$

where $K^\alpha \dfrac{\partial}{\partial x^\alpha}$ is the generator of time translations (Killing vector)

and
$$m\frac{d^2x^\alpha}{d\lambda^2} = e\,F^\alpha{}_\beta\,\frac{dx^\beta}{d\lambda} \qquad (19)$$

$$\frac{dE}{d\lambda} = 0 \qquad (20)$$

(*Exercise:* prove these statements).

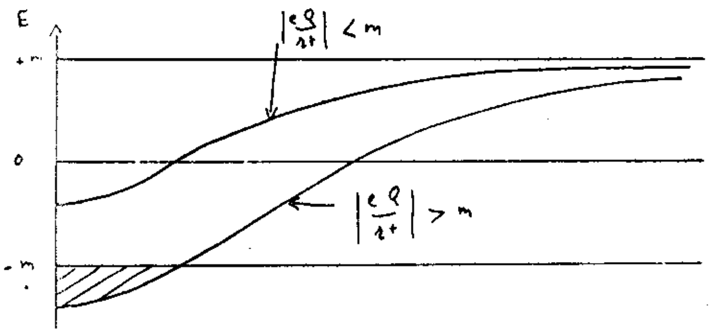
If the particle is at rest : $\dfrac{dx^\alpha}{d\lambda} = \delta_0{}^\alpha(-g_{00})^{-\frac{1}{2}} \qquad (21)$

so
$$E = m\sqrt{-g_{00}} + eA_0 \qquad (22)$$

$$E = m\sqrt{1-\frac{2m}{r}+\frac{Q^2+P^2}{r^2}} + \frac{eQ}{r} \tag{23}$$

$eQ < 0$ E can be negative near the hole.

if $\frac{eQ}{r^+} < -m$ it is energetically possible for pair production to take place. This would discharge the hole.

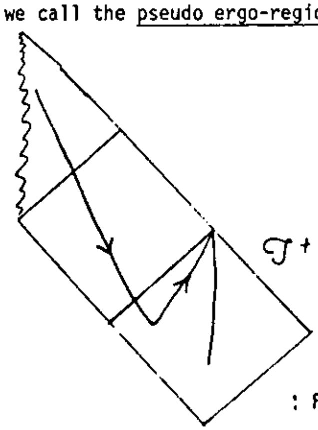


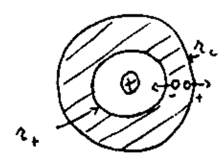
The region of negative energy orbits is where

$K^\alpha + eA^\alpha$ is no longer timelike;this happens at $r < r_c$ where r_c is given by

$$\left|\frac{eQ}{r_c}\right| = \sqrt{1-\frac{2m}{r_c}+\frac{Q^2+P^2}{r_c^2}} \tag{24}$$

which we call the pseudo ergo-region.





: Feynman-Carter Penrose Diagram.

Clearly we need second quantized theory to describe this properly. However, let us consider charge particle accretion so the mass changes from $M \to M + dM$, charge from $Q \to Q + dQ$ area from $A \to A + dA$ etc... as before if ℓ^α is the null generator of the horizon

$$-\ell^\alpha \frac{dx^\alpha}{d\lambda} \geqslant 0 \tag{25}$$

$$\Rightarrow \qquad E - \frac{Qe}{r_+} \geqslant 0 \tag{26}$$

$$dM \geqslant \Phi_H\, dQ \tag{27}$$

$$dM = E \tag{28}$$

$$dQ = e \tag{29}$$

In fact by direct differentiation we find that (*Exercise*)

$$dM = \frac{\kappa\, dA}{8\pi} + \Phi_H\, dQ \tag{30}$$

so again

$$dA \geqslant 0 \tag{31}$$

where

$$A = 4\pi r_+^2 \tag{32}$$

is the area of the event horizon. We can define the irreducible mass M_0 by

$$16\pi\, M_0^2 = A \tag{33}$$

and obtain

$$M = M_0 + \frac{Q^2}{2r^+} \tag{34}$$

which expresses the energy M in terms of a useful amount $\frac{Q^2}{2r^+}$ and an unavailable amount M_0 . This resembles the relation :

$$M^2 = M_0^2 + \rho^2 \tag{35}$$

for kinetic and rest mass energy. Without particle creation, etc., the rest mass energy M_0^2 is not available whereas the contribution ρ^2 is useful. The reader will note the obvious <u>thermodynamic</u> analogies implicit in (30) (31) and (34). These become even more apparent when we consider rotating black holes where we have the kinetic energy of rotation available.

The only axisymmetric stationary neutral black hole is the Kerr black hole (Robinson-Carter Theorem). It has almost been proven (and is physically almost obvious) that any stationary hole must be axisymmetric. This depends on just two parameters -the mass M and angular momentum J - which (analogously to eq. (5)) must satisfy

$$M^4 \geq J^2 \tag{36}$$

The metric is

$$ds^2 = \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + \frac{\sin^2\theta}{\Sigma}\left((r^2+a^2)\,d\varphi - a\,dt\right)^2$$
$$- \frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta\, d\varphi\right)^2 \tag{37}$$

where $\Sigma = r^2 + a^2\cos^2\theta \quad (38)$

$$\Delta = r^2 - 2Mr + a^2 = (r-r_+)(r-r_-) \tag{39}$$

$$r_\pm = M \pm \sqrt{M^2 - a^2} \tag{40}$$

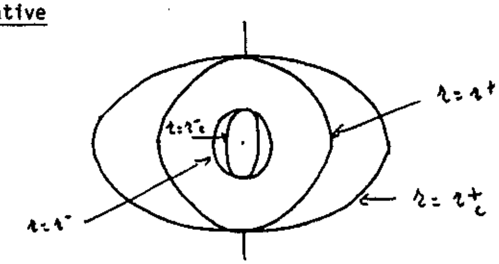
$$J = Ma \tag{41}$$

$$-g_{00} = \frac{\Delta - a^2\sin^2\theta}{\Sigma} \tag{42}$$

which vanishes at the <u>ergo sphere</u>

$$r = r_c^\pm = M \pm \sqrt{M^2 - a^2\cos^2\theta} \tag{43}$$

if $g_{00} > 0$ we are in the <u>ergo region</u> in which the energy of a neutral partic[le] can be <u>negative</u>



The horizons are at $r = r_\pm$

The Kerr metric is invariant under the 2 parameter group $G_{t,\varphi}$ generated by $\frac{\partial}{\partial t} = K^\alpha \frac{\partial}{\partial x^\alpha}$ and $\frac{\partial}{\partial \varphi} = m^\alpha \frac{\partial}{\partial x^\alpha}$ which generate time translations and rotations respectively. The horizon is an exampl[e] of a Killing horizon (stationary null surface or wave front)

1) its null generator ℓ^α coincides with a generator of the group $G_{t,\varphi}$

2) the surface is invariant under $G_{t,\varphi}$. ℓ^α must thus be a linear combination of K^α and m^α

$$\ell^\alpha = K^\alpha + \Omega m^\alpha \tag{44}$$

$$\ell^\alpha \frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi} \tag{45}$$

From the definition $d\varphi$ and dt must lie in a null surface and contain a null direction. One must thus have

$$g_{\varphi\varphi}\, g_{tt} - g_{\varphi t}\, g_{t\varphi} = 0 \tag{46}$$

which shows that the horizons are at $r = r_\pm$ then (44) gives

$$\Omega_\pm = \frac{a}{r_\pm^2 + a^2} \tag{47}$$

The interpretation of $\Omega_\pm$ is that the horizons rotate at a constant angular velocity $\Omega_\pm$ with respect to infinity. This means that relative to the horizon one must introduce co-rotating angular coordinates

$$\tilde{\varphi}_\pm = \varphi - \Omega_\pm t \tag{48}$$

One introduces the Regge-Wheeler coordinate r^* by

$$r^* = \int \frac{dr}{\Delta} \tag{49}$$

$$= \frac{1}{2K_+} \operatorname{Log}\left|\frac{r-r_+}{r_+}\right| + \frac{1}{2K_-} \operatorname{Log}\left|\frac{r-r_-}{r_-}\right| \tag{50}$$

with

$$K_\pm = \frac{|r_+ - r_-|}{2(r_\pm^2 + a^2)} \tag{51}$$

The Finkelstein coordinates are :

$$u = t - r^* \tag{52}$$

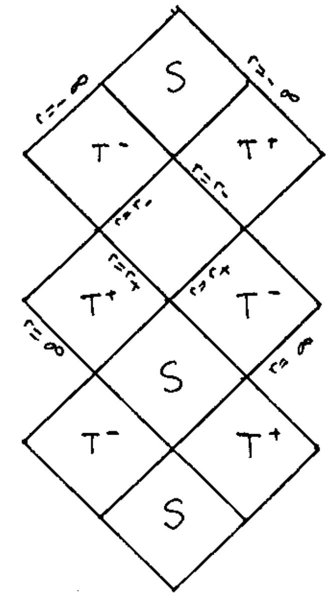
$$v = t + r^* \tag{53}$$

and for each horizon one uses Kruskal coordinates

$$U^\pm = - e^{-K_\pm u} \tag{54}$$

$$V^\pm = e^{K_\pm v} \tag{55}$$

so that $(U^\pm, V^\pm, \varphi^\pm, \theta)$ cover the 2 horizons. Of course (u, r, φ, θ) covers $\mathcal{J}^+$ and (v, r, φ, θ) covers $\mathcal{J}^-$. The Carter-Penrose diagram of the symmetry axis is



Note that $-\infty < r < +\infty$

The singularity is at $\Sigma = 0 \Rightarrow r = 0$ and $\theta = \frac{\pi}{2}$.

This is a <u>ring</u> which is invisible except for null rays lying in the equator plane.

As in the charged case the negative energy regions encourage the su picion that pair creation can reduce the angular momentum. We turn to this later.

<u>ENERGETICS</u> :

The conserved angular momentum of a particle of mass m is (*Exercise*

$$L = m \frac{dx^\alpha}{d\lambda} m_\alpha \qquad (5$$

$$\frac{dL}{d\lambda} = 0$$

using
$$\ell^{\alpha} = K^{\alpha} + \Omega_H m^{\alpha} \tag{57}$$
$$dM = E \tag{58}$$
$$dJ = L \tag{59}$$

we get as before

$$dM - \Omega\, dJ \geqslant 0 \tag{60}$$

in fact we find by differentiating

$$dM = \frac{K\, dA}{8\pi} + \Omega\, dJ \tag{61}$$
$$A = 4\pi(r^2+a^2) = 16\pi M_0^2 \tag{62}$$

and
$$M^2 = M_0^2 + \frac{J^2}{4M_0^2} \tag{63}$$

The most general blak hole is the charged Kerr-Newman Hole for which the metric has the form : (37) but

$$\Delta = r^2 - 2Mr + a^2 + Q^2 + P^2 \tag{64}$$

and the Maxwell field is given by :

$$A = \frac{Qr\,(dt - a\sin^2\theta\, d\phi) + P\cos\theta\,(a\,dt - (r^2+a^2)\,d\phi)}{(r^2 + a^2\cos^2\theta)} \tag{65}$$

$$\Omega = \frac{a}{r^2+a^2} \qquad \text{angular velocity} \tag{66}$$

$$\Phi_H = \frac{Q\, r_+}{r^2+a^2} \qquad \text{electrostatic potential} \tag{67}$$

$$K_\pm = \frac{|r_+ - r_-|}{2(r_+^2+a^2)} \qquad \text{surface gravity} \tag{68}$$

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2 - P^2} \qquad \text{horizon radius} \tag{69}$$

$$A = 4\pi(r_+^2 + a^2) = 16\pi M_0^2 \qquad \text{area and irreducible mass} \tag{70}$$

We have

$$dM = \frac{K\, dA}{8\pi} + \Omega\, dJ + \Phi_H\, dQ \tag{71}$$

(second law)

$$\frac{1}{2} M = \frac{KA}{8\pi} + \Omega J + \frac{1}{2}\Phi_H Q \tag{72}$$

(Smarr mass formula)

$$M^2 = \left(M_0 + \frac{Q^2}{4M_0}\right)^2 + \frac{J^2}{4M_0^2} \tag{73}$$

(Christadoulou-Ruffini irreducible mass formula)

Exercise: use a scaling argument and Euler theorem to prove (72) from (71) and explain the factor $\frac{1}{2}$ in (72).

There is a clear analogy with thermodynamics of some system of energy U, temperature T, entropy S, intensive variables X_i, extensive variables x_i

$$dU = T\,dS + \sum_i X_i\, dx_i$$

$$U = TS + \sum_i X_i\, x_i \qquad \text{(for a homogeneous system).}$$

The No Hair Theorems (i.e. Black Hole uniqueness theorems) show that the outside field of a black hole, no matter from what it was made, depends only on 4 variables - e.g. (A, J, Q, P).

Thus we are tempted to identify

αA with S

$\frac{K}{\alpha\, 8\pi}$ with T

However in <u>classical</u> theory T = 0 since a black hole cannot classically be in equilibrium with a <u>heat bath</u> in a finite box of black body photons or gravitons.

It will absorb them all. Only if a good absorber is also a good <u>radiator</u> of particles is equilibrium possible (cf. Kirchhoff's Law).

In fact, dimensionally α is proportional to $\hbar^{-1}$ which is consistent with this analysis. We are thus led to consider the quantum theory and the production of particles, an effect which we expect to vanish as $\hbar \to 0$.

Before considering particle creation we shall state the <u>4 Laws of Thermodynamics</u> which are

0) K is constant on the horizon

1) $$dM = K\frac{dA}{8\pi} + \Omega\, dJ + \Phi\, dQ$$

2) $$dA \geqslant 0$$

3) K can't be reduced to zero in a finite number of steps.

<u>N.B.1)</u> Law (0) is trivial in the spherical case but a theorem in the non-spherical case. The definition of K in general is that if $\ell^\alpha = \frac{dx^\alpha}{dv}$ where v is a <u>group</u> parameter then (since ℓ^α is geodesic)

$$\ell^\alpha{}_{;\beta}\,\ell^\beta = K\ell^\alpha$$

This shows that V is not an affine parameter, V for which

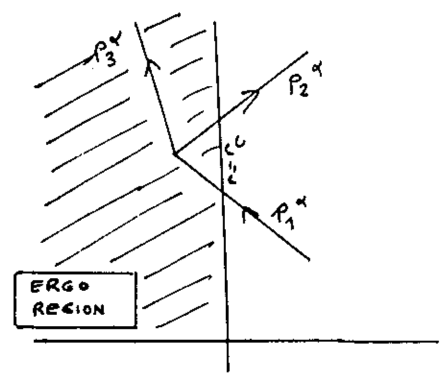
$$\frac{d^2 x^\alpha}{dV^2} = 0$$

In fact K <u>could</u> a priori have varied from generator to generator (e.g. depended on θ). <u>In fact</u> it is constant and the Kruskal coordinates are <u>affine parameters</u> on the appropriate horizons. Thus

- u is affine on $\mathscr{I}^+$
- v is affine on $\mathscr{I}^-$
- U is affine on past horizon H^-
- V is affine on future horizon H^+.

<u>N.B.2)</u> Law 0, 1, 2 are <u>theorems</u>, law 3 is a <u>conjecture</u> which is implied (at least loosely speaking) by the Cosmic Censorship Hypothesis. No counter-example exists.

Finally we mention the phenomenon of <u>Super radiance</u> and the <u>Penrose Process</u>. The latter is the observation that if one has a negative energy region one can allow the following energy extraction process



$$P_1^\alpha = P_2^\alpha + P_3^\alpha$$
$$E_1 = E_2 + E_3$$
$$E_1 > 0$$
$$E_3 < 0$$
$$\Rightarrow \quad E_2 > E_1$$

where $(P_1^\alpha, P_2^\alpha, P_3^\alpha)$ are all <u>future directed timelike</u>.

The Former is the <u>wave analogue</u>.

Let ϕ satisfy

$$-\mathcal{D}_\alpha \mathcal{D}^\alpha \phi + \phi = 0$$

$$\mathcal{D}_\alpha = \nabla_\alpha + ie A_\alpha$$

set $$\phi = e^{-i\omega t} e^{in\varphi} \chi(r,\theta)$$

the conserved current $$J_\alpha = \frac{1}{2i}\,\bar{\phi}\overleftrightarrow{\mathcal{D}_\alpha}\phi$$

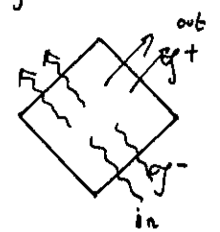
has the form

$$J_\alpha dx^\alpha = (\omega\, dt - n\, d\phi - e A_\alpha dx^\alpha)|\chi|^2$$

and the flux entering the horizon is proportional to

$$J_\alpha \ell^\alpha = (\omega - n\Omega_H - e\Phi_H)\,|\chi|^2$$

Now if $\omega \gg 0$, J_α is timelike at ∞ and thus carries a positive flux through $\mathscr{I}^-$



But if $n\Omega_H + e\Phi_H = \mu$ is sufficiently large, the flux through the future horizon H^+ can be negative. Thus the flux which passes out through $\mathscr{I}^+$ can exceed that sent in through $\mathscr{I}^-$.

REFERENCES FOR LECTURE 3

In addition to those already cited, various articles in the Proceedings of the first Marcel Grossmann Meeting on Recent Developments in the Fundamental of General Relativity,ed. R. Ruffini discuss Black Hole energetics and my own article "Quantum Processes New Black Holes" reviews these aspects of super radiance.

REMARK :

The close analogy between the effects of charge and of rotation can be understood in the Kaluza-Klein formulation if one considers the 5-dimensional metric

$$ds_5^2 = (dx^5 + A_\alpha dx^\alpha)^2 + g_{\alpha\beta}\,dx^\alpha dx^\beta$$

where $\frac{\partial}{\partial x^5}$ is a spacelike Killing vector of the 5 dimensional metric. Spacetime is the space of orbits of the U(1) electromagnetic group generated by $\frac{\partial}{\partial x^5}$. The 5-manifold is a fibre bundle fibred by U(1). The cross term in the metric means that we can formally think of the charged black hole as "rotating" in the 5th direction.

LECTURE 4 : PARTICLE PRODUCTION AND BOGOLIUBOV TRANSFORMATIONS.

We treat a quantum field $\hat{\varphi}$ on some background metric $g_{\alpha\beta}$. $\hat{\varphi}$ satisfies a linear wave equation whose leading term contains the metric e.g. minimally coupled Klein-Gordon

$$(-\nabla_\mu\nabla^\mu + m^2 + \xi R)\,\hat{\Phi} = 0 \tag{1}$$

or Dirac
$$(\gamma^\mu \nabla_\mu + m)\,\hat{\psi} = 0 \tag{2}$$

The equations conserve a current J_μ, e.g.

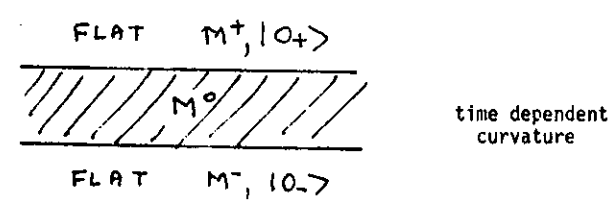
$$J_\mu = \frac{1}{2i}\,\bar{\Phi}\overleftrightarrow{\nabla}_\mu \Phi \tag{3}$$

or
$$J_\mu = \psi^D \gamma_\mu \psi \tag{4}$$

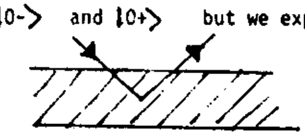
which is used to normalize wave functions

$$\langle \psi, \psi \rangle = -\int_\Sigma J_\mu\, d\Sigma^\mu \tag{5}$$

where Σ is some spacelike Cauchy surface. The conceptually simplest spacetimes are sandwich ones :



In M^- and M^+ we have stable vacua $|0-\rangle$ and $|0+\rangle$ but we expect particles to be created in Mo



We use the standard Stuckelberg-Feynman convention about particles going forward in time and antiparticles backward in time. Note that these particles are out going particles and do not coincide with ingoing particles written as

In fact because of the time dependence in Mc an in going purely negative or positive frequency wave in M^-  appears as a combination of positive and negative frequences in M^+. i.e.

Hole theory indicates that particle creation will thus take place. The correct formulation to handle this in the second quantized theory is that of Bogoliubov Transformations.

In M^- $$\hat{\psi} = \sum_i \hat{a}_i^{in} p_i^{in} + \hat{b}_i^{+in} n_i^{in} \tag{6}$$

in M^+ $$\hat{\psi} = \sum_i \hat{a}_i^{out} p_i^{out} + \hat{b}_i^{+out} n_i^{out} \tag{7}$$

where p_i is positive frequency

n_i is negative frequency

i is a label like momentum.

The normalization is such that (in an obvious matrix notation)

$$\langle (p,n) ; \begin{pmatrix} p \\ n \end{pmatrix} \rangle = G \tag{8}$$

where G is

BOSONS

$$G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

FERMIONS

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{9}$$

Linearity implies

$$\begin{pmatrix} \hat{a} \\ \hat{b}^+ \end{pmatrix}^{in} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{b}^+ \end{pmatrix}^{out} \tag{10}$$

$$(p,n)^{out} = (p,n)^{in} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{11}$$

$$\begin{pmatrix} p \\ n \end{pmatrix}^{out} = \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} \begin{pmatrix} p \\ n \end{pmatrix}^{in} \tag{12}$$

In order that the commutation relations are preserved (or the normalization) we are forced to use commutation relations for bosons and anticommutation relations for fermions and the classical S-matrix $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ (13)

must satisfy

$$S^+ G S = S G S^+ = G \tag{14}$$

or

$$S^{-1} = G S^+ G \tag{15}$$

then

BOSONS

$$S^{-1} = \begin{pmatrix} A^+ , & -C^+ \\ -B^+ , & D^+ \end{pmatrix}$$

FERMIONS

$$S^{-1} = \begin{pmatrix} A^+ & C^+ \\ B^+ & D^+ \end{pmatrix} \tag{16}$$

Thus the number of outgoing particles in the initial vacuum $|0_-\rangle$ is given by

$$N_i^{(+)} = \langle 0_- | \hat{a}_i^+ \hat{a}_i | 0_- \rangle = (C^+ C)_{ii} \tag{17}$$

$$N_i^{(-)} = \langle 0_- | \hat{b}_i^+ \hat{b}_i | 0_- \rangle = (B^+ B)_{ii} \tag{18}$$

A Bogoliubov Transformation with C or B $\neq$ 0 is called "mixing".

To construct the quantum S-matrix from S we define

$$\hat{F} = \sum \hat{a}_i^+ M_{ij} \hat{b}_j^+ \tag{19}$$

BOSONS	FERMIONS
M_{ij} symmetric	M_{ij} antisymmetric

The BakerHausdorf formula gives

$$e^{-\hat{F}} \hat{a}_i e^{\hat{F}} = \hat{a}_i + M_{ij} \hat{b}_j^+ \tag{20}$$

$$e^{-\hat{F}} \hat{b}_i e^{\hat{F}} = \hat{b}_i + M_{ij} \hat{a}_j^+ \tag{21}$$

define $$|B\rangle = e^{-F^{out}} |0_-\rangle \tag{22}$$

and discover that

$$[A\hat{a} + (AM+B)\hat{b}^+] |B\rangle = 0 \tag{23}$$

$$[(\bar{C} + \bar{D}M)\hat{a}^+ + \bar{D}\hat{b}] |B\rangle = 0 \tag{24}$$

There are 2 cases

det A $\neq$ 0 Weak Bogoliubov Transformation

det A = 0 Strong Bogoliubov Transformation

We assume det A $\neq$ 0

Exercise: prove for <u>bosons</u> det A$\neq$0 always using (14) and (15).

We solve (23) (24) by setting

$$|B\rangle = |0_+\rangle \tag{25}$$

$$M = -A^{-1}B \tag{26}$$

and get $$|0_-\rangle = e^{-\mathrm{Im}\,W} e^{\hat{F}} |0_+\rangle \tag{27}$$

$$|\langle 0_+|0_-\rangle|^2 = e^{-2\,\mathrm{Im}\,W} \tag{28}$$

BOSONS	FERMIONS
$\lvert\langle 0_+\lvert 0_-\rangle\rvert^2 = (\det A)^{-2}$	$\lvert\langle 0_+\lvert 0_-\rangle\rvert^2 = (\det A)^{2}$ (29)

<u>REMARKS</u> :

1) $e^{\hat{F}} |0_+\rangle$ contains particle-antiparticle pairs

2) $\langle 0_+|0_-\rangle \neq 0$

3) bosons are <u>always</u> created in pairs.

4) For <u>fermions</u> strong Bogoliubov transformations can produce odd numbers of Fermions -they are not treated here since they are not relevant to Black Holes, see G.W. Gibbons & J. Richer, Phys. Lett. 89B (1980) 338

<u>A SPECIAL CASE</u> is useful in Black Hole problems.

Let each mode P_i carry an extra label $\epsilon = +1$ or $\epsilon = -1$ (e.g. right going and leftgoing respectively) and suppose for <u>BOSONS</u>

$$p_{i\epsilon}^{out} = \cosh\theta_i\, p_{i\epsilon}^{in} + \sinh\theta_i\, n_{i-\epsilon} \tag{30}$$

i.e. S is block diagonal such that

$$\begin{pmatrix} p_{+1} \\ p_{-1} \\ n_{+1} \\ n_{-1} \end{pmatrix}_i^{out} = \begin{pmatrix} \cosh\theta & 0 & 0 & \sinh\theta \\ 0 & \cosh\theta & \sinh\theta & 0 \\ 0 & \sinh\theta & \cosh\theta & 0 \\ \sinh\theta & 0 & 0 & \cosh\theta \end{pmatrix}_i \begin{pmatrix} p_{+1} \\ p_{-1} \\ n_{+1} \\ n_{-1} \end{pmatrix}_i^{in} \tag{31}$$

$$|0_-\rangle = \prod_{i,\epsilon} \frac{1}{\cosh\theta_i} \exp\left(-\tanh\theta_i\, b_{i\epsilon}^+ a_{i-\epsilon}^+\right) |0_-\rangle \tag{32}$$

$$= \prod_i \frac{1}{\cosh\theta_i} \sum_{N,\epsilon} (-\tanh\theta_i)^N |N_{i\epsilon}^+\rangle \otimes |N_{i-\epsilon}^-\rangle \tag{33}$$

$|N_{i\epsilon}^+\rangle$ has N particles in state $\leftrightarrow p_{i\epsilon}$

$|N_{i-\epsilon}^-\rangle$ has N particles in state $\leftrightarrow n_{i-\epsilon}$

i.e. we get all combinations like

$|1^-_{-1}\rangle \otimes |1^+_{+1}\rangle$ $\qquad\qquad$ $|1^+_{-1}\rangle \otimes |1^-_{+1}\rangle$

Suppose the leftgoing states, $\epsilon = -1$ are unobservable we take a trace over these states to get a density matrix for the observable $\epsilon = +1$ states given by

$$\hat{\rho} = \prod_i \sum_N \rho_{iN} \left[|N^+_{i+}\rangle\langle N^-_{i+}| + |N^-_{i+}\rangle\langle N^+_{i+}| \right] \tag{34}$$

where

$$\rho_N = \frac{(\tanh\theta)^{2N}}{(\cosh\theta)^2} \tag{35}$$

if we found that $\tanh\theta_i = \exp - \frac{\omega_i}{2T}$

where ω_i is the energy of the mode ρ^{out}_i and T is independent of the mode, we should get an outgoing thermal Gibbs state at temperature T. Such a Bogoliubov Transformation is called Thermal. If $\tanh\theta_i = \exp - \frac{(\omega_i - \mu_i)}{2T}$ we would have a chemical potential μ_i for the mode ω_i where μ_i would have to be linear in conserved quantum numbers like charge and angular momentum. The situation hypothesised above indeed holds for black holes when $T = \frac{\kappa}{2\pi}$ and where the $\epsilon^{out} = -1$ modes are those that fall through future horizon into the hole and those with $\epsilon^{out} = +1$ escape to $\mathscr{I}^+$.

COMPUTATION OF $A's$ AND $B's$ FOR SCHWARZSCHILD :

To get the correct boundary conditions we must study realistic collapse. Roughly we extrapolate our previous discussion by taking a limit where

$M^- \leftrightarrow \mathscr{I}^-$ $\qquad$ a past Cauchy Surface

$M^+ \leftrightarrow \mathscr{I}^+ \cup H^+$ $\qquad$ a future Cauchy Surface

Defining positive frequency functions on $\mathscr{I}^-$ is easy, we use the advanced time coordinate v

on $\mathscr{I}^-$ $\quad \exp -i\omega v$ is positive frequency if $\omega > 0$

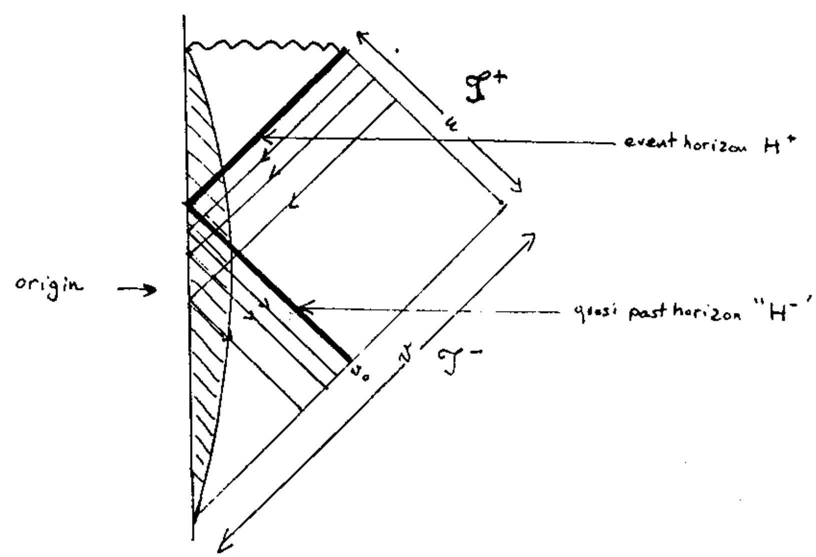
on $\mathscr{I}^+$ $\quad$ – we proceed similarly-exp $-i\omega u$ is positive frequency if $\omega >$

On H^+ it doesn't matter what we choose since we trace over these $\epsilon = -1$ states.

Thus on $\mathscr{I}^+$
$$p^{out} = \frac{e^{-i\omega u}}{\sqrt{2\omega}} Y_{em}(\theta,\phi) f_{em}(r) \tag{36}$$

on $\mathscr{I}^-$
$$p^{in} = \frac{e^{-i\omega v}}{\sqrt{2\omega}} Y_{em}(\theta,\phi) g_{em}(r) \tag{37}$$

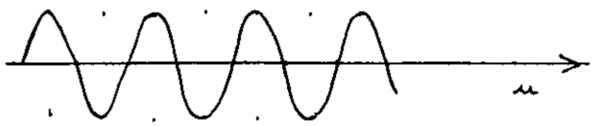
To compute the Bogoliubov Transformations we use geometric optics we follow the wave fronts back to the "quasi past horizon" "H^-" defined as that ingoing null cone which just intersects the origin of the event horizon (see diagram). "H^-" intersects $\mathscr{I}^-$ at v_0



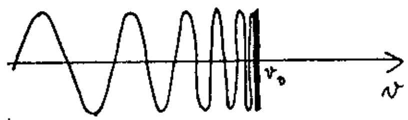
We send p^{out} backwards in time from $\mathscr{I}^+$ and register it on $\mathscr{I}^-$. Then we Fourie pose it into positive and negative frequency parts.

The important point is that no signal sent back from $\mathscr{I}^-$ and bounced off the centre can register on $\mathcal{J}^-$ after advanced time v_0. The wave-fronts must <u>pile up</u> at v_0 since no matter how late they start out $(u \to +\infty)$ they arrive at $\mathcal{J}^-$ no later than the finite time $v = v_0$.

On $\mathcal{J}^+$



on $\mathcal{J}^-$



Let U be an <u>affine</u> parameter on "H^-".

We expect roughly that

$$U = -\exp -\kappa u \tag{38}$$

since near "H^-" the geometry will be close to that of the exact Schwarzschild metric. The signal is thus ∞ blue shifted in getting to "H^-". In being reflected from the origin and scattered back to $\mathcal{J}^-$ it will receive virtually no change in frequency

$$e^{-i\omega u} \quad \text{on } \mathcal{J}^+ \tag{39}$$

$$\text{is} \begin{pmatrix} (-u)^{i\omega/\kappa} & u<0 \\ 0 & u>0 \end{pmatrix} \text{on "}H^-\text{"} \tag{40}$$

and thus $(v_0 - v)^{i\omega/\kappa}\,\theta(v_0 - v)$ on $\mathcal{J}^-$ times some constants independent of v.

$$p_\omega^{out} = \frac{1}{2\pi}\int_0^\infty \alpha_{\omega\omega'}\, d\omega' e^{-i\omega' v} + \frac{1}{2\pi}\int_0^\infty \beta_{\omega\omega'}\, d\omega'\, e^{i\omega' v} \tag{41}$$

where

$$\alpha_{\omega\omega'} = (A^t)_{\omega\omega'} \tag{42}$$

$$\beta_{\omega\omega'} = (C^t)_{\omega\omega'} \tag{43}$$

We Fourier analyze (*Exercise*) to get

$$|\alpha_{\omega\omega'}| = |\beta_{\omega\omega'}| \exp \frac{\omega}{2T} \tag{44}$$

where
$$T = \frac{\kappa}{2\pi} = \frac{1}{8\pi M} \tag{45}$$

If it were really true (as it is in the geometrical optics approximation) that all the wave p^{out} sent from $\mathcal{J}^+$ passes through "H^-" and none is scattered on to $\mathcal{J}^-$ to the future of $v = v_0$ we could immediately apply our previous formalism to deduce that the black hole radiates thermally in each mode with strength

$$\frac{1}{1 - \exp(-\omega\beta)} \tag{46}$$

$$\beta = T^{-1} = 8\pi M \tag{47}$$

More occurately only a fraction Γ of the flux of the wave p^{out} passes through "H^-" and a more detailed argument (see refs) gives a rate

$$\frac{\Gamma(\omega)}{1 - \exp(-\omega\beta)} \tag{48}$$

By turning the Penrose diagram upside down (reversing the sense of time) we see that $\Gamma(\omega)$ has the physical significance of the absorption coefficient for the wave p_ω^{out} . Thus a <u>black hole</u> acts as a <u>black body</u>. It satisfies Kirchhoff's Law.

Clearly the calculation really only uses properties of the exact Kruskal manifold. In this (unphysical) manifold we may define certain <u>states</u> by specifying the definitions of positive frequency on suitable Cauchy surfaces. The states are relevant to the behaviour in the realistic collapse situation. Those most useful are as follows :

STATE	POSITIVE FREQUENCY ON $\mathcal{J}^-$	POSITIVE FREQUENCY ON H^-
BOULWARE	$e^{-i\omega v}$, $\omega > 0$	$e^{-i\omega u}$, $\omega > 0$
HARTLE/HAWKING	$e^{-i\omega V}$, $\omega > 0$	$e^{-i\omega U}$, $\omega > 0$
UNRUH	$e^{-i\omega v}$, $\omega > 0$	$e^{-i\omega U}$, $\omega > 0$

The physical significance is as follows :

STATE	PROPERTIES
BOULWARE	Zero temp at large r , singular on horizon no incoming our outgoing particles at $\mathcal{J}$
HARTLE HAWKING	Finite temperature. Has incoming and outgoing thermal flux at $\mathcal{J}$. Non singular on horizon and Boyer axis.
UNRUH	Zero temp on $\mathcal{J}^-$. Thermal flux at $\mathcal{J}^+$.

Clearly collapse brings about the UNRUH STATE - free radiation into empty space.

Enclosing the hole in a box and allowing equilibrium to establish itself results in the thermal equilibrium described by the Hartle-Hawking state.

The Boulware state would be the natural zero temperature state if the metric for $r < r_s \quad (r_s > 2M)$ were replaced by that of a static star, and no horizon were present. Like the (u, v) coordinates it is singular on the horizon.

N.B. Associated with each state we can associate propagators which bear the same masses.

EFFECTS OF ROTATION AND CHARGE :

1) Rotation : On the past horizon we should use the co-rotating coordinate $\tilde{\phi} = \phi - \Omega t$. A mode of the form $e^{-i\omega t} e^{in\phi}$ becomes

$$e^{-i(\omega - n\Omega)t} e^{in\tilde{\phi}}$$

clearly we can have $\omega > 0$ but $\omega - n\Omega < 0$ for suitable Ω . This shows that even the analogue of the Boulware state would be unstable and radial modes with the same angular momentum as that of the hole, energies lying in the range $0 < |\omega| < |n\Omega|$. These are the super-radiant modes described earlier. When we combine this effect with the thermal redefinition we get an emission rate from the Unruh state of

$$\frac{\Gamma}{1 - \exp[-(\omega - n\Omega)\beta]} \tag{49}$$

2) Charge : On the past horizon we should use the gauge $\hat{A} = \left(\frac{\varphi}{r} - \frac{\varphi}{r_+}\right) dt$. The gauge transformed mode is

$$\tilde{\phi} \propto \exp i(\omega - e\,\phi_H)t$$

Thus even from the analogue of the Boulware state the super-radiant modes such that $\frac{e\varphi}{r_+} > 0$ can carry away charge. If the particles are massive we need $\omega > m$ at $\mathcal{J}^+$ but

$$\omega - \frac{e\varphi}{r_+} < 0 \qquad \text{at } H^-$$

so there is a threshold (cf. Lec. 3 eq.(23)and ff.)

$$\frac{e\varphi}{r_+} > m \tag{50}$$

From the analogue of the Unruh state we have in general an emission rate

$$\frac{\Gamma(\omega)}{1 - \exp - (\omega - \mu)\beta} \tag{51}$$

$$\mu = e\Phi + n\Omega \tag{52}$$

Note that (for bosons) both $\Gamma(\omega)$ and the Boltzmann factor are negative for the super-radiant mode. μ is clearly the chemical potential of the mode.

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SUPER-RADIANCE AND HAWKING EMISSION

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LECTURE 5 : FINITE TEMPERATURE FIELD THEORY APPROACH.

The previous approach cannot easily

1) deal with massive particles

2) deal with interacting particles

3) be extended to quantum gravity.

From a thermodynamic point of view we expect it to be easier to study the possible thermal equilibrium configurations rather than the non equilibrium process consisting of free radiation into empty space. In particular we ask

1) is thermal equilibrium still possible in the presence of interactions ?

2) is the temperature still given by the universal formula $T = \kappa/2\pi$ (independent of particle species) ?

3) What is the origin of this temperature ?

To answer these questions we recall that to study a quantum field theory at finite temperature we must construct Green's functions which are periodic in imaginary time with period $\beta = T^{-1}$ (for bosons and zero chemical potential).

In general for temperature β and chemical potential μ and fields of spin s we require

$$g(\underline{x},\underline{x}',t-t') = e^{\mu\beta}(-1)^{2s}\, g(\underline{x},\underline{x}',t-t'+i\beta) \tag{1}$$

These may be obtained by performing a Euclidean functional integral over fields which are $\begin{pmatrix}\text{periodic}\\ \text{antiperiodic}\end{pmatrix}$ in imaginary time for $\begin{pmatrix}\text{bosons}\\ \text{fermions}\end{pmatrix}$. Geometrically we are working (in the Euclidean approach) not on $\mathbb{R}^4$ (which we do for vacuum→vacuum amplitudes at zero temperature) but on $\mathbb{R}^3 \times S^1$ where the S^1 represents the periodically identified imaginary time coordinate τ. The metric is

$$ds^2 = d\tau^2 + dr^2 + r^2 d\Omega_r^2 \qquad 0 \leq \tau \leq \beta \tag{2}$$

Finite temperature field theory is thus ordinary (Riemannian) Q.F.T. in a space with non trivial topology. If we expand in power of β, the infinite temperature limit corresponds to dimensionally reducing the 4-dimensional theory to three dimensions. Kaluza-Klein theory by the same token can be thought of as the infinite temperature limit of the 5-dimensional theory.

For the present purposes the relevant question is what does the Wick rotation of the Schwarzschild solution look like ? Evidently for $r > 2M$ we get a regular solution

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \tag{3}$$

By setting $t = -i\tau$ (4)

To see what is happening at $r = 2M$ we use the Kruskal form (eqs (12) (13) (15) of Lec. 2) and get ($Y = -i\ T$)

$$ds^2 = \frac{32M^3}{r} e^{r/2M} (dX^2 + dY^2) + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \tag{5}$$

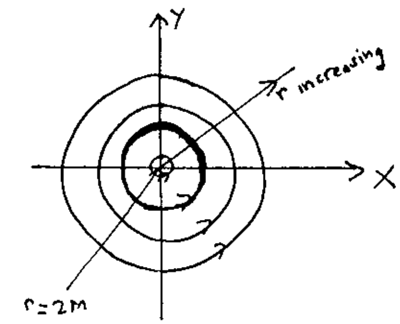
with

$$X = \cos\frac{\tau}{4M} \left|1 - \frac{r}{2M}\right|^{1/2} e^{r/2M} \tag{6}$$

$$Y = \sin\frac{\tau}{4M} \left|1 - \frac{r}{2M}\right|^{1/2} e^{r/2M} \tag{7}$$

and r is an implicit function of $X^2 + Y^2$.

Evidently in imaginary time we have invariance under rotations in the X Y plane. The axis of rotation corresponding to $r = 2M$ (a 2 surface of area $A = 16\pi M^2$)

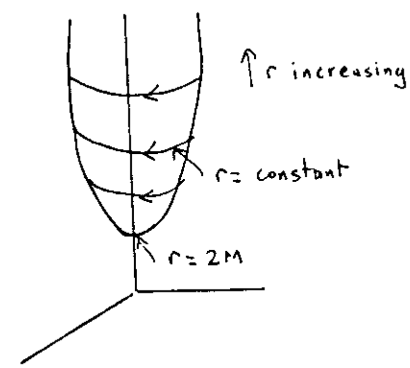


Thus τ is forced to be periodic with period β given by

$$\beta = T^{-1} = 8\pi M \tag{8}$$

This is the geometrical origin of the universal nature of the black hole temperature and indicates that if one constructs Green's functions on this background when analytically continued back to real time they will be finite temperature Green's functions, i.e. expectation values in the interacting analogue of the Hartle-Hawking thermal equilibrium state.

The (r, τ) plane can be isometrically embedded in R^3 and looks like this



Note that the metric is everywhere non singular.

The $r = 0$ singularity is not on the manifold.

As $r \to \infty$ we obtain a metric which tends to the flat metric on $R^3 \times S^1$

described earlier.

The full complex structure of the Schwarzschild solution can be conveniently read off from its representation as an algebraic surface (Lec. 2 eqs (22) (23) (24)). The full complexification has 7 complex coordinates satisfying 3 complex equations giving a 4 complex dimensional surface $\mathcal{M}_c$. By looking at real 4 dimensional submanifolds we get "real sections" if all $(x_1, \ldots x_7)$ are real we get the Real Lorentzian Section (signature -+++) if all $x_1 \ldots x_7$ except x_5 which is pure imaginary we get the Real Euclidean Section (signature ++++) we have just described. This latter is the only non-singular Euclidean real section.

Effects of Charge. The Euclideanization is exactly analogous but we change gauge to get a regular vector potential on the axis

$$\tilde{A} = \left(\frac{\varphi}{r} - \frac{\varphi}{r_+}\right) dt \tag{9}$$

in this gauge the Green's functions satisfy

$$\tilde{g}(r,\theta,r',\theta',\phi,\phi',t-t') = \tilde{g}(r,\theta,r',\theta',\phi,\phi',t-t'+i\beta) \tag{10}$$

$$\beta = \frac{2\pi}{K_+} \tag{11}$$

the gauge transformed Green's functions $g(r,\theta,\phi,r',\theta',\phi',t-t')$ are given by

$$g(r,\theta,\phi,r',\theta',\phi',t-t') = e^{ie\frac{\varphi}{r_+}(t-t')}\tilde{g}(r,\theta,\phi,r',\theta',\phi',t-t') \tag{12}$$

whence

$$g(r,\theta,\phi,r',\theta',\phi',t-t') = e^{\mu\beta} g(r,\theta,\phi,r',\theta',\phi',t-t'+i\beta) \tag{13}$$

$$\mu = \frac{e\varphi}{r_+} \tag{14}$$

The rotating case proceeds in a similar fashion. One sets $t = -i\tau$ and gets a complex metric. One can either work with this since the complex metric (which is also periodic in imaginary time) has no real null vectors one can find unique propagators. Alternatively one can in addition analytically continue in the angular momentum $J = aM$ to get a real Riemannian metric (signature ++++) which is also periodic in imaginary time. Now fix a given angular momentum eigen-state with angular momentum n, look at its Green's function

$$F(r,\theta,r',\theta')\, e^{-i\omega(t-t')} e^{in(\phi-\phi')} = F_n(r,\theta,r',\theta',t-t',\phi-\phi') \tag{15}$$

$$g_n(t-t'+i\beta,\ \tilde{\phi}-\tilde{\phi}') = g_n(t-t',\ \tilde{\phi}-\tilde{\phi}') \tag{16}$$

whence

$$g_n(t-t'+i\beta,\ \phi-\phi') = e^{n\Omega\beta} g_n(t-t',\ \phi-\phi') \tag{17}$$

again we see that

$$\mu = n\Omega \tag{18}$$

is the relevant chemical potential. Note that the box must co-rotate with the hole to maintain equilibrium. It must also be raised to an electrostatic potential equal to that of the hole.

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LECTURE 6 : EUCLIDEAN QUANTUM GRAVITY.

The aim of this lecture is to consider the equilibrium between a hole and some radiation in a finite box of volume V allowing the gravitational field to vary. We note that since black holes (like all gravitating systems) have negative specific heats (the temperature increases as the mass decreases) we should strictly only use the microcanonical ensemble in which we fix the total energy E. The Gibbs ensemble -entailing equilibrium between the system and an infinite heat bath-is unstable. If the hole absorbs some heat it cools down and absorbs even more. Conversely if it loses heat it gets hotter and loses even more heat. Another way of seeing that the canonical ensemble should not strictly exist is to note that

$$Z(\beta) = \int N(E)\, e^{-\beta E}\, dE \tag{1}$$

$$= \int e^{S(E)}\, e^{-\beta E}\, dE \tag{2}$$

where S(E) is the entropy but :

$$S = \frac{1}{4} A = 4\pi E^2 \tag{3}$$

and the integral in (2) cannot possibly converge unless one takes it up the imaginary axis to get an imaginary partition function. We shall proceed formally with the functional integral and see what happens since I know of no alternative functional formulation of the microcanonical ensemble. Note that since we are dealing with quantum gravity, quantum corrections will in general diverge. Around a general Ricci flat background the one loop counterterm is proportional to the Euler number χ. χ = 2 for the Schwarzschild solution. If the constant of proportionality is non zero the theory depends on the arbitrary renormalization length μ and also contains 4-th derivative terms which spoil unitarity. The only presently viable 4 quantum theory of gravity which does not depend on μ at the one-loop level is N = 8 extended supergravity,

provided a particular field representation is chosen.

In general for a partition function Z at temperature $T = \beta^{-1}$ one sums over fields which are periodic in imaginary time with period β. Clearly the Schwarzschild solution is an example of one such solution. In general we define an Asymptotically Flat (A.F.) space as one which outside a compact set of the metric tends to the standard flat metric on $R^3 \times S^1$. That is for large distances the metric becomes :

$$ds^2 = d\tau^2 + dr^2 + r^2\, d\Omega_2^2 + O\left(\frac{1}{r}\right)$$

where $d\Omega_2^2$ is the standard metric on the 2-sphere, S^2, and $0 \leq \tau \leq \beta$

For zero temperature in the vacuum state we wish to sum over metrics which are asymptotically flat not just in 3 directions as above, but in all 4 directions. This is analogous to the instantons in Yang-Mills theory which tend to pure gauge outside some large radius in 4-dimensional Euclidean space. Thus we define an Asymptotically Euclidean metric (A.E.) as one which outside a compact set tends to the standard flat metric on R^4. That is at large distances the metric has the form

$$ds^2 = d\rho^2 + \rho^2\, d\Omega_3^2 + O\left(\frac{1}{\rho^2}\right)$$

where $d\Omega_3^2$ is the standard metric on the 3-sphere, S^3.

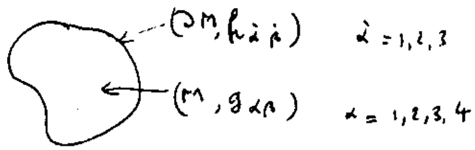
Now, the gravitational action for Riemannian (++++) metrics

i) is not scale invariant

ii) is not positive semi-definite.

This constrasts with Yang-Mills or $\lambda\phi^4$ theory (with no mass) which are both scale invariant and positive semi-definite. To see this recall that the action is given by :

$$I = -\frac{1}{16\pi}\int_M R\sqrt{g}\,d^4x - \frac{1}{8\pi}\int_{\partial M}(K-K_0)\sqrt{h}\,d^3x$$

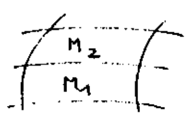


(6)

M is the manifold (metric $g_{\alpha\beta}$) ∂M its boundary and $h_{\alpha\beta}$ the metric induced on ∂M from $g_{\alpha\beta}$. K is the trace of the second fundamental form (or extrinsic curvature) of ∂M $\quad h = \det h_{ij}$

$$K = \frac{1}{2}\frac{h^{ij}}{\sqrt{h}}\frac{\partial h_{ij}}{\partial n} \qquad (7)$$

where $\frac{\partial}{\partial n}$ is differentiation along the normals n of ∂M .
<u>Physically</u> $\int K\sqrt{h}\,d^3x$ is designed to remove second derivatives from the action so that over spacetime regions, M_1 , M_2 , and for continuous metrics



$$I(M_1) + I(M_2) = I(M_1 \cup M_2)$$

<u>Geometrically</u>

$$\int_{\partial M} K\sqrt{h}\,d^3x = \frac{\partial}{\partial n}\int\sqrt{h}\,d^3x = \frac{\partial V}{\partial n} \qquad (8)$$

where V is the 3-volume of ∂M . K_0 is a term designed to render the action of flat space (on R^4 or $R^3 \times S^1$) zero.

A.E. R^4 FLAT SPACE	A.F. $R^3\times S^1$ FLAT SPACE
$\partial M = S^3$	$\partial M = S^2\times S^1$
$h_{ij}\,dx^i dx^j = R^2\,d\Omega_3^2$	$h_{ij}\,dx^i dx^j = R^2\,d\Omega_2^2 + d\tau^2$
$V = 2\pi^2 R^3$	$V = 4\pi R^2\beta$
$\frac{\partial V}{\partial n} = 6\pi^2 R^2$	$\frac{\partial V}{\partial n} = 8\pi R\beta$

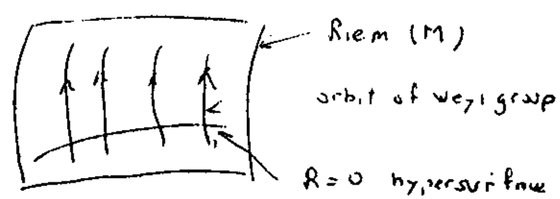
The action of flat space would be ∞ (as $R \to \infty$) unless a contribution of the form $\frac{1}{8\pi}\int K_0\sqrt{h}\,d^3x$ were subtracted off. We wish to weight the functional integral with the difference between the action of flat space and the non-flat metric we are interested in. Now if we rescale $g_{\alpha\beta}$ by a <u>Weyl rescaling</u>

$$g_{\alpha\beta} \to \Omega^2 g_{\alpha\beta} \qquad (9)$$

we get

$$I[\Omega^2 g_{\alpha\beta}] = -\frac{1}{16\pi}\int_M (R\Omega^2 + 6\nabla_\alpha\Omega\nabla^\alpha\Omega)\sqrt{g}\,d^4x - \frac{1}{8\pi}\int_{\partial M}\Omega^2(K-K_0)\sqrt{h}\,d^3x \qquad (10)$$

This shows that with a suitable Ω, I can be made as negative as we please. Consider the space of all metrics on M : Riem (M), it is fibred by the Weyl group



Along the orbits I decreases. We therefore fix the conformal freedom by setting

$$R = 0 \tag{11}$$

This is a conformal gauge fixing. We integrate over conformal factors Ω by setting $\Omega = 1 + i y$ with y real. The question is then

1) whether I is positive in the transverse directions

2) can one always rescale $g_{\alpha\beta}$ to set $R = 0$?

2) depends on the operator $-\nabla^\alpha \nabla_\alpha + \frac{1}{6} R = \Delta_c$ acting on scalars. If $\int \psi \Delta_c \psi \sqrt{g}\, d^4x > 0 \;\forall \psi$ then we can, but if Δ_c can have negative modes we can't. In general Δ_c will have negative modes.

1) depends on the ∂M . For A.E. metrics, we have the Positive Action Theorem (Schoen and Yau) which states that if a metric is given asymptotically by

$$ds^2 = \left(1 + \frac{\alpha}{\rho^2}\right)^2 \left[d\rho^2 + \rho^2 d\Omega_3^2 \right] + O\left(\frac{1}{\rho^3}\right)$$

and satisfies $R = 0$, then the action I satisfies

i) $I \geq 0$

ii) $I = 0 \iff g_{\alpha\beta}$ is flat

iii) $R_{\alpha\beta} = 0 \iff I = 0$

This shows that flat space is the <u>only</u> A.E. instanton and that the conformally invariant part of the gravitational action is indeed positive semi-definite for A.E. metrics. On the other hand, for A.F. metrics this is not so.

Clearly, the Schwarzschild solution is a gravitational instanton satisfying these boundary conditions. Its action I is found to be (exercise)

$$I = \frac{1}{2} M \beta \tag{12}$$

This is good because we have

$$Z = e^{-F\beta} = e^{-I_c}$$

where F is the <u>free energy</u> and I_c the classical action. We thus have

$$F = \frac{M}{2}$$

Now

$$F = M - TS$$

but for black holes (with $J = \varphi = P = 0$) using the Smarr relation

$$\frac{1}{2} M = TS$$

and we obtain perfect consistency with our previous thermodynamic arguments.

On the other hand, the gravitational contributions at <u>one loop</u> are found to be

$$Z_1 = \frac{\det \Delta_1}{(\det \Delta_L)^{1/2}}$$

where $\Delta_1 = -\nabla_\alpha \nabla^\alpha$ acts on the <u>vector ghost</u> and Δ_L (the Lichnerowicz Laplacian) acts on symmetric tensors $\delta g_{\alpha\beta}$. Explicitly Δ_L is given by

$$\Delta_L(\delta g_{\alpha\beta}) = -\nabla_\sigma \nabla^\sigma (\delta g_{\alpha\beta}) - 2 R_\alpha{}^\sigma{}_\beta{}^\rho (\delta g_{\sigma\rho})$$

To regularize these determinants we use <u>zeta-function-regularization</u>. For an operator D define

$$\zeta_D(s) = \sum_n \frac{\mu^{2s}}{\lambda_n^s}$$

where real part of $s > 2$ and λ_n are the eigenvalues of D. If D is second order, $\zeta_D(s)$ has a meromorphic extension to all s which is analytic at $s = 0$ (Seeley's theorem). We thus define

$$\det D = \left(\exp - \zeta_D'(0)\right) \mu^{2\zeta_D(0)} \tag{20}$$

Note that $\det D$ is explicitly μ-dependent if $\zeta_D(0) \neq 0$. $\zeta_D(0)$ is essentially the one-loop counter-term of dimensional reduction and may be evaluated using an extension of the 't Hooft algorithm. One finds for extended supergravity (ignoring boundary terms and using background field equations)

$$\zeta_D(0) = A \frac{\int |R_{abcd}|^2 \sqrt{g}\, d^4x}{32\pi^2} + B \frac{\int \Lambda^2 \sqrt{g}\, d^4x}{12\pi^2} \tag{21}$$

where	A	B
N = 0	106/45	- 87/10
N = 1	41/24	- 77/12
N = 2	11/12	- 13/3
N = 3	0	- 5/2
N = 4	- 1	- 1
N = 5	- 2	0
N = 6	- 3	0
N = 7	- 5	0
N = 8	- 5	0

N = 0 is Einstein's theory

Remarkably if one uses a different field content -67 scalars, 7 2-forms and 1 3-form- Siegel discovered that for N = 8 A = 0. The new field content comes naturally from dimensional reduction from 11 dimensions.

To return to Schwarzschild, the operator Δ_L has a spectrum which must be investigated numerically. Perry has shown that in addition to the expected

zero modes which may be handled using collective coordinates, Δ_L has just one negative mode. This mode is

i) time independent

ii) spherically symmetric

iii) has $\lambda = -\frac{0.9}{M^2}$

It is tempting to speculate that this single negative mode is related to the instability of the canonical ensemble for gravity though no very precise relation has yet been worked out. What is clear however (and follows from earlier work of Page) is that the Schwarzschild solution is <u>not</u> a local minimum of the conformally invariant part of the gravitational action.

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