Part III: Applications of Differential Geometry to Physics

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These foregoing notes consist of an introductiion to differential geometry at the level needed to understand the course, followed by more material on the content of the lectures in roughly the order it will be lectured. The introductory material will almost all have been covered in the Michaelmas term Part III General Relativity Course. Similar material, although in a somewhat different style, is covered in the Michaelmas term pure mathematics course on Part III Differential Geometry. For an elementary account of General Relativity in old fashioned Tensor Calculus notation, the reader may consult my Part II lecture notes which are available on my Web site.

The lecture notes are the result of combining and extending previous lecture notes and are not in a final polished form. As a result various topics are treated in a less than optimal order and not all mistakes and typo's have yet been eliminated. Thus they are not designed as a subsitute for taking notes during the lectures, merely as a reference. In fact, a transcript of a previous year's lectures was prepared by one of the audience using microsoft word. This is available on my web page

http://www.damtp.cam.ac.uk/user/gr/members/gwg.html

to users inside DAMTP.

Surgeries

I shall be available in my office (B1.24) from 2.00pm to 3.00pm on most Tuesdays to answer queries. A sign-up sheet for supervisions will be circulated during lecture 3.

2 Some Textbooks

In preparing the course I have used among others:

- 1 Y Choquet Bruhat, C DeWitt-Morette & M Dilliard Bleick *Analysis Manifolds and Physics*, North Holland
- 2 H Flanders *Differential Forms*, Dover
- 3 B O'Neil *Semi-Riemannian Geometry*, Academic Press
- 4 B Dubrovin, S Novikov & A Fomenko *Modern Geometry*, Springer
- 5 T Eguchi, P Gilkey and A J Hanson *Physics Reports* **66** (1980) 213-393
- 6 V Arnold *Mathematical Methods of Classical Mechanics*, Springer
- 7 N M J Woodhouse *Geometric Quantization*, Oxford

This list does not exhaust the set of good textbooks on this subject at the level at which it will be lectured. In particular many speak highly of

- 8 C Nash and S Sen *Topology and Geometry for physicists* Academic Press
- 9 M Nakahara, *Geometry, Topology and Physics* Institute of Physics
- 10 R Darling *Differential Forms and Connections* Cambridge University Press
- 11 T Fraenkel *The Geometry of Physics* Cambridge University Pres

There is a great deal of relevant material on particular topics in

- 12 J Marsden and R Ratiu, *Introduction to Mechanics and Symmetry*
- 13 R Gilmore *Lie Algebra Lie Groups and some of their Applications*
- 14 R Aldrovandi and J G Perira *An Introduction to Geometrical Physics* World Scientific
- 15 B Felsager *Geometry Particles and Physics*
- 16 C J Isham *Modern Differential Geometry for Physicists* World Scientific
- 17 V Guillemin and S Sternberg *Symplectic Techniques in Physics* Cambridge University Press
- 18 R Abraham and J Marsden *Foundations of Mechanics*
- 19 A S Schwarz *Topology for Physicists* Springer
- 20 A Visconti *Introductory Differential Geometry for Physicists*

Perhaps the best book covering almost all the course is probably number 1 or number 4. I found 2 extremely useful for a first look at differential forms. Finally much information and many relevant examples are contained in

- 21 C Misner, K Thorne and J Wheeler, *Gravitation* Freeman

3 Manifolds

The reader is expected to have encountered already some of the more elementary ideas of Einstein's theory of General Relativity, and with it the basic concepts of tensor analysis. For advanced work, however, rather more is required and one often needs to consider general, topologically non-trivial manifolds that may not be directly related to spacetime, the target spaces of non-linear sigma models for example. What follows is a brief introduction to the necessary machinery. The underlying theme is to start with a bare manifold with no further structure and then to introduce such things as connections and metrics.

It is customary to utter the following incantation:

Spacetime *is a connected, Hausdorff, differentiable pseudo-Riemannian manifold of dimension 4 whose points are called events.*

What does this mean? We need to introduce some ideas.

3.1 Topological spaces

A *topological space* (X, Θ) consist of a point set X where $\Theta = \{U_i\}$, $i \in I$, where I is an index set labelling members of a a collection of special subsets of X called *open sets*, such that

(i) All unions of open sets are open,

(ii) finite intersections of open sets are open, and

(iii) X and the empty set $\varnothing$ are open.

Different collections of subsets Θ may endow the same point set X with a different topology. A *basis for a topology* is a subset of all possible open sets which, by intersections and unions, can generate all possible open sets. A possible basis for the real line $\mathbb{R}$ is the collection of open intervals. A possible basis for $\mathbb{R}^n$ is the collection of open *balls* indexed by centre $\mathbf{x}_0$ and radius r such that $|\mathbf{x} - \mathbf{x}_0| < r$. An *open cover* $\{U_i\}$ of X is a collection of open sets such that every point $x \in X$ is contained in at least one U_i. X is *compact* if every open cover has a finite sub-cover. X is *Hausdorff* if every pair of disjoint points is contained in a disjoint pair of open sets. Any open set containing a point $x \in X$ is also called a *neighbourhood* of x.

A function from one topological space X to another Y, $f\colon X \to Y$ is *continous* if the inverse image of every open set is open. Two topological spaces are *homeomorphic* if there is 1–1 map ϕ from X onto Y (a *bijection*) such that ϕ and ϕ^{-1} are continuous. By Leibniz's principle of the **Identity of Indiscernibles**, two homeomorphic topological spaces are usually thought to be same.

3.2 Manifolds

We define a *smooth*[1] *n-manifold* with a smooth 'atlas' of 'charts' as

(i) a topological space X,

(ii) an open cover $\{U_i\}$ called 'patches', and

(iii) a set (called an *atlas*) of maps $\phi_i\colon U_i \to \mathbb{R}^n$, called *charts*, which are injective, homeomorphisms onto their images and whose images are open in $\mathbb{R}^n$,

such that

(iv) if two patches U and $\tilde{U}$ intersect, then on $U \cap \tilde{U}$, both $\tilde{\phi} \circ \phi^{-1}$ and $\phi \circ \tilde{\phi}^{-1}$ are smooth maps from $\mathbb{R}^n$ to $\mathbb{R}^n$.

We write $\phi(x) = x^\mu$, $\mu = 1, 2, \ldots, n$. x^μ is called a *local coordinate* on X. In the usual notation we have $\phi \circ \tilde{\phi}^{-1}$ given by $\tilde{x}^\mu(x^\nu)$ and $\tilde{\phi} \circ \phi^{-1}$ given by $x^\mu(\tilde{x}^\nu)$.

Note that we are here taking the *passive* viewpoint of coordinate transformations, thinking of the points of the manifold as fixed and only their *labels* as changing.

Two atlases are said to be *compatible* if, where defined, the coordinates are smooth functions of one another.

[1]Smooth means as smooth as we need. When in doubt take it to mean C^∞

A *smooth n-manifold* with complete atlas is the maximal equivalence class consisting of all possible compatible atlases.

We shall denote such a manifold by M or M^n. As we have defined it, a manifold need not be Hausdorff, and so this as an additional requirement.

3.2.1 Example: S^2 and stereographic projection

Consider the unit two-sphere which we may think as the surface

$$x^2 + y^2 + z^2 = 1 \tag{3.1}$$

embedded $\mathbb{R}^3$. We could use as coordinates spherical polars θ, ϕ. However, these break down at the north and south poles where $\theta = 0$ and $\theta = \pi$, respectively, because one cannot assign to these points unique values of ϕ. However, we can use stereographic projection to define two charts

$$U_+ = S^2 \setminus \text{south pole} \; ; \qquad \phi_+ \colon (\theta, \phi) \mapsto x + iy = e^{i\phi} \tan \frac{\theta}{2} \tag{3.2}$$

and

$$U_- = S^2 \setminus \text{north pole} \; ; \qquad \phi_- \colon (\theta, \phi) \mapsto x' + iy' = e^{-i\phi} \cot \frac{\theta}{2} \; . \tag{3.3}$$

On the overlap

$$x + iy = \frac{1}{x' + iy'} \tag{3.4}$$

This is a two-chart atlas. We get a six-chart atlas by considering orthogonal projection with respect to all three axes. We think of S^2 as $x^2 + y^2 + z^2 = 1$ and define

$$\phi_3^+ \colon (x, y, z),\ z > 0,\ \mapsto (x, y) \; , \tag{3.5}$$

$$\phi_3^- \colon (x, y, z),\ z < 0,\ \mapsto (x, y) \; , \tag{3.6}$$

etc.

3.2.2 Example: Two incompatible atlases for $\mathbb{R}$

Let z be the standard coordinate on $\mathbb{R}$. We get a one-chart atlas by taking as ϕ to be the identity map $\phi : z \mapsto x$, say, and another one-chart atlas by taking for $\tilde{\phi}$ the cubic map $\tilde{\phi} : z \mapsto \tilde{x}^3$. Now $\tilde{x} = x^{1/3}$ is not smooth at $x = 0$.

3.2.3 Example: A non-Hausdorff 1-manifold

Let $X = \mathbb{R} \times \{0\} \cup \mathbb{R} \times \{1\}$ with $(x, 0)$ and $(x, 1)$ identified if $x > 0$. The points $(0, 0)$ and $(0, 1)$ are distinct, but every pair of neigbourhoods intersects. This is a toy model for a 'trouser universe' which splits into two. The only one-dimensional Hausdorff manifolds are S^1 and $\mathbb{R}$.

3.2.4 Functions on manifolds and orientability

We can now define a *real valued, smooth function* $f \colon M \to \mathbb{R}$ as one which is smooth in all coordinate systems, i.e. $f \circ \phi^{-1} = f(x^\alpha)$ is smooth in the usual sense. The set of all smooth functions $C^\infty(M)$ on a manifold forms a commutative ring, since we can define addition and multiplication by pointwise addition and multiplication of the values. We can also think of the functions as a commutative algebra, and we shall then denote them by $\mathfrak{F}(M)$.

A manifold M is said to *orientable* if it admits an atlas such that for all overlaps the Jacobian satisfies

$$\det \frac{\partial x^\alpha}{\partial \tilde{x}^\beta} > 0 \; . \tag{3.7}$$

3.2.5 Example: Projective spaces

Consider first the real projective plane $\mathbb{RP}^2$. This is the set of directions through the origin in $\mathbb{R}^2$, i.e. triples x, y, z such that $(x, y, z) \equiv (\lambda x, \lambda y, \lambda z)$, $\lambda \neq 0$. We get a three-chart atlas as follows:

$$\text{Set} \begin{cases} a^1 = \frac{x}{z},\ a^2 = \frac{y}{z} & \text{if } z \neq 0\ , \\ b^1 = \frac{z}{y},\ b^2 = \frac{x}{y} & \text{if } y \neq 0\ , \\ c^1 = \frac{y}{x},\ a^2 = \frac{z}{x} & \text{if } x \neq 0\ . \end{cases} \tag{3.8}$$

If $zy \neq 0$, we have $b^1 = 1/a^2$, $b^2 = a^1/a^2$. Computing the Jacobian shows that this atlas is not oriented. In fact, no oriented atlas exists: $\mathbb{RP}^2$is not orientable. A similar calculation for $\mathbb{RP}^3$ *does* give an oriented atlas, wherefore $\mathbb{RP}^3$ is orientable. The general results is that $\mathbb{RP}^n$ is orientable if and only if n is odd. This may be understood by from the fact that $\mathbb{RP}^n \equiv S^n/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by the antipodal map. This is orientation reversing if n is even and orientation preserving if n is odd.

There is only one one-dimensional compact Hausdorff manifold, and hence the simplest projective space $\mathbb{RP}^1$ is just S^1. We need two charts with coordinates x and $\tilde{x} = -\frac{1}{x}$. $\mathbb{RP}^1$ is a homogeneous space of $PSL(2, \mathbb{R})$ which acts by fractional linear transformations

$$x \mapsto \frac{ax + b}{cx + d}, \qquad ad - bc = 1\ . \tag{3.9}$$

These transformations are also known as *projective transformations* and arise in string theory as a finite dimensional subgroup of the infinite dimensional group of all diffeomorphisms of the circle S^1.

3.3 Tangent vectors

A *smooth curve* c in M is a smooth map $c : \mathbb{R} \to M$. In local coordinates, $c : \lambda \mapsto x^\mu(\lambda)$, where x^μ is a smooth function of λ. A *closed curve* is a map from S^1 to M. A curve is *simple* if it is 1–1 onto its image. For us a *path* is the image of a curve, i.e. it is a point set. Thus a curve contains information about the parameterization that a path does not contain[2]. If we orient $\mathbb{R}$, we can give a privileged direction to the associated equivalence class of curves and indicate this by attaching an arrow to the curve. In other words, two curves $x^\mu(\lambda)$ and $x^\mu(\tilde{\lambda})$ have the same orientation if, say, $\tilde{\lambda}$ is a monotonically increasing function of λ, $d\tilde{\lambda}/d\lambda > 0$. If M is a spacetime a curve, its path is called a *world-line* and corresponds to a *particle*. The curve with opposite orientation $x^\mu(-\lambda)$ corresponds to its *antiparticle*[3].

Given a curve c and a function f we can compose them to get a map $f \circ c : \mathbb{R} \to \mathbb{R}$ given in local coordinates by $f(x^\alpha(\lambda))$, and we can differentiate it. By the chain rule

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda}\ . \tag{3.10}$$

If we look at this at a point $p \in M$ and vary the curves passing through that point, we get a map $T : C^\infty(M) \to \mathbb{R}$ taking smooth functions on M to the reals:

$$T : f \mapsto Tf = \left.\frac{df}{d\lambda}\right|_{\lambda=0}\ , \text{ where } x^\alpha(0) = p\ , \tag{3.11}$$

which satisfies

(i) $T(f + g) = Tf + Tg$ (linearity), and

(ii) $T(fg) = (Tf)g + f(Tg)$ (the Leibniz rule).

Such a map is called a *tangent vector at p*. For clarity we could have written T_p but have not done in order to eliminate clutter. Since T is a linear operator, we can add two such vectors according to the formula

$$(T_1 + T_2)f = T_1 f + T_2 f, \tag{3.12}$$

[2]This terminology is not quite standard. Many books use the word path for what we call a curve and appear to have no special term for what we call a path.

[3]This is independent of whether the spacetime itself has a time-orientation.

and we can multiply by constants. Thus the space of tangent vectors at a point $p \in M$ is a vector space, which we call the *tangent space* and denote by $T_p(M)$[4]. Its dimension is n, as may be seen by by using Taylor's Theorem in local coordinates:

$$f(x) = f(p) + x^\alpha \partial_\alpha|_p f + \dots \ . \tag{3.13}$$

Thus if

$$Tx^\alpha = T^\alpha \quad \text{, then} \tag{3.14}$$

$$Tf = T^\alpha \partial_\alpha f|_p \ , \tag{3.15}$$

and thus $\frac{\partial}{\partial x^\alpha}$ is a basis, and we may write

$$T = T^\alpha \frac{\partial}{\partial x^\alpha} \ . \tag{3.16}$$

The basis $\frac{\partial}{\partial x^\alpha}$ is called a *coordinate basis* or sometimes *natural basis*. If T is the tangent vector to a curve c, we then have from (3.10)

$$T^\alpha = \frac{dx^\alpha}{d\lambda} \ . \tag{3.17}$$

In a new coordinate system the components will change. If

$$T = T^\alpha \frac{\partial}{\partial x^\alpha} = \tilde{T}^\alpha \frac{\partial}{\partial \tilde{x}^\alpha} \ , \tag{3.18}$$

then chain rule gives

$$\tilde{T}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} T^\beta \ . \tag{3.19}$$

This is the elementary definition of a *contravariant vector*.

We can now introduce the idea of a *vector field*, which is a continuous assignment of a vector $V(p) \in T_p(M)$ to each point p in the manifold M. In local coordinates

$$V \equiv V^\mu(x^\nu) \frac{\partial}{\partial x^\mu} \ . \tag{3.20}$$

The set of all vector fields on M is denoted by $\Gamma(TM)$ or $\mathfrak{X}(M)$.

Given a vector field $V \in \mathfrak{X}(M)$ we have, at least locally, the associated *integral curves* defined as the solutions of the non-linear o.d.e

$$\frac{dx^\mu}{d\lambda} = V^\mu(x) \ , \tag{3.21}$$

whose tangent vectors coincide with the vector field at every point in M. In general a family of curves, one passing through each point p of M, is called a *congruence of curves*.

We can also introduce the concept of the *tangent bundle* $T(M)$ or *velocity space* as the space of all possible vectors at all possible points, i.e.

$$T(M) = \bigcup_{p \in M} T_p(M) \ . \tag{3.22}$$

This is an $2n$-dimensional manifold with local coordinates (x^μ, V^ν), where $V \equiv V^\nu \frac{\partial}{\partial x^\nu}$. One may think of a vector field as a sort of n-dimensional surface in $T(M)$. In the terminology of vector bundles, which we will introduce in more detail later, this surface is called a *section*.

3.4 Non-coordinate bases

It is often useful to use for $T_p(M)$ a *non-coordinate basis* $\{\mathbf{e}_a\}$, $a = 1, 2, \dots, n$, called variously a *tetrad*, *vierbein*, *vielbein*, *4-leg* or *repère mobilie*, *moving frame* etc. Thus

$$T = T^a \mathbf{e}_a. \tag{3.23}$$

[4]Many authors omit the brackets and write T_pM.

With respect to a coordinate basis $\frac{\partial}{\partial x^\alpha}$ we have

$$\mathbf{e}_a = e_a{}^\alpha(x)\frac{\partial}{\partial x^\alpha} \ . \tag{3.24}$$

It is customary to refer to α as a *world index* and to a as a *tangent space index*. This is because now we are allowed not only coordinate transformations which induce a change of the basis $\frac{\partial}{\partial x^\alpha}$, but also position dependent changes of the basis $\mathbf{e}_a$

$$\mathbf{e}_a \to \tilde{\mathbf{e}}_a = \Lambda_a{}^b(x)\mathbf{e}_b \ . \tag{3.25}$$

This induces the change of components

$$T^a \to \tilde{T}^a = \Lambda^a{}_b(x)T^b \ , \text{ where} \tag{3.26}$$

$$\Lambda^a{}_b\Lambda_c{}^b = \delta^a_c \ . \tag{3.27}$$

In a matrix notation in which the components T^a form a column vector, also denoted by T, and the basis vectors $\mathbf{e}_a$ a row vector, denoted by $\mathbf{e}$,

$$\Lambda^a{}_b = (\Lambda)^a{}_b \ , \tag{3.28}$$

$$\Lambda_c{}^b = (\Lambda^{-1})^b{}_c \ , \tag{3.29}$$

$$T \to \Lambda T, \qquad \mathbf{e} \to \mathbf{e}\Lambda^{-1}, \qquad \Lambda \in GL(n,\mathbb{R}) \ . \tag{3.30}$$

A coordinate transformation induces a change of the natural basis with

$$\Lambda^\alpha{}_\beta = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} \quad \text{and} \tag{3.31}$$

$$\Lambda_\alpha{}^\beta = \frac{\partial x^\beta}{\partial \tilde{x}^\alpha} \ . \tag{3.32}$$

However, this is not the most general local frame rotation, since

$$\partial_{[\gamma}\Lambda^\alpha{}_{\beta]} = 0 \ . \tag{3.33}$$

3.5 Co-vectors

Given any finite dimensional vector space V, we define its *dual space* $V^\star$ as the space of linear maps $V \to \mathbb{R}$. We write for $\omega \in V^\star$ and for $U, W \in V$

$$\omega(U) = \langle\omega|U\rangle = \langle\omega, U\rangle \in \mathbb{R} \ , \tag{3.34}$$

such that

$$\langle\omega|U+W\rangle = \langle\omega|U\rangle + \langle\omega|W\rangle \quad \text{and} \tag{3.35}$$

$$\langle\omega|\lambda U\rangle = \lambda\,\langle\omega|U\rangle \ \forall\ \lambda \in \mathbb{R} \ . \tag{3.36}$$

This is a vector space of the same dimension as V, and the dual of $V^\star$ is the original vector space V, $\left(V^\star\right)^\star \cong V$.

Given a basis $\{\mathbf{e}_a\}$ for V we set $\omega_a = \omega(\mathbf{e}_a)$, and we denote the *dual* or *reciprocal* basis by $\{\mathbf{e}^a\}$ such that

$$\langle\mathbf{e}^a|\mathbf{e}_b\rangle = \delta^a_b \ . \tag{3.37}$$

Thus

$$\omega(U) = \omega_a U^a \ , \tag{3.38}$$

where U^a are the components of U in the basis $\{\mathbf{e}_a\}$.

3.5.1 Examples of dual vector spaces

In homely three dimensions, the dual space corresponds to momentum or wave vector space ($\mathbf{x}\cdot\mathbf{k}$ is a scalar). Given a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, not necessarily orthonormal, the reciprocal basis is given by $\mathbf{e}^3 = \frac{\mathbf{e}_1\times\mathbf{e}_2}{(\mathbf{e}_1\times\mathbf{e}_2)\cdot\mathbf{e}_3}$ etc. This construction is frequently used in crystallography when $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the basis vectors of a lattice. The reciprocal basis gives the reciprocal lattice in momentum space.

3.6 Change of basis

Under a change of basis,

$$\mathbf{e}^a \rightarrow \Lambda^a{}_b \mathbf{e}^b \ , \tag{3.39}$$

$$\omega_a \rightarrow \Lambda_a{}^c \omega_c = \omega_c (\Lambda^{-1})^c{}_a \ . \tag{3.40}$$

Geometrically, while vectors define directions through the origin of V, one may think of co-vectors as hyperplanes or or co-normals to planes through the origin. This is because for fixed ω, the set of points $U \in V$ such that $\omega(X) = 0$ is a hyperplane, and ω and $\lambda\omega$ define the same hyperplane. The duality is then the standard one of points and hyperplanes in elementary projective geometry.

Now at every point p in a manifold M one defines the *cotangent space* $T_p^\star(M)$ as the dual space of the tangent space $T_p(M)$. Then a *covector field* is one for which

$$\omega(fU) = f\omega(U) \ , \tag{3.41}$$

where now f is a function of x.

An example of a co-vector is a *differential* 1*-form.* We consider a function $f : M \rightarrow \mathbb{R}$ and define its *differential*, *exterior derivative* or *gradient df* by

$$\langle df, U \rangle = Uf \ , \quad \text{for all } U \in \mathfrak{X}(M) \ . \tag{3.42}$$

If we call $\Omega^1(M)$ the space of 1-forms on a manifold, and $\Omega^0(M)$ the real-valued functions on M, i.e. what, thought of as a commutative ring, we called $C^\infty(M)$ and $\mathfrak{F}(M)$ earlier, then we have a map $d : C^\infty(M) = \Omega^0(M) \rightarrow \Omega^1(M)$ that is Leibnizian:

$$d(fg) = dfg + gdf, \quad f, g \in \Omega^0(M) \ . \tag{3.43}$$

In a coordinate basis

$$df = \partial_\mu f(x) dx^\mu, \qquad U^\mu(x) \frac{\partial}{\partial x^\mu} \ , \tag{3.44}$$

and

$$\langle df, U \rangle = Uf = U^\mu(x) \partial_\mu f(x) \ , \tag{3.45}$$

$$df_\mu U^\mu = U^\mu \partial_\mu f \ . \tag{3.46}$$

By arguments similar to those given for vector fields, we deduce that dx^μ gives a basis for $T_p^\star(M)$ dual to the coordinate basis of vectors:

$$\left\langle dx^\mu, \frac{\partial}{\partial x^\nu} \right\rangle = \delta^\mu_\nu \ . \tag{3.47}$$

Thus if $\omega = \omega_\mu dx^\mu$, under a coordinate transformation

$$\omega = \omega_\mu \frac{\partial x^\mu}{\partial \tilde{x}^\nu} d\tilde{x}^\nu \ , \tag{3.48}$$

and hence

$$\tilde{\omega}_\nu = \omega_\mu \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \ , \tag{3.49}$$

which corresponds to the elementary definition of a covariant vector. Of course not all 1-forms ω are of the form df. A necessary condition is that

$$\partial_\mu \omega_\nu - \partial_\nu \omega_\mu = 0 \ . \tag{3.50}$$

We shall have more to say about sufficient conditions when we discuss the Poincaré Lemma.

The *geometrical interpretation* of *df* is as follows: Locally at least $f(x^\alpha) =$ const. $= a$ defines the a *level sets* of the function, i.e. the $(n-1)$-dimensional surface Σ_a on which the function takes the constant value a. Now along a curve $c(\lambda)$ intersecting Σ_a, we have

$$\frac{df}{d\lambda} = Tf = \langle df, T \rangle \ , \text{ where} \tag{3.51}$$

$$T = \frac{dx^\mu}{d\lambda} \frac{\partial}{\partial x^\mu} = T^\mu \frac{\partial}{\partial x^\mu} \tag{3.52}$$

is the tangent to the curve $c(\lambda)$. Now if $c(t)$ lies in Σ_a, $\left.\frac{df}{d\lambda}\right|_{\lambda=t} = 0$, and so the tangent vector T lies in $T(\Sigma)$, and thus

$$\langle df, T\rangle = 0 \ . \tag{3.53}$$

Thus $n_\alpha = \frac{\partial f}{\partial x^\alpha}$ should be thought of as the *co-normal* of the surface Σ_a. Note that in the absence of a natural map from co-vectors to vectors a surface has no natural unique normal vector. Such a map is provided by a *metric* on M, which we will introduce in a later section.

Just as we defined the tangent bundle $T(M)$ we can define the $2n$-dimensional *co-tangent bundle* $T^\star(M)$ as the space of a possible co-vectors at all possible points:

$$T^\star(M) = \bigcup_{p\in M} T^\star_p(M) \ . \tag{3.54}$$

This is also called the *phase space* or *momentum space* space of the manifold. It has as local coordinates (x^α, p_β), where p_β are the components of an arbitrary 1-form $p \equiv p_\beta dx^\beta$.

3.7 Tensor algebra

Having defined vectors and co-vectors we can go on to define the associated tensor product spaces. Thus a co-tensor Ω of rank q can be thought of as a multi-linear map from the q-fold Cartesian product $T_p(M) \times T_p(M) \ldots \times T_p(M)$ taking its values in $\mathbb{R}$. We write this as $\Omega(U, V, \ldots, W)$ or $\langle \Omega | U, V, \ldots, W\rangle$. For co-tensor *fields* the linearity is over *functions*, i.e.

$$\langle \Omega | fU, gV, \ldots, hW\rangle = fgh \, \langle \Omega | U, V, \ldots, W\rangle \quad \forall \ f, g, \ldots, h \in C^\infty(M) \ . \tag{3.55}$$

In a basis we have

$$\Omega = \Omega_{ab\ldots c} \ \mathbf{e}^a \otimes \mathbf{e}^b \otimes \ldots \otimes \mathbf{e}^c \ , \text{ with} \tag{3.56}$$

$$\Omega_{ab\ldots c} = \Omega\left(\mathbf{e}_a, \mathbf{e}_b, \ldots, \mathbf{e}_c\right) \ . \tag{3.57}$$

We can also use a coordinate basis in which the components of Ω are

$$\Omega_{\mu\nu\ldots\rho} = \Omega_{ab\ldots c} e^a_\mu e^b_\nu \ldots e^c_\rho \ . \tag{3.58}$$

Similarly, we can define contravariant and mixed tensors of type $\binom{p}{q}$.

The usual operations of contraction, symmetrization and anti-symmetrization can be introduced. Special interest attaches to the latter since we can also introduce a product called the *wedge product*. Since the construction is both useful and universal and works for any vector space we devote the next subsection to it.

3.8 Exterior or Grassmann algebra

Given any n-dimensional vector space V a p-form ω is a totally antisymmetric multilinear map $V \times V \times \ldots \times V \to \mathbb{R}$. We call $\Lambda^p(V)$ the *vector space of p-forms*, and $\dim \Lambda^p(V) = \frac{n!}{p!(n-p)!}$. By convention, $\Lambda^0(V) = \mathbb{R}$. We denote the direct sum of the vector spaces $\Lambda^p(V)$ by

$$\Lambda^*(V) = \bigoplus_{p=0}^{n} \Lambda^p(V) \ . \tag{3.59}$$

Thus $\dim \Lambda^*(V) = 2^n$.

We now turn $\Lambda^*(V)$ into an *algebra* over $\mathbb{R}$ by defining a product called the *wedge* or *exterior product*

$$\wedge : \Lambda^p(V) \times \Lambda^q(V) \to \Lambda^{p+q}(V) \tag{3.60}$$

which satisfies the following properties:

(i) $(f\alpha) \wedge \beta = f(\alpha \wedge \beta), \quad f \in \mathbb{R}$ (linearity),

(ii) $(\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma$ (distributivity),

(iii) $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) = \alpha \wedge \beta\gamma$ (associativity), and

(iv) $\alpha_p \wedge \beta_q = (-1)^{pq} \beta_q \wedge \alpha_p$ (graded commutativity).

In rule (iv), $\alpha_p \in \Lambda^p(V)$ and $\beta_q \in \Lambda^q(V)$. Note that $\Lambda^*(V)$ is what is called a *graded vector space*, that is it splits into a sum of vector spaces labelled by a degree p, and wedge multiplication respects the grading. A coarser grading, so called $\mathbb{Z}_2$-grading, is obtained by lumping together even and odd degree forms, $\Lambda^*(V) = \Lambda(V)^+ \oplus \Lambda^-(V)$. Exterior multiplication also respects this grading.

The explicit form of the wedge product is given by in components by

$$\boxed{(\alpha_p \wedge \beta_q)_{a_1 \dots a_p a_{p+1} \dots a_{p+q}} = \frac{(p+q)!}{p!q!} \, \alpha_{[a_1 \dots a_p} \beta_{a_{p+1} \dots a_{p+q}]} \,,} \tag{3.61}$$

where, if $\{\mathbf{e}_b\}$ and $\{\mathbf{e}^a\}$ are mutually dual bases for V and V^*, respectively, such that $\langle \mathbf{e}^a | \mathbf{e}_b \rangle = \delta^a_b$ and

$$\alpha\left(\mathbf{e}_{a_1}, \dots, \mathbf{e}_{a_p}\right) = \alpha_{a_1 \dots a_p} \,, \tag{3.62}$$

then

$$\alpha = \alpha_{a_1 \dots a_p} \mathbf{e}^{a_1} \otimes \mathbf{e}^{a_2} \otimes \dots \otimes \mathbf{e}^{a_p} = \frac{1}{p!} \alpha_{a_1 \dots a_p} \mathbf{e}^{a_1} \wedge \mathbf{e}^{a_2} \wedge \dots \wedge \mathbf{e}^{a_p} \,. \tag{3.63}$$

Thus

$$\mathbf{e}^1 \wedge \mathbf{e}^2 = \mathbf{e}^1 \otimes \mathbf{e}^2 - \mathbf{e}^2 \otimes \mathbf{e}^1 \,, \tag{3.64}$$

and for example, the Faraday tensor is given by

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = -E_i dt \wedge dx^i + \frac{1}{2} \epsilon_{ijk} B_k dx^i \wedge dx^j \,. \tag{3.65}$$

As an algebra, $\Lambda^*(V)$ is generated by the relations

$$\mathbf{e}^a \wedge \mathbf{e}^b + \mathbf{e}^b \wedge \mathbf{e}^a = 0 \,. \tag{3.66}$$

In this context, $\{\mathbf{e}^a\}$ are often called *Grassmann numbers*, and one calculates with them using the usual rules of algebra, taking into account the fact that they anti-commute. An arbitrary real valued function of these Grassmann variables is a polynomial of at most degree n and is just another name for an element of $\Lambda^*(V)$. The polynomials of degree p are just p-forms, i.e. elements of $\Lambda^p(V)$.

4 Inner products and pseudo-Riemannian manifolds

In this section we shall introduce the idea of an inner product or 'metric' on a manifold. This is a smooth assignment to the tangent space at each point of the manifold of an inner product or bilinear form which is linear over functions. To see what this means, recall that an *inner product* or *metric* on a finite dimensional vector space V is a real valued, non-degenerate, symmetric bilinear form g, i.e. a function $g : V \times V \to \mathbb{R}$ satisfying

(i) $g(U, W) = g(W, U)$ for all vectors $U, W \in V$ (symmetry),

(ii) $g(f_1 U, f_2 W) = f_1 f_2 g(U, W)$ for all $U, W \in V$, $f_1, f_2 \in \mathbb{R}$ (linearity),

(iii) $g(U, W) = 0 \; \forall \; W$ iff $U = 0$ (non-degeneracy).

In (ii), f_1, f_2 are arbitrary real numbers. To gain insight, we introduce a basis $\{\mathbf{e}_a\}$ in which the metric has components, which we may think of as a symmetric matrix

$$g_{ab} = g\left(\mathbf{e}_a, \mathbf{e}_b\right) = g_{ba} \,. \tag{4.67}$$

Condition (iii) implies that

$$\det g_{ab} \neq 0 \,. \tag{4.68}$$

We may therefore diagonalize g, and it will have s positive eigenvalues and t negative eigenvalues. By suitably rescaling we can find a *pseudo-orthonormal basis* in which g is diagonal with entries $+1$ s times and -1 t times. One says that the metric has *signature* (s, t), although sometimes $s - t$ is called the signature. Of course $\dim V = s + t$.

4.1 The musical isomorphism: Index raising and lowering

Given a bilinear form g (which need be neither symmetric nor non-degenerate) we get a natural map $\flat : V \to V^\star$ called *index lowering*, since if U is a vector, we can define a 1-form $U_\flat$ by

$$\langle U_\flat, W\rangle = g(U, W) \quad \forall\, W \in V \ . \tag{4.69}$$

In components,

$$\left(U_\flat\right)_b = U^a g_{ab} \ . \tag{4.70}$$

If g is non-degenerate, this map is invertible and we get an isomorphism of V and $V^\star$ called the *musical isomorphism*. The inverse map $\sharp : V^\star \to V$ is called *index raising*. Thus if

$$g^{-1}\left(\mathbf{e}^a, \mathbf{e}^b\right) = g^{ab} \ , \quad g_{ab} g^{bc} = \delta^c_b, \tag{4.71}$$

then given a 1-form ω we obtain a vector $\omega^\sharp$ by

$$\left\langle \sigma, \omega^\sharp \right\rangle = g^{-1}(\sigma, \omega) \quad \text{for all 1-forms } \sigma \ , \tag{4.72}$$

or

$$\left(\omega^\sharp\right)^a = g^{ab}\omega_b \ . \tag{4.73}$$

If a given metric is understood, it is usual to drop the $\sharp$'s and $\flat$'s and use the same kernel letter for vectors or tensors related by index raising and lowering. Indeed, if g is *positive* it is customary to restrict bases to be orthonormal in which

$$g_{ab} = \delta_{ab} \ , \tag{4.74}$$

and the distinction between contravariant and covariant tensors is dropped. All indices are usually written downstairs as in elementary Cartesian tensors and vectors in Euclidean 3-space $\mathbb{E}^3$. If g is *indefinite*, one typically, but not always, adopts a pseudo-orthonormal basis for which

$$g_{ab} = \eta_{ab} \ , \tag{4.75}$$

with $\eta_{ab} = \mathrm{diag}(+1, +1, \ldots, +1, -1, -1, \ldots, -1)$. However, index position must still be retained since factors of ± 1 arise on raising and lowering.

In four-dimensional general relativity it is sometimes convenient to adopt a null tetrad $(l, n, m, \bar{m})$ for which

$$g_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \ , \tag{4.76}$$

with $m = \frac{1}{\sqrt{2}}(a + ib)$, with the real dyad a, b unit and orthogonal to one another and to the real null vectors l and n.

4.1.1 Moduli space of metrics

One may clearly endow a given vector space V with many metrics. It suffices to give a dual basis $\{\mathbf{e}^a\}$ and deem it to be pseudo-orthonormal. Of course, $SO(s,t)$-rotations of our basis that preserve η_{ab} will give the same metric. If we fix an initial basis, any other basis may be specified by giving the element of $GL(n, \mathbb{R})$ needed to pass to the new basis, i.e. by the matrix with elements

$$e = \left(e^a_\mu\right) \ . \tag{4.77}$$

The metric is thus

$$g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu \ , \quad g = e^t \eta e \ . \tag{4.78}$$

Now e and Se with $S \in SO(s,t)$ give the same metric, so the set of different metrics is the coset space $SO(s,t)\backslash GL(n, \mathbb{R})$.

In fact, the space of metrics itself carries a one-parameter family of $GL(n, \mathbb{R})$-metrics given by

$$ds^2 = \mathrm{Tr}\left(e^{-1} de e^{-1} de\right) + \lambda \left(\mathrm{Tr}\left(e^{-1} de\right)\right)^2 \ . \tag{4.79}$$

Metrics on spaces of metrics of this kind arise frequently in general relativity, particularly in connection with dimensional reduction. Another example arises in the *Hamiltonian* or $(3+1)$-formulation of general relativity, in which the kinetic term may be considered a metric on the space of 3-metrics, called in this context the *DeWitt metric* (for which $\lambda = -\frac{1}{2}$).

4.1.2 Lattices

Lattices arise in many places in physics, including crystallogrophy, super-gravity and string theory and dimensional reduction, and the foregoing theory is applicable to all of them. Although the underlying manifolds are flat, and hence locally trivial, there are still a number of subtleties.

To begin with, we must distingush between a lattice L without metric and one with metric. The latter we shall call a *geometric lattice*. Group-theoretically an n-dimensional lattice L, i.e one of rank n, is a free abelian group with n generators $\{\mathbf{e}_a\}$. If we think of $\{\mathbf{e}_a\}$ as a basis for an affine space modelled on a vector space V whose coordinates are x^μ, then the lattice translations consist of all integer combinations of the basis

$$n^a \mathbf{e}_a \ , \quad n^a \in \mathbb{Z}^n \ . \tag{4.80}$$

One gets a torus T^n by identifying points related by a lattice translation, so that $T^n = \mathbb{R}^n/L$. Thus if we introduce coordinates for V such that

$$\mathbf{e}_a = e^\mu_a \frac{\partial}{\partial x^\mu} \ , \tag{4.81}$$

then x^μ are periodic coordinates with unit period. Clearly, *as a manifold* our torus is just the product manifold $\left(S^1\right)^n$ with some privileged coordinates. As yet, it carries no metric.

The *dual* or *reciprocal lattice* $L^\star$ lives in $V^\star$ and is generated by the dual basis $\{\mathbf{e}^a\}$. Thus if p_μ are coordinates for $V^\star = T^\star(V)$, then

$$\mathbf{e}^a = e^a_\mu \frac{\partial}{\partial p_\mu} \ , \tag{4.82}$$

and the coordinates p_μ must be identified with period unity.

Note that two bases related by an matrix with integer entries whose inverse has integer entries will generate the same lattice. Such a matrix must have unit determinant, and so up to ± 1 it is given by an element of $SL(n,\mathbb{Z})$.

Now suppose that the lattice L is equipped with a metric g, with components g_{ab} in our basis. Thus we have a flat metric on T^n invariant under an $\left(S^1\right)^n$ isometry group. In local coordinates we have

$$ds^2 = g_{ab}\, e^a_\mu dx^\mu \, e^b_\nu dx^\nu \ . \tag{4.83}$$

In the same local coordinates, the dual lattice $L^\star$ carries the metric

$$ds^2 = g^{ab} e^\mu_a dp_\mu \, e^\nu_b dp_\nu \ . \tag{4.84}$$

In general these two geometric lattices or tori are not isometric. But now we have a way of mapping $L^\star$ into V using the map $\sharp$. The question is then whether $(L^\star)^\sharp$ lies on or coincides with L. The basis for $(L^\star)^\sharp$ is

$$\left\{ g_{ab}\, \mathbf{e}^b \right\} \ , \tag{4.85}$$

and thus $(L^\star)^\sharp$ will lie on L if the matrix g_{ab} has integer entries. Such a lattice is called *integral*. The lattice is called *even* if all of its entries are even integers. The lattice $(L^\star)^\sharp$ will coincide with L if the matrix g_{ab} is invertible over the integers, in which case it is called *self-dual*. As noted above this means that the matrix g_{ab} must have unit determinant, and such lattices are called *unimodular*. Anomaly cancellation in string theory requires that the string move in even self-dual lattices.

Finally, we note that the moduli space of geometric lattices of flat metrics on T^n a double co-set $SO(s,t)\backslash GL(n,\mathbb{R})/SL(n,\mathbb{Z})$. If we fix the volume, then this becomes of flat metrics on T^n a double co-set $SO(s,t)\backslash SL(n,\mathbb{R})/SL(n,\mathbb{Z})$. If $n = 2$ and $t = 0$, such lattices are parametrized by a *modular parameter* τ lying in the upper half plane and such that τ and $\frac{a\tau+b}{c\tau+d}$ are identified, where the integers a, b, c, d satisfy $ad - bc = 1$. The square lattice corresponds to $\tau = i$ and the triangular lattice to $\tau = e^{\frac{i\pi}{3}}$, which are singular ('orbifold') points of the moduli space.

The natural metric on these lattices is

$$ds^2 = \frac{d\tau d\bar{\tau}}{\left(\Im\tau\right)^2} \ . \tag{4.86}$$

Further very important examples of the foregoing theory, which arise in particle physics and string and supergravity theory, are the *maximal tori* of compact Lie groups and their relation to *root lattices*. Geometrically, one may think of the maximal torus as an r-dimensional submanifold of a Lie group G of rank r. The torus inherits a flat metric from the ambient bi-invariant *Killing metric* on G. In an appropriate basis, the metric is given by the integer valued *Cartan matrix*.

The dual or reciprocal lattice is the *weight lattice*, since it gives the *characters* or *charges* of the representations of the T^r-group generated by a maximal set of commuting operators.

4.2 Local pseudo-orthonormal frames

Given a metric on a manifold, its components in a local coordinate basis are are

$$g_{\mu\nu}(x) = g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) \ . \tag{4.87}$$

We can always introduce locally a set of frames $\{e_a^\mu(x)\}$. and in that basis the components of the metric are in general position dependent:

$$g_{\mu\nu} e_a^\mu e_b^\nu = g_{ab}(x) \ . \tag{4.88}$$

Inverting gives formulae such as

$$g^{\mu\nu} = e_a^\mu e_b^\nu g^{ab}(x) \ , \text{etc.} \tag{4.89}$$

It is often convenient, but not compulsory, to adopt a local pseudo-orthonormal frame in which

$$g_{ab} = \eta_{ab} \ . \tag{4.90}$$

There is a partial converse. If one has a global frame field $\{\mathbf{e}_a\}$ with dual basis $\{\mathbf{e}^a\}$ and a constant matrix g_{ab}, one may define a global metric by

$$g = g_{ab}\mathbf{e}^a \otimes \mathbf{e}^b \ . \tag{4.91}$$

This construction arises in the case of Lie groups, which, as will be explained later, admit a globally defined field of left-invariant vectors L_a and right-invariant vectors R_a and allow us to construct left- or right-invariant metrics on Lie groups.

4.3 Symplectics and symplectic manifolds

The existence of a musical isomorphism does not depend on any symmetry property of the bilinear form. It particular it holds for a non-degenerate skew-symmetric form, i.e. a 2-form ω such that

$$\omega(U, W) = -\omega(W, U) \ . \tag{4.92}$$

A vector space equipped with such a form is called a *symplectic vector space*. An important set of examaples are provided by various spin spaces in various dimensions. They are often equipped with such symplectic forms, which arise physically as charge conjugation matrices.

Because

$$\det \omega = \det\left(\omega^t\right) = (-1)^n \det \omega \ , \tag{4.93}$$

with $n = \dim V$, the vector space must be even-dimensional. One may then show that there exists a basis, unique up to an $Sp(2, \mathbb{R})$-transformation, in which

$$\omega = \mathbf{e}^1 \wedge \mathbf{e}^2 + \mathbf{e}^3 \wedge \mathbf{e}^4 + \dots \ . \tag{4.94}$$

A *symplectic manifold* is a smooth manifold M equipped with a non-degenerate 2-form $\omega \in \Omega^2(M)$ which is closed, i.e.

$$d\omega = 0 \ . \tag{4.95}$$

The basic example of a symplectic manifold is the co-tangent bundle $T^\star(M)$ of any manifold M for which, in local coordinates, the *Liouville symplectic form* is

$$\omega = dp_\mu \wedge dx^\mu \in \Omega^2(T^\star M) \ . \tag{4.96}$$

Other examples arise from Kähler and hyper-Kähler manifolds.

4.4 Volume forms

A *volume form* η on a vector space V of dimension n is just another name for an n-form. All such volume forms are proportional (note that $\dim \Lambda^n V = 1$). Thus in any basis

$$\eta = \mu \mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \ldots \wedge \mathbf{e}^n \ . \tag{4.97}$$

Usually we demand that the volume form be everwhere non-vanishing, in which case it induces an orientention on the manifold M. If the manifold is not orientable, one may still introduce a volume element using the idea of a density. We shall show how later.

4.4.1 Example: Symplectic and Kähler manifolds

A symplectic manifold, and hence any Kähler manifold, of dimension $n = 2m$ has a natural volume form and a natural orienatation given by

$$\eta = -1^m \frac{\omega^m}{m!} \ , \tag{4.98}$$

where ω^m means the m-th exterior power, and ω is the symplectic form.

4.5 Change of basis

In another basis such that $\mathbf{e}^a = \Lambda^a{}_b \, \tilde{\mathbf{e}}^b$ one has

$$\begin{aligned} \eta &= \mu \det \Lambda \ \tilde{\mathbf{e}}^1 \wedge \tilde{\mathbf{e}}^2 \wedge \ldots \wedge \tilde{\mathbf{e}}^n & (4.99) \\ &= \tilde{\mu} \ \tilde{\mathbf{e}}^1 \wedge \tilde{\mathbf{e}}^2 \wedge \ldots \wedge \tilde{\mathbf{e}}^n \ , & (4.100) \end{aligned}$$

where we have used the fact that

$$\tilde{\mathbf{e}}^a \wedge \tilde{\mathbf{e}}^b \wedge \ldots \wedge \tilde{\mathbf{e}}^c = \epsilon^{ab\ldots c} \, \tilde{\mathbf{e}}^1 \wedge \tilde{\mathbf{e}}^2 \wedge \ldots \wedge \tilde{\mathbf{e}}^n \ , \tag{4.101}$$

where $\epsilon^{ab\ldots c}$ is $+1$ if $(a, b, \ldots, c)$ is an even permutation of $(1, 2, \ldots, n)$, -1 if it is an odd permutation and 0 otherwise. Thus

$$\mu \rightarrow \tilde{\mu} = \mu \det \Lambda \ . \tag{4.102}$$

One says that μ transforms as a *density of weight one.* If the factor had been $(\det \Lambda)^w$, we would have had a density of weight w. One may also introduce *tensor-densities of type* $\binom{p}{q}$ *and weight* w.

Given a metric g on V, we can pick a pseudo-orthonormal basis and set $\mu = 1$ in the basis to define the volume form. Thus, in such a basis,

$$\eta = e^1_\mu dx^\mu \wedge e^2_\nu dx^\nu \wedge \ldots \wedge e^n_\rho dx^\rho \ . \tag{4.103}$$

But

$$dx^\mu \wedge dx^\nu \wedge \ldots \wedge dx^\rho = \epsilon^{\mu\nu\ldots\rho} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \ . \tag{4.104}$$

Thus

$$\eta = \det \left(e^a_\mu \right) dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \ . \tag{4.105}$$

But

$$\det \left(e^a_\mu \right) = \sqrt{|\det \left(g_{\mu\nu} \right)|} \ , \tag{4.106}$$

so that

$$\eta = \sqrt{|\det \left(g_{\mu\nu} \right)|} \ dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \ . \tag{4.107}$$

As we have defined it, η is a covariant tensor with components in a natural basis $\eta_{\mu\nu\ldots\sigma} = (-1)^t \sqrt{|\det \left(g_{\mu\nu} \right)|} \, \epsilon_{\mu\nu\ldots\sigma}$, where $\epsilon_{\mu\nu\ldots\sigma}$ takes the values $\pm(-1)^t$ or zero, depending on whether or not $(\mu, \nu, \ldots, \sigma)$ is an even (upper sign) or odd (lower sign) permutation of $(1, 2, \ldots, n)$. Because $\sqrt{\det \left(g_{\mu\nu} \right)}$ is a scalar density of weight one, $\epsilon_{\mu\nu\ldots\sigma}$ is an n-th rank totally anti-symmetric covariant tensor density of weight -1. Then $\epsilon^{\mu\nu\ldots\sigma}$ is an n-th rank totally anti-symmetric contravariant tensor density of weight $+1$. Further

$$\epsilon_{\mu\nu\ldots\sigma} = g_{\mu\alpha} g_{\nu\beta} \ldots g_{\sigma\gamma} \epsilon^{\alpha\beta\ldots\gamma} \ , \tag{4.108}$$

and

$$\epsilon^{\alpha\beta\ldots\gamma} \epsilon_{\mu\nu\ldots\sigma} = (-1)^t \left(\delta^\alpha_\mu \delta^\beta_\nu \ldots \delta^\gamma_\sigma + \ldots \right) \ . \tag{4.109}$$

5 Hodge duality

Suppose the vector space V is equipped with an inner product or metric (not necessarily positive definite) g and that $\{\mathbf{e}_a\}$ is an ordered (pseudo-)orthonormal basis with $g(\mathbf{e}_a, \mathbf{e}_b) = 0, \pm 1$, with t negative signs and $n - t$ positive signs. Thus if $t = 1$ or $t = n - 1$ we have a Lorentzian metric. The *volume form* is given by

$$\eta = \mathbf{e}^1 \wedge \mathbf{e}^2 \ldots \wedge \mathbf{e}^n = \frac{1}{n!} \eta_{a_1 a_2 \ldots a_n} \mathbf{e}^{a_1} \wedge \mathbf{e}^{a_2} \wedge \ldots \wedge \mathbf{e}^{a_n} \ , \tag{5.110}$$

where $\eta_{a_1 a_2 \ldots a_n} = 0, \pm 1$, depending on whether $(a_1, a_2, \ldots, a_n)$ is an even or odd permutation of $(1, 2, \ldots, n)$. Choosing an ordering for the basis is equivalent to choosing an orientation for the vector space V. The metric g induces an inner product on each $\Lambda^p(V)$ by

$$(\alpha, \beta) = (\beta, \alpha) = \frac{1}{p!}\, \alpha^{a_1 a_2 \ldots a_p} \beta_{a_1 a_2 \ldots a_p} \ , \quad \alpha, \beta \in \Lambda^p(V) \tag{5.111}$$

and may be extended to $\Lambda^*(V)$ such that the summands $\Lambda^p(V)$ are orthogonal, where the indices on $\alpha^{a_1 a_2 \ldots a_p}$ are raised using the inverse metric g^{ab}. One may now introduce a linear map

$$* : \Lambda^p(V) \to \Lambda^{n-p}(V) \tag{5.112}$$

by requiring

$$\boxed{\lambda \wedge \mu = (*\lambda, \mu)\eta \ , \quad \forall\ \mu \in \Lambda^{n-p}(V) \ .} \tag{5.113}$$

In components we have

$$\frac{n!}{p!(n-p)!} \lambda_{[a_1 a_2 \ldots a_p} \mu_{a_{p+1} \ldots a_n]} = \frac{*\lambda^{b_1 b_2 \ldots b_q}}{q!} \mu_{b_1 b_2 \ldots b_q} \eta_{a_1 a_2 \ldots a_n} \ . \tag{5.114}$$

But

$$\eta^{a_1 a_2 \ldots a_n} \eta_{a_1 a_2 \ldots a_n} = (-1)^t n! \ , \tag{5.115}$$

thus

$$\boxed{*\lambda^{b_1 b_2 \ldots b_q} = \frac{(-1)^t}{p!} \eta^{a_1 a_2 \ldots a_p b_1 b_2 \ldots b_q} \lambda_{a_1 a_2 \ldots a_p} \ .} \tag{5.116}$$

Now

$$\eta_{a_1 a_2 \ldots a_n} \eta^{b_1 b_2 \ldots b_n} = (-1)^t \sum_\sigma (-1)^{\pi(\sigma)} \delta^{\sigma(b_1)}_{a_1} \delta^{\sigma(b_2)}_{a_2} \cdots \delta^{\sigma(b_n)}_{a_n} \ , \tag{5.117}$$

where the summation is over all permutations σ of the indices $b_1, b_2, \ldots, b_n$ and $\pi(\sigma)$ is 0 or 1 for even and odd permutations, respectively. We now deduce by contraction and $pq = p(n-p)$ index interchanges that

$$\boxed{**\lambda = (-1)^t (-1)^{p(n-p)} \lambda \ .} \tag{5.118}$$

If one now sets $\mu = \alpha$ and $\lambda = *\beta$ and uses the graded commutativity property, one finds that

$$\boxed{\alpha \wedge \beta = (-1)^t (\alpha, \beta) \eta \ .} \tag{5.119}$$

5.0.1 Example: The cross product in $\mathbb{R}^3$

We set $n = 3$ and $t = 0$ and find $** = 1$ for all forms. Hodge duality takes 2-forms to vectors and *vice versa*, thus we can define the *cross product* by

$$\mathbf{a} \times \mathbf{b} = *(\mathbf{a} \wedge \mathbf{b}) \ , \tag{5.120}$$

which endows $\mathbb{R}^3$ with a *non-associative* algebra structure. Because

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} . \mathbf{a})\mathbf{b} - (\mathbf{b} . \mathbf{c})\mathbf{a} \ , \tag{5.121}$$

the Jacobi identity holds:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} + (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} = 0 \ , \tag{5.122}$$

and so we have a Lie algebra with $[\mathbf{a}, \mathbf{b}] = \mathbf{a} \times \mathbf{b}$. This Lie algebra is of course isomorphic to $\mathfrak{so}(3) \cong \mathfrak{su}(2)$.

We also have

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = *\, [\mathbf{a}, \mathbf{b}, \mathbf{c}] \ , \tag{5.123}$$

where $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = (\mathbf{a} \times \mathbf{b}) . \mathbf{c}$ is the *triple scalar product.*

5.0.2 Example: Quaternions and octonions

One may identify $\mathbb{R}^3$ with the purely imaginary part $\Im\mathbb{H}$ of the *associative* but *non-commutative* division algebra of Hamilton's *quaternions* $\mathbb{H}$, and ϵ_{ijk}, or equivalently the volume form, are the structure constants. In the usual basis

$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji} \ , \quad \mathbf{i}^2 = -1 \ , \quad \text{etc.}, \tag{5.124}$$

where the quaternion product is indicated by juxtaposition. A matrix representation may be obtained using Pauli-matrices

$$\mathbf{i} = \frac{1}{\sqrt{-1}}\,\sigma_1 \ , \quad \text{etc.} \tag{5.125}$$

5.0.3 Example: The Faraday 2-form

This has $n = 4$, $p = 2$ and $t = 1$. In Minkowski spacetime $\mathbb{E}^{3,1}$ define $\eta_{ijk0} = \epsilon_{ijk}$, which implies that $\eta = dx \wedge dy \wedge dz \wedge dt$, and

$$(dy \wedge dz, dy \wedge dz) = +1 = -(dt \wedge dx, dt \wedge dx) \ , \tag{5.126}$$

and so

$$*\,(dt \wedge dx) = -dy \wedge dz \ , \quad *\,(dy \wedge dz) = dt \wedge dx \ . \tag{5.127}$$

Thus if

$$F = -E_x dt \wedge dx + B_x dy \wedge dx + \cdots \ , \quad \text{then} \tag{5.128}$$

$$*F = B_x dt \wedge dx + E_x dy \wedge dz + \cdots \ , \tag{5.129}$$

so

$$*\mathbf{E} = -\mathbf{B} \ , \quad *\mathbf{B} = \mathbf{E} \ , \tag{5.130}$$

that is

$$**F = -F \ . \tag{5.131}$$

Now

$$(F, F) = \frac{1}{2}\left(2F_{0i}F^{0i} + F_{ij}F^{ij}\right) = \mathbf{B}^2 - \mathbf{E}^2 \ , \quad \text{and} \tag{5.132}$$

$$F \wedge *F = \left({E_x}^2 - {B_x}^2\right) dx \wedge dy \wedge dz \wedge dt + \cdots = \left(\mathbf{E}^2 - \mathbf{B}^2\right)\eta \ , \tag{5.133}$$

that is

$$F \wedge *F = -(F, F)\eta \ . \tag{5.134}$$

Later we will see that the Maxwell action can be written as

$$\frac{1}{2}\int F \wedge *F = \frac{1}{2}\int \left(\mathbf{E}^2 - \mathbf{B}^2\right) d^4x \ . \tag{5.135}$$

5.1 Geometrical interpretation

Every one-form ω defines an $(n-1)$-plane Π_ω through the origin in V given by those vectors $v \in V$ such that

$$\omega(v) = \omega_a v^a = 0 \ . \tag{5.136}$$

Evidently ω and $f\omega$, $f \in \mathbb{R} \setminus \{0\}$ correspond to the same $(n-1)$-plane. One refers to ω as a *co-normal* of the $(n-1)$-plane Π_ω. Similarly

$$\omega_1 \wedge \omega_2 = \frac{1}{2}\left(\omega_1 \otimes \omega_2 - \omega_2 \otimes \omega_1\right) \tag{5.137}$$

corresponds to the $(n-2)$-dimensional intersection $\Pi_{\omega_1} \cap \Pi_{\omega_2}$ of the two $(n-1)$-planes Π_{ω_1} and Π_{ω_2} since

$$\omega_1 \wedge \omega_2(v, \ ) = 0 \tag{5.138}$$

implies $v \in \Pi_{\omega_1} \cap \Pi_{\omega_2}$. A two-form which is the wedge product of two one-forms is called *simple*, and all such simple two-forms define an $(n-2)$-plane. Similarly a simple p-form defines an $(n-p)$-plane. In a basis the normal p-form of an $(n-p)$-plane Π spanned by $\{\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^p\}$ may be taken to be $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_p$.

If V is equipped with a positive definite metric (i.e. $t = 0$) we may consider an orthonormal basis. Then if ω is the normal p-form of Π, the Hodge dual $*\omega$ is the normal $(n-p)$-form of the orthogonal $(n-p)$-plane $\Pi^\perp$. If the metric is indefinite, then care must be taken with p-planes which contain null directions.

We remark here that the space of all p-planes in V is a compact manifold, called a *Grassmannian* $\text{Grass}(V,p)$, and may be identified with a homogeneous space of $O(n)$, the coset space $\frac{O(n)}{O(p)\times O(n-p)}$. The stabilizer group consists of rotations in the p-plane and rotations on the orthogonal $(n-p)$-plane.

Hodge duality interchanges a p-plane with its orthogonal $(n-p)$-plane, and hence $\text{Grass}(V,n-p) \cong \text{Grass}(V,n-p)$. Of course $\text{Grass}(\mathbb{R}^n,1)$ is the projective space $\mathbb{RP}^n$.

5.1.1 Example: Grassmannians and spaces of geodesics

As an example, the geodesics or *great circles* on $S^{n-1} \subset \mathbb{E}^n$ correspond to the intersection of a 2-plane through the origin of $\mathbb{E}^n$. Thus the set of great circles in S^{n-1} is the Grassmannian $\text{Grass}(\mathbb{E}^n,2) \cong \frac{O(n)}{O(n-2)\times O(2)}$. The other Grassmannians correspond to totally geodesic S^{p-1}'s in S^{n-1}. They are totally geodesic because they are fixed point sets of an involution corresponding to total inversion in the orthogonal $(n-p)$-plane.

These results have Lorentzian analogues. The set of timelike geodesics in$(n-1)$-dimensional de-Sitter spacetime dS_{n-1} or anti-de-Sitter spacetime AdS_{n-1} correspond to timelike or totally timelike, respectively, 2-planes through the origin of $\mathbb{E}^{n-1,1}$ or $\mathbb{E}^{n-2,2}$, respectively. These are, respectively, $\frac{O(n-1,1)}{O(1,1)\times O(n-2)}$ and $\frac{O(n-2,2)}{O(n-2)\times O(2)}$.

5.2 Interior multiplication

For all $v \in V$ we define a map

$$i_v : \Lambda^p(V) \to \Lambda^{p-1}(V) \tag{5.139}$$

such that

(i) $i_v f = 0$ for all $f \in \Lambda^0(V)$,

(ii) $i_v(\alpha_p \wedge \beta_q) = (i_v\alpha_p) \wedge \beta_q + (-1)^p \alpha_p \wedge (i_v\beta_q)$ for all $\alpha_p \in \Lambda^p(V)$ and $\beta_q \in \Lambda^q(V)$, and

(iii) $i_v\omega = \langle\omega|v\rangle$ for all $\omega \in \Lambda^1(V)$.

In components, we contract v^a on the first index on the form

$$(i_v\lambda)_{b\ldots c} = v^a\lambda_{ab\ldots c} \ . \tag{5.140}$$

We have

$$i_v^2 = 0 \ , \quad i_{v_1}i_{v_2} + i_{v_2}i_{v_1} = 0 \ . \tag{5.141}$$

5.2.1 Example: Analogy with fermion annihilation and creation operators

We think of

- $\Lambda^0(V)$ as the ground state,
- $\Lambda^1(V)$ as a one-fermion state,
- $\Lambda^2(V)$ as a two-fermion state, etc.

Thus $\omega\wedge : \Lambda^p(V) \to \Lambda^{p+1}(V)$ creates a fermion and $i_v : \Lambda^p(V) \to \Lambda^{p-1}(V)$ destroys a fermion.

If V is equipped with a positive definite metric g and we use an orthonomal basis $\{\mathbf{e}_a\}$ and dual basis which may be identified as usual, we can define annihilation and creation operators:

$$a_i = i_{\mathbf{e}_i} \ , \quad a_i^\dagger = \mathbf{e}_i \wedge \ . \tag{5.142}$$

We leave the reader to check that acting on Λ^* one obtains the standard anti-commutation relations for fermions.

$$a_ia_j + a_ja_i = 0 = a_i^\dagger a_j^\dagger + a_j^\dagger a_i^\dagger \ , \tag{5.143}$$

$$a_ia_j^\dagger + a_j^\dagger a_i = \delta_{ij} \ . \tag{5.144}$$

Moreover $a_i^\dagger$ is the adjoint of a_i with respect to the inner product $(\,\cdot\,,\cdot)$ on $\Lambda^*(V)$ induced from g. Thus if $\omega = v_\flat$, then

$$(\omega \wedge \beta_{p-1}, \alpha_p) = (\beta_{p-1}, i_v\alpha_p) \ . \tag{5.145}$$

5.3 Exterior differentation

Suppose that M is an orientable manifold. At each point $x \in M$ we take $V = T_x(M)$, the tangent space at x, and we consider fields of p-forms. Note that a 0-form is a C^∞-function $f\colon M \to \mathbb{R}$, i.e. $f \in \mathfrak{F}(M)$.

We define a gradient operator d by

$$Xf = \langle df|X\rangle \quad \forall\, X \in T_x(M)\ . \tag{5.146}$$

Since locally

$$Xf = X^\mu \frac{\partial f}{\partial x^\mu} \quad \text{and} \quad \langle df|X\rangle = X^\mu df_\mu\ , \tag{5.147}$$

it follows that

$$df_\mu = \frac{\partial f}{\partial x^\mu}\ . \tag{5.148}$$

The Leibniz rule holds

$$d(fg) = df g + f dg \quad \forall\, f, g \in C^\infty(M)\ . \tag{5.149}$$

We extend d to act on p-forms to give a map $d : \Omega^p(M) \to \Omega^{p+1}(M)$ with the properties

(i) $d(\alpha + \beta) = d\alpha + d\beta$, (linearity)

(ii) $d(\alpha_p \wedge \beta_q) = (d\alpha_p) \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q$, (graded Leibnizian)

(iii) $d^2\alpha = 0$,

(iv) df is the usual gradient of a function, and

(v) $d\alpha$ is 'local' in that it depends only on the first partial derivatives of the components of α.

In components one has

$$\boxed{(d\omega^p)_{\mu_1\mu_2\ldots\mu_{p+1}} = \frac{(p+1)!}{p!}\partial_{[\mu_1}\omega^p{}_{\mu_2\ldots\mu_{p+1}]}\ .} \tag{5.150}$$

That is,

$$\boxed{d\omega = \partial \wedge \omega\ .} \tag{5.151}$$

One way of establishing the tensorial nature of d uses some concepts to be introduced formally later. If ∇ is a free connection with components $\Gamma_{\mu}{}^{\nu}{}_{\sigma} = \Gamma_{\sigma}{}^{\nu}{}_{\mu}$, then

$$(d\omega^p)_{\mu_1\mu_2\ldots\mu_{p+1}} = \frac{(p+1)!}{p!}\nabla_{[\mu_1}\omega^p{}_{\mu_2\ldots\mu_{p+1}]}\ , \tag{5.152}$$

or

$$d\omega = \nabla \wedge \omega\ . \tag{5.153}$$

5.3.1 Maxwell's equations and generalizations

In Minkowski spacetime, one sets

$$A = -\phi dt + A_i dx^i\ , \tag{5.154}$$

and

$$F = dA = -(\partial_i\phi + \partial_t A_i) dx^i \wedge dt + \partial_k A_i dx^k \wedge dx^i\ . \tag{5.155}$$

Thus

$$\begin{aligned} F_{0i} = -E_i = \partial_t A_i + \partial_i \phi\ , &\qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi\ , \\ F_{ij} = \partial_i A_j - \partial_j A_i\ , &\qquad \mathbf{B} = \nabla \times \mathbf{A}\ . \end{aligned} \tag{5.156}$$

These equations may immediately be carried over to any curved spacetime, whether it has a metric or not. (However, one obviously cannot introduce the curl notation.)

The *Bianchi idendity* or *Faraday's law* reads

$$F = dA \Rightarrow \boxed{dF = 0} \Leftrightarrow F_{[\mu\nu,\sigma]} = 0\ . \tag{5.157}$$

In flat spacetime these read

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \ , \quad \nabla . \mathbf{B} = 0 \ . \tag{5.158}$$

The *Field equations* or *Ampère's and Gauss' law* in the absence of sources are

$$\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} \ , \quad \nabla . \mathbf{E} = 0 \ . \tag{5.159}$$

They actually follow from the electric-magnetic duality $(\mathbf{E}, \mathbf{B}) \to (-\mathbf{B}, \mathbf{E})$, i.e. $F \to *F$ and suggest using in a general spacetime

$$\boxed{d *F = 0 \ .} \tag{5.160}$$

This suggests the obvious generalization for a p-form F in arbitrary dimensions:

(i) $F = dA \ \Rightarrow \ dF = 0$, and

(ii) $d *F = 0$.

The case $p = 2$ is Maxwell theory generalized to all dimensions. The case $p = 1$ is scalar field theory, the case $p = 3$ arises in string theory and the case $p = 4$ in eleven dimensional supergravity (but the field equation becomes non-linear, $d *F \propto F \wedge F$).

In $n = 4$ the cases $p = 1$ and $p = 3$ are equivalent under Hodge duality. They correspond to the axion.

5.3.2 Self-duality

The Bianchi identity $dF = 0$ and field equation $d *F$ follow from one another if the self-duality condition $F = \pm *F$ holds. This is possible if and only if

$$n = 2p \ , \text{ and } \ * * = (-1)^t (-1)^{p(n-p)} = +1 \ . \tag{5.161}$$

Thus if t is odd, e.g. 1, this requires $n = 4k + 2$, $k = 0, 1, 2, \dots$.

The case $n = 2$ is essentially string theory.

$$\begin{aligned} \partial_\mu A = \pm \epsilon_\mu^{\ \lambda} \partial_\lambda A \quad &\Leftrightarrow \quad \partial_t = \pm \partial_z A \\ &\Leftrightarrow \quad A = A(t \pm z) \ . \end{aligned} \tag{5.162}$$

Thus a self-dual solution is a *right-mover* or a *left-mover*. A self-dual or anti-self-dual vector in dimension $(1 + 1)$ is a left-pointing or right-pointing null vector. The fact that a string world-sheet admits a null congruence of, say, right-pointing vectors means that classically, if the world-sheet remains non-singular, the string cannot change topology, i.e. split and join in a continuous fashion.

Self-duality is possible in 6 and 10 dimensions, and both cases are of interest in supergravity and string theory. The case $n = 10$, $p = 5$ arises in particular in Type IIB string theory.

Finally the case $n = 4$, $t = 0$ allows self-dual 2-forms and has proved to be very important in Yang-Mills theory and Donaldson's approach to 4-manifolds.

5.3.3 Action principles

Here we shall use the formalism of *densities*. Later we will use differential forms. We will ignore surface terms in the varations. An appropriate *action functional* is

$$S = -\frac{1}{2p!} \int F^{\mu_1 \mu_2 \dots \mu_p} F_{\mu_1 \mu_2 \dots \mu_p} \sqrt{-g} d^n x \ . \tag{5.163}$$

The variation is

$$\begin{aligned} \delta S &= -\frac{1}{(p-1)!} \int F^{\mu_1 \mu_2 \dots \mu_p} \nabla_{\mu_1} (\delta A)_{\mu_2 \dots \mu_p} \sqrt{-g} d^n x \\ &= \frac{1}{(p-1)!} \int \nabla_{\mu_1} F^{\mu_1 \mu_2 \dots \mu_p} (\delta A)_{\mu_2 \dots \mu_p} \sqrt{-g} d^n x \ . \end{aligned} \tag{5.164}$$

Thus

$$\delta S = 0 \quad \Rightarrow \quad \boxed{\nabla_{\mu_1} F^{\mu_1 \mu_2 \dots \mu_p} = 0} \ . \tag{5.165}$$

In order to see the equivalence with the equation $d *F = 0$, we introduce a covariantly constant tensor density field $\eta_{\mu_1\mu_2\ldots\mu_n} = \eta_{[\mu_1\mu_2\ldots\mu_n]}$ such that

$$\eta_{\mu_1\mu_2\ldots\mu_n}\eta^{\nu_1\nu_2\ldots\nu_n} = (-1)^t \sum_\sigma (-1)^{\pi(\sigma)} \delta^{\sigma(\nu_1)}_{\mu_1}\delta^{\sigma(\nu_2)}_{\mu_2}\ldots\delta^{\sigma(\nu_n)}_{\mu_n} \ , \tag{5.166}$$

where the sum is over all permutations σ of the indices $\nu_1, \nu_2, \ldots, \nu_n$, and $\pi(\sigma) = 0, 1$ is the parity of the permutation.

Now we need the fact that η is covariantly constant with respect to the Levi-Civita connection. To see this, note that if not, then $\nabla_X \eta = h\eta$ for some function $h \in \mathfrak{F}(M)$ for all $X \in \mathfrak{X}(M)$. But

$$(\eta, \eta) = (-1)^t \quad \Rightarrow \quad 2(\nabla_X\eta, \eta) = 0 = 2(h\eta, \eta) = 2h(-1)^t \quad \Rightarrow \quad h = 0 \ . \tag{5.167}$$

Now if $q = n - p$, then

$$d *F = 0 \quad \Rightarrow \quad \nabla_{[\mu_1}\eta_{\mu_2\mu_3\ldots\mu_{q+1}]\nu_1\nu_2\ldots\nu_p}F^{\nu_1\nu_2\ldots\nu_p} = 0 \ , \tag{5.168}$$

Now contract with $\eta^{\mu_1\mu_2\ldots\mu_{q+1}\sigma_{q+1}\ldots\sigma_n}$ to get a multiple of $\nabla_{\mu_1}F^{\mu_1\sigma_{q+1}\ldots\sigma_n}$. Thus

$$\boxed{d *F = 0 \quad \Rightarrow \quad \nabla_{\mu_1}F^{\mu_1\sigma_{q+1}\ldots\sigma_n} = 0 \ .} \tag{5.169}$$

For examaple, if $p = 1$ and $F_\mu = \partial_\mu A$, where A is a scalar field, then

$$d *F = 0 \quad \Rightarrow \quad \nabla^\mu F_\mu = 0 \quad \Rightarrow \quad \nabla^2 A = 0 \ , \tag{5.170}$$

and we get the massless scalar wave equation.

5.3.4 The Poincaré Lemma

If $U \subset M$ is a *star-shaped domain*, in particular if U is contractible, then if ω^p is a *closed* p-form, i.e. $d\omega^p = 0$, then there exist a $(p-1)$-form ω^{p-1} such that $\omega^p = d\omega^{p-1}$. Of course ω^{p-1} is not unique, because ω^{p-1} and $\omega^{p-1} + d\omega^{p-2}$ will give the same ω^p. Note that if M is topologically non-trivial, then ω^{p-1} may not exist globally.

For example, consider the circle $M = S^1$ with local coordinate θ ranging between 0 and 2π. Locally, the volume form is $\eta = d\theta$, but this is not d of any function, because θ is not a globally defined, single-valued function on the circle, i.e. $\theta \notin \Omega^0(S^1) = C^\infty(S^1)$.

The same is true if $M = S^2$, the 2-sphere. The volume form in spherical polar coordinates is $\eta = \sin\theta\, d\theta \wedge d\phi = d(-\cos\theta\, d\phi)$. But the 1-form $-cos\theta\, d\phi$ is singular at the north and south poles where $\theta = 0$ and $\theta = \pi$. When we discuss Stokes' theorem, we shall see that pathologies will show up in any other choice of the 1-form.

6 Maps between manifolds

In addition to the exterior derivative d, there is another type of derivative that one may define on a manifold without any further structure, which is called the *Lie Derivative.* To define it, we need a smooth map $\psi : M \to N$ from a smooth manifold M^m to another smooth manifold N^n, which need not be of the same dimension, i.e. $m \neq n$. Of course an interesting special case will correspond to taking $M = N$, but for clarity — and also with other applications in mind — we keep M and N distinct. By a smooth map we mean one given by smooth functions in every smooth coordinate chart. Thus if x^α are local coordinates for M and y^β for N, then ψ takes a point p to $q = \psi(p)$, and if p has coordinates x^α, $\alpha = 1, 2, \ldots, m$ and q has coordinates y^β, $\beta = 1, 2, \ldots, n$, then $y^\beta = y^\beta(x^\alpha)$.

Associated with ψ are two maps,

- the *push-forward* (or *derivative*) $\quad \psi_* : T_p(M) \to T_q(N)$,
- and the *pull-back* $\quad \psi^* : T^*_p(M) \leftarrow T^*_q(N)$.

The push-forward acts on curves, vectors and contravariant tensors, the pull-back acts on functions, co-vectors, p-forms and covariant tensor fields.

6.1 The push-forward map

Consider a curve in M, i.e. a map $c : \mathbb{R} \to M$ given by $x^\alpha(\lambda)$. The pushed-forward a curve c_* is just obtained by composition, in other words $c_* = \psi \circ c$, or $y^\beta(\lambda) = y^\beta(x^\alpha(\lambda))$. The chain rule now allows us to push-forward the tangent vector T:

$$\frac{dy^\beta}{d\lambda} = \frac{\partial y^\beta}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda} \ , \tag{6.171}$$

or, if $\psi_* T = T_*$,

$$T_*^\beta(q) = \frac{\partial y^\beta}{\partial x^\alpha} T^\alpha(p) \ . \tag{6.172}$$

Clearly we can extend ψ_* to arbitrary contravariant tensor fields.

6.2 The pull-back map

Consider a function on N, i.e. a map $N \to \mathbb{R}$ given by $f(y^\beta)$. The pull-back function f^* on M is just obtained by composition, in other words $f^* = \psi^* f = f \circ \psi$, or $f^*(x^\alpha) = f(y^\beta(x^\alpha))$. The chain rule now allows us to pull back the gradient co-vector $\omega = df$:

$$\frac{\partial f^*}{\partial x^\alpha} = \frac{\partial y^\beta}{\partial x^\alpha}\frac{\partial f}{\partial y^\beta} \ , \tag{6.173}$$

or, if $\psi^*\omega = \omega^*$,

$$\omega^*_\alpha(p) = \frac{\partial y^\beta}{\partial x^\alpha}\omega_\beta(q) \ . \tag{6.174}$$

Clearly we can extend ψ_* to arbitrary covariant tensor fields.

Clearly push-forwards and pull-backs are related. One could define one in terms of the other by

$$(V_* f)_q = (V f)_p \quad \text{or} \quad \langle \omega^* | V \rangle_p = \langle \omega | V_* \rangle_q \ , \tag{6.175}$$

where $V \in \mathfrak{X}(M)$, $\omega \in \Omega^1(N)$ and $f \in C^\infty(N)$.

6.3 Exterior derivative commutes with pull-back

The formula given above is identical to that used earlier to show that the exterior derivative takes p-forms to $p+1$-forms. In the present context that calculation shows that d commutes with pull-back, i.e.

$$\boxed{d(\psi^*\omega) = \psi^*(d\omega) \ .} \tag{6.176}$$

For example, consider the map

$$\psi : \mathbb{R}^3 \setminus z\text{-axis} \ \to S^1 \ , \quad (x, y, z) \mapsto \theta = \tan^{-1}\left(\frac{y}{x}\right) \ . \tag{6.177}$$

We can pull back the volume form on S^1:

$$\psi^* d\theta = \frac{-y\,dx + x\,dy}{x^2 + y^2} = B \ , \quad dB = 0 \ . \tag{6.178}$$

Physically, B is the magnetic field $\mathbf{B}$ due to a current flowing along the z-axis, and $dB = 0 \ \Leftrightarrow \ \nabla \times \mathbf{B} = 0$. Note that

$$\oint_\gamma \mathbf{B}.\,\mathbf{dx} = 2\pi = \oint_{S^1} d\theta \ , \tag{6.179}$$

where γ is any closed curve encircling the z-axis once.

Another example is provided by the map

$$\psi : S^2 \to \mathbb{R}^3 \ , \quad (\phi, \theta) \mapsto (x, yz) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \ , \tag{6.180}$$

which embeds the 2-sphere into three-dimensional Euclidean space. Let

$$\omega = z\,dx \wedge dy + x\,dy \wedge dz + y\,dz \wedge dx \ , \quad \text{so} \ \ d\omega = 3\,dx \wedge dy \wedge dz \ . \tag{6.181}$$

We have

$$\psi^*\omega = \sin\theta \, d\theta \wedge d\phi \ , \quad \text{so} \quad d\left(\psi^*\omega\right) = 0 = \psi^*\left(d\omega\right) \ . \tag{6.182}$$

The last equality follows because $\psi^* d\omega$ is a 3-form in 2-dimensions.

Our last example is related to the Dirac monopole in the same way that the first was related to the vortex:

$$\psi : \mathbb{R}^3 \setminus \{0\} \to S^2 \ , \quad (x,y,z) \mapsto \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) \ , \tag{6.183}$$

with $r = \sqrt{x^2+y^2+z^2}$. If

$$F = \frac{1}{r^3}\left(z\, dx\wedge dy + x\, dy \wedge dz + y\, dz\wedge dx\right) = \frac{1}{2}\epsilon_{ijk}\frac{x^k}{r^3}\, dx^i \wedge dx^j \ , \tag{6.184}$$

then one has $F = \psi^*\eta$, where $\eta = \sin\theta \, d\theta \wedge d\phi$ is the volume form on S^2, and so F is closed, $d\eta = 0 \ \Rightarrow dF = 0$. Thus if $\mathbf{B}$ is the magnetic field, then $\nabla \times \mathbf{B} = 0$, and

$$\int_\Sigma \mathbf{B}.\, \mathbf{dS} = 4\pi \ , \tag{6.185}$$

for any closed surface Σ which encloses the origin once.

This calculation easily generalizes to arbitrary dimensions, and it can be applied to the Ramond-Ramond and Neveu-Schwarz charges of various p-brane solutions in supergravity and string theory.

6.4 Diffeomorphisms: Active versus passive viewpoint

In terms of diagrams we are just following the arrows. Analytically we only need the Jacobian matrix $\left(\frac{\partial y^\beta}{\partial x^\alpha}\right)$, we never need what may not exist, i.e. $\frac{\partial x^\alpha}{\partial y^\beta}$. However, if $M = N$ and ψ is, at least locally, invertible, so that ψ is a local *diffeomorphism*, then $\left(\frac{\partial x^\alpha}{\partial y^\beta}\right)$ will exist, and what we have been doing is identical to what we earlier thought of as a local *coordinate transformation* or *change of chart* from ϕ say to $\phi' = \phi \circ \psi$. This is just a change of viewpoint: initially one may have thought of ψ **actively** as moving the points of the manifold M, while we usually think of coordinate transformations **passively** as relabeling the same points. The two concepts are essentially interchangeable.

6.5 Invariant tensor fields

We can now give a condition that a contravariant tensor field S of rank k is *invariant* under the action of a diffeomorphism ψ: It must equal its push-forward S_* under ψ, i.e.

$$\psi_* S(p) = S\big(\psi(p)\big) \ , \tag{6.186}$$

or, in components,

$$S^{\beta_1\beta_2\ldots\beta_k}(y) = \frac{\partial y^{\beta_1}}{\partial x^{\alpha^1}}\frac{\partial y^{\beta_2}}{\partial x^{\alpha^2}}\cdots\frac{\partial y^{\beta_k}}{\partial x^{\alpha^k}} S^{\alpha_1\alpha_2\ldots\alpha_k}(x) \ . \tag{6.187}$$

Similarly, a covariant second rank tensor field g, for example the metric tensor, is invariant under the action of ψ if g equals its pull-back:

$$g(q) = \psi^* g(p) \ , \tag{6.188}$$

that is

$$g_{\alpha\beta}(x) = g_{\mu\nu}(y)\frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta} \ . \tag{6.189}$$

It follows that

$$ds^2(x) = ds^2(y) \ , \tag{6.190}$$

and thus infinitesimal lengths are preserved by such diffeomorphisms, which are called *isometries*. Later we will discuss infinitesimal isometries and Killing vector fields.

6.6 Immersions and embeddings

The derivative map or *Jacobian* of a map $\psi : M \to N$

$$\psi_* : T_p(M) \to T_{\psi(p)}(N) \tag{6.191}$$

is a linear map, explicitly the matrix $\left(\frac{\partial y^\beta}{\partial x^\alpha}\right)$. Its *rank* is the number of linearly independent rows. Then

- ψ_* is *injective* if $\operatorname{rank}\psi_* = \dim M$, i.e. no non-zero vector in $T_p(M)$ is mapped to zero, and
- ψ_* is *surjective* if $\operatorname{rank}\psi_* = \dim N$, i.e. every vector in $T_{\psi(p)}(N)$ is the image of some vector in $T_p(M)$.

The map ψ is called a smooth *immersion* if locally the inverse ψ^{-1} exists. The *Implicit Function Theorem* states that ψ is an immersion if and only if ψ_* is injective. In that case we say that $\psi(M)$ is an *immersed submanifold*. A smooth *embedding*[5] is a smooth immersion which is injective and a homeomorphism onto its image[6].

For example an embedding of S^1 into $\mathbb{R}^2$ is a circle or an ellipse or any simple smooth curve which does not intersect itself. An example of an immersion would be a figure-of-eight curve. For each point on S^1 there is a unique tangent vector, but at two points S^1 with the same image in $\mathbb{R}^2$ there are two distinct tangent vectors.

A *hypersurface* N in M, for example a $t =$ constant surface in spacetime, is an embedded submanifold of co-dimension one, i.e. $\dim N = \dim M - 1$.

In this language, a smooth diffeomorphism is such that ψ is bijective and ψ^{-1} is smooth. The Implicit Function Theorem then asserts that this is true if ψ_* is both surjective and injective. Thus

$$\operatorname{rank}\psi_* = \dim M = \dim N \ . \tag{6.192}$$

It follows that $(\psi_*)^{-1} = (\psi^{-1})_*$ are *isomorphisms* between $T_p(M)$ and $T_q(N)$ and that the two tensor algebras are also isomorphic.

Thus in the case of a diffeomorphism the distinction between push-forward and pull-back is to some extent lost. We can, by convention, define the push-forward of a covariant tensor field Ω by the formula

$$\psi_*\Omega(y) = (\psi^{-1})^*\Omega(x) \ . \tag{6.193}$$

Thus on functions, for example, $(f^*)_* = f$.

7 One-parameter families of diffeomorphims: Lie derivatives

Suppose $\psi_t : \mathbb{R} \times M \to M$ is one parameter family of smooth maps such that

$$\psi_s \circ \psi_t = \psi_{t+s} \tag{7.194}$$

and ψ_0 is the identity map. Then

$$\psi_{-t} = \psi_t^{-1} \ , \tag{7.195}$$

and we have an *action* of the additive group $\mathbb{R}$ on M. One should have in mind the example of time translations. The *orbit* $\psi_t(p)$ of ψ_t through a point $p \in M$ with coordinates x^α is a curve $y^\alpha(t)$ with $y^\alpha(0) = x^\alpha$, the coordinates of the point p. The curves $\psi_t(p)$ have tangent vectors $T(t)$ and define a vector field $K(x)$ on M. Conversely, given a vector field K we obtain a one-parameter family of diffeomorphisms ψ_t by moving the points of M up the integral curves of K by an amount t. Thus in local charts, ψ_t is given by

$$\psi_t : x^\alpha \mapsto y^\alpha(t) \ , \tag{7.196}$$

where $y^\alpha(t)$ is a solution of

$$\frac{dy^\alpha}{dt} = K^\alpha\big(y(t)\big) \ , \quad \text{with} \ \ y^\alpha(0) = x^\alpha \ . \tag{7.197}$$

[5]Sometimes called an *imbedding*.
[6]This is equivalent to demanding that the inverse image of any compact set be compact. If M is compact, every injective immersion is an embedding.

Infinitesimally, for small t we have

$$y^\alpha(t) = x^\alpha + K^\alpha(x)t + \mathcal{O}(t^2) \ . \tag{7.198}$$

Given ψ_t we can *Lie-drag* curves and tensor fields along the integral curves using the push-forward map $(\psi_t)_*$. Thus, for a vector V, $V_* = (\psi_t)_* V$ is given by

$$V^\alpha_*\big(y(t)\big) = \frac{\partial y^\alpha(t)}{\partial x^\beta} V^\beta(x) \ . \tag{7.199}$$

We now define the *Lie Derivative* $\mathcal{L}_K S$ of any contravariant tensor field S, say, by

$$\mathcal{L}_K S(x) = -\frac{d}{dt}\left((\psi_t)_*\, S(x)\right)\Big|_{t=0} = \lim_{t\to 0}\left(\frac{1}{t}\big(S(x) - (\psi_t)_* S(x)\big)\right) \ . \tag{7.200}$$

The origin of the minus sign is as follows: We are evaluating the Lie derivative of S at the point x. To do so, we compare $S(x)$ with the value obtained by dragging S from the point $\psi_{-t}(x) = \psi_t^{-1}(x)$ to the point x.

As mentioned above, for a (local) diffeomorphism we can extend the push-forward map to covariant tensor fields, and the definition of the Lie Derivative extends as well. Alternatively one could define it by using the pull-back map, but one now gets a plus sign:

$$\mathcal{L}_K \Omega(x) = +\frac{d}{dt}\left((\psi_t)^*\, \Omega(x)\right)\Big|_{t=0} = \lim_{t\to 0}\left(\frac{1}{t}\big((\psi_t)^*\Omega(x) - \Omega(x)\big)\right) \ . \tag{7.201}$$

7.0.1 Example: Functions

The simplest case to consider is that of a function $f \in C^\infty(M)$:

$$\mathcal{L}_K f = \lim_{t\to 0} \frac{f(x) - f(y^\alpha(-t))}{t} \ . \tag{7.202}$$

Thus

$$\mathcal{L}_K f = \lim_{t\to 0} \frac{f(x) - f(x - tK(x))}{t} = K^\alpha \frac{\partial f}{\partial x^\alpha} = Kf \ . \tag{7.203}$$

Doing it the other way we get the same answer:

$$\mathcal{L}_K f = \lim_{t\to 0} \frac{f(x + tK(x)) - f(x)}{t} = K^\alpha \frac{\partial f}{\partial x^\alpha} = Kf \ . \tag{7.204}$$

A slightly symbolic but nevertheless illuminating notation makes use of the *exponential map* e^{tK} of a vector field. Geometrically, this takes points an amount t up the integral curves of the vector field K:

$$e^{tK} x^\alpha = y^\alpha(t) \ . \tag{7.205}$$

On functions we have therefore:

$$e^{tK} f = f^* = (\psi_t)^* f \ . \tag{7.206}$$

The push-forward is given by

$$e^{-tK} f = f_* = (\psi_t)_* f \ . \tag{7.207}$$

7.0.2 Example: Vector fields

We need

$$(\mathcal{L}_K V)^\alpha(x) = -\frac{d}{dt}\left(V^\beta\big(y(-t)\big)\frac{\partial y^\alpha(-t)}{\partial x^\beta}\right) \ . \tag{7.208}$$

Since

$$\frac{\partial y^\alpha(-t)}{\partial x^\beta}(x) = \delta^\alpha_\beta - tK^\alpha{}_{,\beta}(x) + \mathcal{O}(t^2) \ , \tag{7.209}$$

we obtain

$$\boxed{(\mathcal{L}_K V)^\alpha = V^\alpha{}_{,\beta} K^\beta - K^\alpha{}_{,\beta} V^\beta = [K, V]^\alpha \ .} \tag{7.210}$$

This calculation can also be done using the exponential notation:

$$(V_* f)(q) = (V^* f)(p) \quad \Rightarrow \quad (V^* f)(p) = (V^* f)^*(p) \ , \tag{7.211}$$

thus

$$(\psi_t)_* V = e^{-tK} V e^{tK} \quad \Rightarrow \quad -\frac{d}{dt}(\psi_t)_* V\Big|_{t=0} = [K, V] \ . \tag{7.212}$$

It is straightforward to see how to take the Lie derivative of higher rank contravariant tensor fields. One gets terms with a partial derivative of K for each upper index. For example, for a second rank contravariant tensor field S we have

$$(\mathcal{L}_K S)^{\alpha\beta} = K^\gamma \partial_\gamma S^{\alpha\beta} - \partial_\gamma K^\alpha S^{\gamma\beta} - \partial_\gamma K^\beta S^{\alpha\gamma} \ . \tag{7.213}$$

7.0.3 Lie derivative of covariant tensors

We have

$$(\mathcal{L}\omega)_\alpha = -\frac{d}{dt}\left(\omega_\beta\big(y(-t)\big)\frac{\partial y^\beta(-t)}{\partial x^\alpha}\right)\Bigg|_{t=0} \tag{7.214}$$

Using the fact that

$$\frac{\partial y^\beta(-t)}{\partial x^\alpha} = \delta^\beta_\alpha - t\partial_\alpha K^\beta + \mathcal{O}(t^2) \ , \tag{7.215}$$

one gets

$$\boxed{(\mathcal{L}_K \omega)_\alpha = K^\beta \partial_\beta \omega_\alpha + \partial_\alpha K^\beta \omega_\beta \ .} \tag{7.216}$$

For a second rank covariant tensor, such as the metric tensor g, one has

$$\boxed{(\mathcal{L}_K g)_{\alpha\beta} = K^\gamma \partial_\gamma g_{\alpha\beta} + \partial_\alpha K^\gamma g_{\gamma\beta} + \partial_\beta K^\gamma g_{\alpha\gamma} \ .} \tag{7.217}$$

Note that we have not assumed a particular symmetry for $g_{\alpha\beta}$, but if we had, that symmetry would be inherited by the Lie derivative.

7.0.4 Example: Adapted coordinates, stationary and static metrics

From its definition it is clear that the Lie derivative takes tensor fields to tensor fields. The presence of the terms involving $\partial_\beta K^\alpha$ guarantees that. However, it is always possible locally to introduce *adapted coordinates* t, x^i such that

$$K^\alpha \text{ “=” } \delta^\alpha_t \ , \qquad \Leftrightarrow \quad K \text{ “=” } \frac{\partial}{\partial t} \ . \tag{7.218}$$

In these coordinates, for any tensor field,

$$\mathcal{L}_K \text{ “=” } \frac{\partial}{\partial t} \ . \tag{7.219}$$

Thus if K is a *Killing vector field*, that is if

$$\mathcal{L}_K g = 0 \ , \tag{7.220}$$

where g is the metric tensor, then in adapted coordinates the metric is independent of the coordinate t:

$$ds^2 = g_{00}(x^k)\, dt^2 + 2g_{0i}(x^k)\, dt\, dx^i + g_{ij}(x^k)\, dx^i\, dx^j \ . \tag{7.221}$$

The existence of adapted coordinates may be shown as follows: We introduce an initial hypersurface nowhere tangent to Σ so that (at least in a local neighbourhood U) the integral curves of K intersect Σ once and only once. The coordinates x^i are chosen on $\Sigma \cap U$ and then Lie-dragged along the integral curves, in other words they are defined to be constant along the integral curves. Points on Σ are assigned the coordinate $t = 0$. We then assign to each point p in $U \subset M$ the coordinates t, x^i, where x^i labels the integral curve passing through p, and t is the parameter necessary to reach p starting from Σ.

There is thus clearly some “gauge freedom” in choosing the hypersurface Σ. Changing Σ to some other hypersurface Σ' will alter the origin of the t-parameter along each integral curve. If Σ' intersects the integral curve labelled by x^i at $t = \tau(x^i)$, then the new coordinate t' will be given by

$$t' = t - \tau(x^i) \ . \tag{7.222}$$

In physical applications, if the Killing vector is timelike, one calls the metric *stationary* unless there is an additional *time reversal symmetry*, in which case one may put $g_{0i} = 0$. The metric is then said to be *static*. For example, a non-rotating Schwarzschild black hole has gives rise to a static metric. A rotating Kerr black hole has a stationary metric.

7.1 Lie derivative and Lie bracket

So far we have not verified explicitly that the Lie derivative $\mathcal{L}_V W$ is a vector field. This may be seen directly by introducing the *bracket* or *commutator*

$$\boxed{[V,W] = -[W,V]} \tag{7.223}$$

of two vector fields V and W by its action on a function $f \in \mathfrak{F}(M)$:

$$[V,W]f = V(Wf) - W(Vf) \ . \tag{7.224}$$

The bracket of two vector fields is itself a vector field. To see this, one needs to check that the Leibniz property holds:

$$[V,W](fg) = g[V,W]f + f[V,W]g \ . \tag{7.225}$$

This is a simple, albeit tedious, calculation.

Moreover, a short calculation in local coordinates reveals that

$$[V,W]^\alpha = (\mathcal{L}_V W)^\alpha \ . \tag{7.226}$$

7.1.1 The Jacobi identity

In the same vein, one easily verifies that the Lie bracket on functions satisfies the *Jacobi identity*:

$$\boxed{\big[U,[V,W]\big] + \big[V,[W,U]\big] + \big[W,[U,V]\big] = 0 \ .} \tag{7.227}$$

It follows from (7.223) and (7.227) that the set of vector fields $\mathfrak{X}(M)$ forms an infinite dimensional *Lie algebra*. This is sometimes denoted diff(M).

7.1.2 Example: diff(S^1) is called the Virasoro algebra

It crops up in string theory. If $\theta \in (0, 2\pi]$ is a coordinate on S^1, one defines a basis indexed by $n \in \mathbb{Z}$ by

$$D_n = ie^{in\theta}\frac{\partial}{\partial\theta} \ . \tag{7.228}$$

The brackets are

$$[D_n, D_m] = (n-m)D_{m+n} \ . \tag{7.229}$$

There is a finite sub-algebra spanned by D_{-1}, D_0, D_{-1} that is the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of the projective group $SL(2,\mathbb{R})$.

7.2 The Lie bracket and closure of infinitesimal rectangles

Suppose that we move from O to A by an amount t along the integral curves of U and then from A to B by an amount s along the integral curves of V. We may compare our final position B with what would have happened if we had first moved from O to C by an amount s along the integral curves of V and then from C to D by an amount t along the integral curves of U. A short calculation in local coordinates using Taylor's theorem shows that to lowest non-trivial order it is:

$$\boxed{x^\mu_B - x^\mu_D = [U,V]^\alpha st + \cdots \ .} \tag{7.230}$$

Thus, if the bracket vanishes, the infinitesimal rectangle $DCOAB$ will close. More generally it may be shown that if the bracket vanishes, then the two vector fields lie in a 2-surface and conversely. It is not necessary that the bracket vanish for the vector fields to lie in a 2-surface. It suffices that the bracket $[U,V]$ is a linear combination of U and V with coefficients $a(x)$ and $b(x)$, which may depend on x. This is a special case…

7.3 Frobenius' Theorem

Suppose that k non-vanishing vector fields U_a, $a = 1, 2, \ldots, k$ satisfy

$$\boxed{[U_a, U_b] = -D_{a\,b}^{\,c} U_c \ ,} \tag{7.231}$$

where the functions $D_{a\,b}^{\,c} \equiv D_{a\,b}^{\,c}(x)$ are called *structure functions*, then the vector fields are tangent to some l-dimensional submanifold with $0 < l \leq k$.

Note that if the structure functions are independent of position, they become the *structure constants* of some Lie algebra.

7.3.1 Example: The simplest non-trivial Lie algebra

Let

$$U_1 = x\frac{\partial}{\partial x} \quad \text{and} \quad U_2 = \frac{\partial}{\partial x} \ . \tag{7.232}$$

Here $k = 2, l = 1$ and $D_{2\,1}^{\,2} = -D_{1\,2}^{\,2} = 1$ and all others zero. Changing the $x\frac{\partial}{\partial x}$ to $a(x)\frac{\partial}{\partial x}$ would lead to structure functions rather than structure constants, but the two vector fields would still be tangent to a one-dimensional manifold.

7.3.2 Example: Two commuting translations

Let

$$U_1 = \frac{\partial}{\partial x} \quad \text{and} \quad U_2 = \frac{\partial}{\partial y} \ . \tag{7.233}$$

Here $k = 2$ and $l = 2$ and $D_{a\,b}^{\,c} = 0$ for all a, b, c.

7.3.3 Example: Kaluza or torus reductions

Following the initial idea of Kaluza one could imagine a $4+m$ dimensional spacetime invariant under the action of the abelian torus group $T^m = (S^1)^m$. There will be m mutually commuting Killing fields K_a, $a = 1, \ldots, m$:

$$[K_a, K_b] = 0 \ . \tag{7.234}$$

By a combination of the Frobenius theorem and the discussion above about adapted coordinates, we may introduce coordinates $\{y^a\}$ such that

$$K_a = \frac{\partial}{\partial y^a} \tag{7.235}$$

and the metric takes the form

$$ds^2 = g_{ab}(x^\lambda)(dy^a + A^a_\mu(x^\lambda)dx^\mu)(dy^b + A^b_\nu(x^\lambda)dx^\nu) + g_{\mu\nu}(x^\lambda)dx^\mu dx^\nu \ . \tag{7.236}$$

One may interpret $g_{\mu\nu}(x^\lambda)$ as the metric on our lower-dimensional spacetime with coordinates $\{x^\lambda\}$. The co-vector fields $A_\mu(x^\lambda)$ may be interpreted as m abelian gauge fields which are called *gravi-photons*. This works because the coordinates $\{y^a\}$ are determined only up to a coordinate transformation of the form

$$y^a \to \tilde{y}^a = y^a + \Lambda^a(x^\lambda) \ , \tag{7.237}$$

under which the 1-forms $A^a_\mu(x^\lambda)$ undergo a gauge transformation

$$A^a_\mu(x^\lambda) \to A^a_\mu(x^\lambda) - \partial_\mu \Lambda^a(x^\lambda) \ . \tag{7.238}$$

The fields $g_{ab}(x^\lambda)$ are invariant under both gauge transformations, and $(u(1)^m$ gauge-transformations. They are scalar fields, called *gravi-scalars*, which take their values in the space of symmetric matrices or m-dimensional metrics which, as we shall see, may be identified with the symmetric space $GL(m, \mathbb{R})/SO(m)$. It is sometimes convenient to split off the determinant $\det(g_{ab})$ and to consider unimodular matrices, in which case the remaining scalars take their values in $SL(m, \mathbb{R})/SO(m)$. Some multiple of the logarithm of $\det(g_{ab})$ is often called the *dilaton*.

What has been achieved here is to embed the local $U(1)^m$ gauge group as a subgroup of the the diffeomorphism group of the higher dimensional spacetime. The analogue of time reversal is *charge conjugation symmetry*, under which $y^a \to -y^a$. If it holds for one of the A^a_μ's, then we may set $A^a_\mu = 0$, and the $U(1) \equiv SO(2)$ contribution to the isometry group is augmented to $O(2)$.

7.3.4 Example: $\mathfrak{so}(3)$

Let

$$L_z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \quad \text{and cyclic permutations .} \tag{7.239}$$

Here

$$[L_x, L_y] = -L_z \quad \text{and cyclically,} \tag{7.240}$$

and we get the Lie algebra of the rotation group $SO(3)$. If $r^2 = x^2 + y^2 + z^2$, one verifies that

$$L_x r^2 = L_y r^2 = L_z r^2 = 0 \ . \tag{7.241}$$

Thus $k = 3$, but $l = 2$ — all three vector fields are tangent to 2-spheres centred on the origin of three-dimensional Euclidean space $\mathbb{E}^3$. Note that while $\{L_x, L_y, L_z\}$ generates rotations in the positive sense in the three orthogonal coordinate planes, the structure constants are the negative of the standard structure constants for $\mathfrak{so}(3)$, in other words

$$[L_i, L_j] = -\epsilon_{ijk} L_k \ . \tag{7.242}$$

This turns out to be a universal sign reversal, as will be explained later.

7.4 Lie derivative and exterior derivative

It is important that d commutes with the Lie derivative $\mathcal{L}$, i.e.

$$\mathcal{L}_U(d\omega) = d(\mathcal{L}_U\omega) \ , \tag{7.243}$$

for any p-form ω and vector field U. This is no more than the fact that the exterior derivative commutes with pull-back, but it may be proved more concretely by writing out the relevant expressions in local coordinates. Another important fact is that one may swap the exterior derivative for the bracket. Thus, for a 1-form ω,

$$\langle d\omega | U, V\rangle = U\,\langle\omega|V\rangle - V\,\langle\omega|U\rangle - \langle\omega|[U,V]\rangle \ . \tag{7.244}$$

7.5 Cartan's formula: Lie derivative and interior product

In the same spirit one has the extremely useful formula

$$\boxed{\mathcal{L}_U\omega = i_U d\omega + d(i_U\omega) \ ,} \tag{7.245}$$

where ω is a 2-form and U a vector field.

7.5.1 Example: Electrostatic potentials

Suppose that $\omega = F$, a Maxwell 2-form, which by Faraday's law is closed (i.e. $dF = 0$), and suppose that F is invariant under time-translation or some other symmetry, generated by a vector field U, which could be a Killing vector field. We have:

$$\mathcal{L}_U F = 0 \quad \Rightarrow \quad d(i_U F) = 0 \quad \Rightarrow \quad i_U F = d\Phi \tag{7.246}$$

for some function Φ, which plays the role of an an electrostatic or magnetostatic potential. Thus in the Schwarzschild solution, for example, if

$$F = Q\frac{dt\wedge dr}{r^2} \quad \text{and} \quad U = \frac{\partial}{\partial t} \ , \tag{7.247}$$

then

$$\Phi = -\frac{Q}{r} \ . \tag{7.248}$$

This example can readily be generalized to higher rank forms, which appear in higher dimensions in supergravity and superstring theory.

8 Affine connections

So far we have defined no extra structure on our manifold M. We want now to define the structure needed to 'parallely transport rq, or 'parallely'propagate a vector W along a curve $c(\lambda)$. We can't use the Lie derivative because it depends on a *family* of curves or alternativly it depends on the *derivative* of the tangnet vector $T = \frac{dc}{d\lambda}$. We proceed by axiomatizing the requiremnt that it be possible to parallely transport W along c . We say that W is parallely progated along c in direction, if

$$T.\nabla W = f(\lambda)W \tag{8.249}$$

where $T.\nabla : T(M) \times T(m) \to T(M)$ so that $T.\nabla W$ is a vector for all W and T and if $f, g \in \Omega^0(M)g$ then

$$(i) \qquad (fT + fS).\nabla W = f(T.\nabla W) + g(S.\nabla W) \qquad \text{linearity} \tag{8.250}$$

$$(ii) \qquad T.\nabla(fW) = (Tf)W + f(T.\nabla W) \qquad \text{Leibniz}\,. \tag{8.251}$$

Rather than $T.\nabla$, it is sometimes convenient to adopt the notation ∇_T. The former notation brings out the fact that one may introduce a *covariant derivative operator* $\nabla : T(M) \to T^\star(M) \otimes T(M)$ by

$$\langle U, \nabla V\rangle = \nabla_U V = U.\nabla V\,. \tag{8.252}$$

In a basis one frequently writes

$$(\nabla V)^\alpha_\beta = \nabla_\beta V^\alpha\,, \tag{8.253}$$

and

$$(\nabla_U V)^\alpha = U^\beta \nabla_\beta V^\alpha\,. \tag{8.254}$$

A covariant derivative operator is often called an *affine connection* and a manifold equipped with such structure is said to be *affinely connected.*

Once we know how to differentiate vectors we can extend the action of the entire tensor algebra by replacing (i) with

$$(iii) \qquad T.\nabla \qquad \text{commutes with tensor products} \tag{8.255}$$

and demanding that

$$(iv) \qquad T.\nabla \qquad \text{commutes with contraction} \tag{8.256}$$

and on functions $f \in \Omega^0$

$$\nabla_T f = Uf = \langle df|U\rangle\,. \tag{8.257}$$

Thus in a natural basis

$$\nabla_\alpha(W_\beta V^\beta) = \partial_\alpha(W_\beta V^\beta) = W_\beta(\nabla_\alpha V^\beta) + V^\beta(\nabla_\alpha W_\beta)\,. \tag{8.258}$$

8.1 Introduction of a basis: Moving Frames

For practical calculations we sually have to introduce a basis. We could use a natural basis for one forms dx^α and its dual basis for vector fields $\frac{\partial}{\partial x^\alpha}$. However it is usually more convenient to introduce what is sometimes called a *non-holonomic basis* for which, by Froebenius' theorem no such coordinates can be found because the bracket of the basis of vectors

$$[\mathbf{e}_a, \mathbf{e}_b] = D_a{}^c{}_b \mathbf{e}_c\,, \tag{8.259}$$

is non vanishing.The quantities $D_a{}^c b$ are in general be position dependent, are called *structure functions*. Of course, if they were constants then we would have an $n-$ dimensional Lie algebra.

Using the formula

$$\langle d\omega|U, V\rangle = U\langle\omega - V\langle\omega|U\rangle - \langle\omega|[U, V]\rangle\,. \tag{8.260}$$

one deduces that the dual basis of one-forms satisfies

$$d\mathbf{e}^a = -\frac{1}{2} D_b{}^a{}_c\, \mathbf{e}^b \wedge \mathbf{e}^c\,. \tag{8.261}$$

Cartan called such a basis of this type *repère mobile* or moving frame and in physics they often referrred to as an *n-bein,* as in *vier-bein* where n is the dimension of M. In English, Dirac tried to introduce the *n-leg* but the Greek term *n-ad* as in *dyad, triad* and *tetrad* is usually preferred. At this point in our discussion, we have not yet introduced a metric and so these frames are not in general orthonormal. We are therefore free to rotate them by an arbitary postion dependent element of $GL(n, \mathbb{R}$.

It is helpful to reserve Greek indices for the components of a tensor in a coordinate basis and Latin indices for the more general case. Greek indices are often called *world indices* and Latin indices *tangent space indices*, and they envisaged as transforming under different 'gauge groups', that is diffeomorphisms and local changes of basis of the tangent or co-tangent space respectively. While not being very precise as stated, the ideas intended can and have been be made on many occasions into a systematic, if often confusing formal theory under the rubric of the guage theory of gravity the e physical utility of which is unclear and whose adumbration here, time does not permit. Suffice it to say, that since diffeomorphisms lift in a natural way to the tangent and co-tangent spaces they will induce a change of basis via the push forward and pull-back maps.

Given a basis and any affine connection ∇ , we define the *components of the affine connection* by

$$\nabla_{\mathbf{e}_a} \mathbf{e}_c = \Gamma_a{}^b{}_c \, \mathbf{e}_b \,. \tag{8.262}$$

or equivalently

$$\nabla \mathbf{e}_c = \mathbf{e}_b \otimes \mathbf{e}^a \Gamma_a{}^b{}_c = \mathbf{e}_b \otimes \Gamma^b{}_c \,, \tag{8.263}$$

where the $\mathfrak{g}(n, \mathbb{R})$-valued *conection one-form* $\Gamma^b{}_c = (\Gamma)^b{}_c$

$$\Gamma^b{}_c = \mathbf{e}^a \Gamma_a{}^b{}_c. \tag{8.264}$$

Note that we have introduced a number of conventions about index positions which are not universal. In particular our diffentiating index is always the left-most. Now for an arbitrary vector V with components V^a we have form our general axioms

$$\nabla(V^c \mathbf{e}_c) = (dV^c) \otimes \mathbf{e}_c + \mathbf{e}_b \otimes \Gamma^b{}_c V^c \tag{8.265}$$

or more succinctly

$$\nabla V^c = dV^c + \Gamma^c{}_b V^b \,. \tag{8.266}$$

In a natural or coordinate basis we just pass to world indices

$$\nabla_\alpha V^\beta = \partial_\alpha V^\beta + \Gamma_\alpha{}^\beta{}_\sigma V^\sigma \,. \tag{8.267}$$

For covariant vectors we have

$$\nabla_\alpha W_\beta = \partial_\alpha W_\beta - \Gamma_\alpha{}^\sigma{}_\beta W_\sigma \,. \tag{8.268}$$

8.2 Covariant Derivative of an arbitrary Representation $GL(n, \mathbb{R})$.

A uniform approach to covariant differentiation for all tensor fields, or more generally any representation of $GL(n, \mathbb{R})$ is often convenienent.A simple extension of the idea will allow us to deal with spinor fields.

Let Φ^A be the components of some representation such that if $\omega^a{}_b$ is an infinitesimal $\mathfrak{g}(n, \mathbb{R})$ transformation, then the infinitesimal transformation of Φ^A is

$$\delta\Phi^A = \omega^a{}_b \Sigma_a{}^{bA}{}_B \Phi^B \,. \tag{8.269}$$

One now has

$$\nabla\Phi^A = d\Phi^A + \Gamma^a{}_b \Sigma_a{}^{bA}{}_B \Phi^B \,. \tag{8.270}$$

8.3 Effect of a change of basis

The connection coefficents are not the components of a tensor field of type $\binom{1}{2}$. They transform inhomogeneously. To see this it is convenient to use a matrix notation which brings out the analogy with Yang-Mills theory. Thus under a change of basis

$$\mathbf{e}_a \rightarrow \Lambda_a{}^b \mathbf{e}_b = \tilde{\mathbf{e}}_a \,, \tag{8.271}$$

$$\mathbf{e}^c \rightarrow \Lambda^c{}_d \mathbf{e}^d = \tilde{\mathbf{e}}^c \,, \tag{8.272}$$

with

$$\Lambda_a{}^d \Lambda^c{}_d = \delta^c_a. \tag{8.273}$$

Thus if

$$\Lambda^a{}_b = (\Lambda)^a{}_b \qquad \Lambda_a{}^b = (\Lambda^{-1})_a{}^b\,. \tag{8.274}$$

One then sees that under a change of basis

$$\Gamma \to \tilde{\Gamma} = \Lambda\Gamma\Lambda^{-1} + \Lambda d\Lambda^{-1}\,. \tag{8.275}$$

In a natural basis one has

$$\tilde{\Gamma}_\beta{}^\alpha{}_\gamma = \frac{\partial \tilde{x}^\alpha}{\partial x^\epsilon}\frac{\partial x^\phi}{\partial \tilde{x}^\beta}\frac{\partial x^\chi}{\partial \tilde{x}^\gamma}\Gamma_\phi{}^\epsilon{}_\chi + \frac{\partial \tilde{x}^\alpha}{\partial x^\epsilon}\frac{\partial^2 x^\epsilon}{\partial \tilde{x}^\beta \partial \tilde{x}^\gamma}\,. \tag{8.276}$$

On deduces immediately that

(i) $\Gamma_\alpha{}^\beta{}_\gamma$ are not the components of a tensor.

(ii) $\Gamma_{[\beta}{}^\alpha{}_{\gamma]}$ are the components of a tensor because $\frac{\partial^2 x^\epsilon}{\partial \tilde{x}^{[\beta}\partial \tilde{x}^{\gamma]}} = 0$.

(iii) The difference of the components of two affine connections is a tensor field.

(iv) $\Gamma_{(\alpha}{}^\beta{}_{\gamma)}$ can always be set to zero at an arbitrary point $0 \in M$.

The proof of (iv) follows by considering the coordinate transformation

$$\tilde{x}^\alpha = x^\alpha + \frac{1}{2}\Gamma_\beta{}^\alpha{}_\gamma(0)x^\beta x^\gamma \Rightarrow x^\alpha = \tilde{x}^\alpha + \frac{1}{2}\Gamma_\beta{}^\alpha{}_\gamma(0)\tilde{x}^\beta\tilde{x}^\gamma + \ldots \tag{8.277}$$

Local coordinates in which $\Gamma_{(\alpha}{}^\beta{}_{\gamma)} = 0$ at a point are called *normal coordinates*. Note that in general one cannot set $\Gamma_{(\alpha}{}^\beta{}_{\gamma)} = 0$ everywhere.

The proof of (iii) also follows from the formula, valid for all $f \in F(M)$

$$(\nabla - \tilde{\nabla})(fV) = f(\nabla - \tilde{\nabla})V\,. \tag{8.278}$$

8.4 The Torsion Tensor of an Affine connection

One may check that the following formula defines a vector valued two-form called the *torsion tensor of the afine connection*

$$T(V,U) = \nabla_U V - \nabla_V U - [U,V]\,. \tag{8.279}$$

An affine connection whose torsion tensor vanishes is said to be *torsion-free* or to be *symmetric*.

In a general basis

$$T_b{}^a{}_c = \Gamma_b{}^a{}_c - \Gamma_c{}^a{}_b - D_b{}^a{}_c \tag{8.280}$$

and in a natural basis one passes form Latin to Greek and drops the commutator term.

$$T_\beta{}^\alpha{}_\gamma = \Gamma_\beta{}^\alpha{}_\gamma - \Gamma_\gamma{}^\alpha{}_\beta \tag{8.281}$$

In a basis of one forms one obtains what is called *Cartan's First Structural Equation*

$$d\mathbf{e}^a = -\Gamma^a{}_b \wedge \mathbf{e}^b + T^a \tag{8.282}$$

where

$$T^a = \frac{1}{2}T_b{}^a{}_c\mathbf{e}^b \wedge \mathbf{e}^c\,. \tag{8.283}$$

On a function $f \in F(M)$ one finds that

$$(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla\alpha) = T_\alpha{}^\sigma{}_\beta\partial_\sigma f. \tag{8.284}$$

Thus for a connection with torsion, covariant differentiation fails to commute even on scalars. This has an interesting physical application to what is called *minimal gravitational coupling*. Suppose one wants to introduce electromagnetism on a an affinely connected manifold. In flat spacetime the Faraday tensor $F_{\alpha\beta}$is given in terms of a local vector potential A_α by

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha\,. \tag{8.285}$$

Now the minimal coupling principle is that in a gravitational field one should replace partial derivatives with covariant derivatives

$$F_{\alpha\beta} \to \nabla_\alpha A_\beta - \nabla_\beta A_\alpha\,. \tag{8.286}$$

If the connection is torsion free then

$$\nabla_\alpha A_\beta - \nabla_\beta A_\alpha = \partial_\alpha A_\beta - \partial_\beta A_\alpha\,, \tag{8.287}$$

but if the torsion is no zero, this is not so. Even worse, if one uses (8.286) one finds that the resulting Faraday two-form is not even invariant under an electromagnetic gauge transformation $A_\alpha \rightarrow A_\alpha - \partial_\alpha \Lambda$. The situation is similar for the higher rank forms in higher dimensions that arise in supergravity and superstring theories. For that reason it is customary to define curvatures using exterior differentiation. In fact, for a symmetric connection, one has for any p-form A

$$dA = \partial \wedge A = \nabla \wedge A\,. \tag{8.288}$$

8.5 Geometrical Interpretation of Torsion

Consider two vector fields U and V so that U is parallely propagated along the integral curves of V and *vice versa* with rspect to a connection whose torsion is no zero.. Thus $\nabla_U V = 0 = \nabla_V otvanish U$. It follows their Lie bracket $[U, V]$ cannot be zero. But the Lie bracket measures to extent to which infinitesimal rectangles fail to close.

8.5.1 Example: The right and left connections on a Lie Group

We deem a vector to parallely transported with respect the left connection if its components in the basis L_b are constants. Thus in this basis $\Gamma_a{}^b{}_c = 0$ and one calculates that the torsion $T_a{}^b{}_c = -C_a{}^b{}_c$.

We deem a vector to parallely transported with respect the right connection if its components in the basis R_b are constants. Thus in this basis $\Gamma_a{}^b{}_c = 0$ and one calculates that the torsion $T_a{}^b{}_c = +C_a{}^b{}_c$.

In both cases we have a distant parallelism and the curvature vanishes. More generally we could use the L_b basis and define a one parameter family of connections ∇_λ, $\lambda \in \mathbb{R}$ such that

$$\nabla_{\lambda L_a} L_b == \lambda [L_a, L_b] \tag{8.289}$$

One finds that

$$T(L_a, L_b) = (2\lambda - 1) C_a{}^c{}_b L_c\,. \tag{8.290}$$

Note that the connection is torsion free if $\lambda = \frac{1}{2}$

One use of the Jacobi identity, the curvature works out to be

$$R(L_a, L_b) L_c = \lambda(1-\lambda)[L_a, [L_b, L_c]]\,. \tag{8.291}$$

This vanishes if $\lambda = 0$, the left connection, or if $\lambda = 1$, the right connection.

One may also check that

$$R^d{}_{eab} = C_a{}^f{}_b C_f{}^d{}_e (\lambda^2 - \lambda) \tag{8.292}$$

and that the Ricci tensor, obtained by contraction of the first and third indices is

$$R_{eb} = (\lambda^2 - \lambda) C_d{}^f{}_b C_f{}^d{}_e\,. \tag{8.293}$$

The left and right connections arise in string theory and supergravity. In particular in so-called Wess-Zumino-Witten models when a string moves on a group manifold G with a 3-form back ground given by structure constants. That construction requires a bi-invariant metric on the Lie group G which we shall discuss shortly.

8.6 The Levi-Civita connection

The reader is assumed to be familiar with the elmentary idea of a metric and its associated geodesics.

$$\boxed{\frac{d^2 x^\alpha}{d\tau^2} + \left\{ {\alpha \atop \gamma\ \delta} \right\} \frac{dx^\gamma}{d\tau} \frac{dx^\delta}{d\tau} = 0\ ,} \tag{8.294}$$

with

$$\boxed{\left\{ {\alpha \atop \gamma\ \delta} \right\} = \frac{1}{2} g^{\alpha\epsilon} \left(\frac{\partial g_{\epsilon\delta}}{\partial x^\gamma} + \frac{\partial g_{\gamma\epsilon}}{\partial x^\delta} - \frac{\partial g_{\gamma\delta}}{\partial x^\epsilon} \right)\ .} \tag{8.295}$$

The rather strange collection of objects $\left\{ {\alpha \atop \gamma\ \delta} \right\}$ are called *Christoffel symbols*. It is a striking fact that

(i) $\left\{ {\alpha \atop \gamma\ \delta} \right\} = \left\{ {\alpha \atop \delta\ \gamma} \right\}$, and

(ii) $\left\{ {\alpha \atop \gamma\ \delta} \right\}$ transform *precisely* as a symmetric affine connection.

We call this connection the *Levi-Civita connection* or the *metric connection*. It has the following remarkable property:

$$\boxed{\nabla_\alpha g_{\beta\gamma} = \partial_\alpha g_{\alpha\beta} - \Gamma^{\ \epsilon}_{\alpha\ \beta}\ g_{\epsilon\gamma} - \Gamma^{\ \epsilon}_{\alpha\ \gamma}\ g_{\epsilon\beta} = 0\ ,} \tag{8.296}$$

where we now write $\Gamma^{\ \alpha}_{\gamma\ \delta} = \left\{ {\alpha \atop \gamma\ \delta} \right\}$. In fact, a stronger statement is true

The Fundamental Theorem of Differential Geometry

The Levi-Civita connection $\left\{ {}_\beta{}^\alpha{}_\gamma \right\}$ *is the unique affine connection s.t.*

(i)

$$\Gamma_\beta{}^\alpha{}_\gamma = \Gamma_\gamma{}^\alpha{}_\beta, \qquad \text{i.e. is symmetric} \tag{8.297}$$

(ii)

$$\nabla_\alpha g_{\gamma\beta} = 0, \qquad \text{the metric is covariantly constant.} \tag{8.298}$$

Proof: We write out the covariant constancy condition three times, cyclically permuting the three indices and then take a suitable linear combination. The symmetry of $\Gamma_\alpha{}^\beta{}_\gamma$ leads to cancellations:

$$(i) \qquad \nabla_\alpha g_{\beta\gamma} = \partial_\alpha g_{\beta\gamma} - \Gamma_\alpha{}^\epsilon{}_\beta g_{\epsilon\gamma} - \Gamma_\alpha{}^\epsilon{}_\gamma g_{\epsilon\beta} = 0. \tag{8.299}$$

$$(ii) \qquad \nabla_\gamma g_{\alpha\beta} = \partial_\gamma g_{\alpha\beta} - \Gamma_\gamma{}^\epsilon{}_\alpha g_{\epsilon\beta} - \Gamma_\gamma{}^\epsilon{}_\beta g_{\epsilon\alpha} = 0. \tag{8.300}$$

$$(iii) \qquad \nabla_\beta g_{\gamma\alpha} = \partial_\beta g_{\gamma\alpha} - \Gamma_\beta{}^\epsilon{}_\gamma g_{\epsilon\alpha} - \Gamma_\beta{}^\epsilon{}_\alpha g_{\epsilon\gamma} = 0. \tag{8.301}$$

Now take $(i) - (ii) - (iii)$ and use the symmetry of the connection to get

$$2g_{\epsilon\alpha}\Gamma_\beta{}^\epsilon{}_\gamma + \partial_\alpha g_{\beta\gamma} - \partial_\gamma g_{\alpha\beta} - \partial_\beta g_{\alpha\gamma} = 0, \tag{8.302}$$

which gives

$$\Gamma_\beta{}^\epsilon{}_\gamma = \frac{1}{2} g^{\epsilon\sigma}\big(\partial_\beta g_{\sigma\gamma} + \partial_\gamma g_{\sigma\beta} - \partial_\sigma g_{\beta\gamma}\big)\,. \tag{8.303}$$

8.6.1 Example: metric-preserving connections with torsion

Repeat the above exercise for a connection with torsion to find that the connection coefficients are now given by

$$\Gamma_\beta{}^\epsilon{}_\gamma = \frac{1}{2} g^{\epsilon\sigma}\big(\partial_\beta g_{\sigma\gamma} + \partial_\gamma g_{\sigma\beta} - \partial_\sigma g_{\beta\gamma}\big) + T_\beta{}^\epsilon{}_\gamma - T_{\alpha\gamma}{}^\epsilon - T_{\gamma\alpha}{}^\epsilon. \tag{8.304}$$

The additional term

$$K_\beta{}^\epsilon{}_\gamma = T_\beta{}^\epsilon{}_\gamma - T_{\alpha\gamma}{}^\epsilon - T_{\gamma\alpha}{}^\epsilon \tag{8.305}$$

is sometimes called the *contorsion tensor*.

8.6.2 Example:Weyl Connections

These preserve angles under parallel transport but not the metric. Thus

$$\nabla_a g_{bc} = A_a g_{bc}, \tag{8.306}$$

for some one-form A_a. Thus

$$\Gamma_b{}^c{}_d = \left\{ {}_b{}^c{}_d \right\} - \frac{1}{2}\big(\delta^c_b A_d + \delta^c_d A_b - g_{db} g^{ce} A_e\big). \tag{8.307}$$

Note that, if $g_{ab} = \Omega^2 \tilde{g}_{ab}$, then $A_a = 2\frac{\partial_a \Omega}{\Omega}$. Conversely, if $A_a = 2\frac{\partial_a \Omega}{\Omega}$, then we can find a conformally related metric which is invariant under parallel transport. A Weyl connection of that type is rather trivial because it can be eliminated by a conformal 'gauge'transformation. A gauge-invariant measure of triviality in this sense is the second contraction

$$S_{db} = R^a{}_{adb} = n\big(\partial_d A_b - \partial_b A_d\big). \tag{8.308}$$

Weyl suggested that one might generalize Einstein's theory by endowing spacetime with what we now call a Weyl connection in which measuring rods moved from A to B along a path γ suffer parallel transport. Einstein pointed out that if it were non-trivial, then the measuring rods would not return to their original size on returning to A by some other curve unless the curvature $R^a{}_{abd} = S_{bd}$ vanishes. This contradicts the observed fact that measuring rods are made from atoms and atoms have quite definite sizes.

In fact, atomic units of length and time are constructed from the Bohr radius

$$R_B = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2}. \tag{8.309}$$

Even if the charge on the electron e or the mass of the electron m_e varied with position, under parallel transport the Bohr radius, and hence the metric would undergo a trivial conformal change which could be eliminated by a conformal rescaling.

Although Weyl rapidily abandoned his theory that the equations of physics should be gauge-invariant in the sense of independent of units of length, the idea of gauge-invariance resurfaced soon after when quantum mechanics was discovered and it was realized that in the presence of an electromagnetic field, one should replace $\partial_a\Psi$ by $\left(\partial_a - i\frac{e}{\hbar}A_a\right)\Psi$ in the Schrödinger equation, where now A_b is the electromagnetic vector potential. Parallel transport of an electron now results in a change of the phase of its wave function Ψ. It is now believed that the equations of physics are exactly gauge-invariant in this sense.

8.7 Holonomy and Autoparallels

If $c(\lambda)$ is a curve given by $x^\alpha(\lambda)$ and $T^\alpha = \frac{dx^\alpha}{d\lambda}$ its tangent vector we define the *absolute derivative* of a vector V^α along $c(\lambda)$ by

$$\boxed{\frac{DV^\alpha}{D\lambda} = T^\beta\nabla_\beta V^\alpha\,.} \tag{8.310}$$

We often denote $\nabla_\beta V^\alpha$ by $V^\alpha{}_{;\beta}$ and so

$$\frac{DV^\alpha}{D\lambda} = V^\alpha_{;\beta}T^\beta\ . \tag{8.311}$$

We say that V^α is *parallely transported along* γ if

$$\boxed{\frac{DV^\alpha}{D\lambda} = 0.} \tag{8.312}$$

That is

$$\frac{dV^\alpha}{d\lambda} + \left(T^\beta\Gamma_\beta{}^\alpha{}_\gamma\right)V^\gamma = 0\ , \tag{8.313}$$

or, infinitesimally,

$$dV^\alpha = -\Gamma_\beta{}^\alpha{}_\epsilon V^\epsilon dx^\beta \tag{8.314}$$

along $c(\lambda)$. This is a linear o.d.e. along $c(\lambda)$ and has a unique solution given the initial value of the vector, $V^\alpha(0)$. However *parallel transport is path dependent*, it depends on $c(\lambda)$ and two curves c and c' joining the same two points in M will have different vectors V^α at the end points.

Note that we could have demanded the apparently weaker condition

$$\frac{DV^\alpha}{D\lambda} = f(\lambda)V^\alpha \tag{8.315}$$

along c for some function $f(\lambda)$, but if

$$V^\alpha = fU^\alpha, \tag{8.316}$$

we have

$$\dot{g}U^\alpha + g\frac{DU^\alpha}{D\lambda} = fgU^\alpha, \tag{8.317}$$

and so, by setting $\frac{\dot{g}}{g} = f$, we get

$$\frac{DU^\alpha}{D\lambda} = 0. \tag{8.318}$$

8.7.1 Holonomy Group of an Affine Connection

Consider smooth closed curves in M, i.e.smooth maps from $S^1 \to M$, which start and finish at some base point $p \in M$. . For convenience choose $\lambda \in (0,1]$ as the curve passes round the circle. Two such curves c and c' can be multiplied, in other words

$$cc'(\lambda) = c'(2\lambda) \qquad \lambda \in (0, \frac{1}{2}] \tag{8.319}$$

$$= c(2\lambda - 1) \qquad \in (\frac{1}{2}, 1]\,. \tag{8.320}$$

Note that we first go around c' and then we go around c. This law of multiplication provides the space of loops based at p which we call $\mathrm{Loop}_p(M)$ with the structure of a non-abelian group with inverse $c^{-1}(\lambda) = c(1-\lambda)$. Parallel transport around each loop gives a representation of this group into $GL(n,\mathbb{R})$ because if we start with vector $V(0)$ and finish with vector $V(1)$ then

$$V(1) = S(c)V(0) \tag{8.321}$$

and clearly

$$S(cc') = S(c)S(c'). \tag{8.322}$$

This representation of $\mathrm{Loop}_p(M)$ is called the holonomy representation $\mathrm{Hol}_p(M,\nabla)$. The dependence on the base point p is not really significant because if we pass to a new base point q then $\mathrm{Hol}_q(M,\nabla)$ is conjugate to $\mathrm{Hol}_p(M,\nabla)$.

For a general affine conection ∇ , the holonomy group will be all of $GL(n,\mathbb{R})$, or at least its connected component. For special connections it may be a proper subgroup. For example for a metric preserving conection such as the Levi-Civita connection the Holonomy group lies in $SO(p,q)$, where the metric has signature (p,q). As we shall see later, the concept of a reduced Holonomy group plays an important role in supergravity theories. A connection is flat if its holonomy group is a discrete subgroup of $GL(n,\mathbb{R})$. If M is simply connected, a the holonomy of a flat connection is completely trivial.

8.8 Autoparallel curves

These are curves along which the tangent vector T^α is parallely transported

$$\boxed{\frac{DT^\alpha}{D\lambda} = 0} \tag{8.323}$$

or

$$\boxed{\frac{d^2x^\alpha}{d\lambda^2} + \Gamma_\beta{}^\alpha{}_\gamma \frac{dx^\gamma}{d\lambda}\frac{dx^\beta}{d\lambda} = 0\,.} \tag{8.324}$$

For the Levi-Civita connection, $\Gamma_\alpha{}^\beta{}_\gamma = \{{}_\alpha{}^\beta{}_\gamma\}$ we recover our old definition of a *geodesic*.

Note that we could have demanded the apparently weaker condition that

$$\frac{DT^\alpha}{D\lambda} = f(\lambda)T^\alpha, \tag{8.325}$$

for some $f(\lambda)$. However if we change parameter

$$\lambda \to \tilde\lambda = \tilde\lambda(\lambda), \tag{8.326}$$

we find

$$T^\alpha = g\tilde T^\alpha, \tag{8.327}$$

where

$$g = \frac{d\tilde\lambda}{d\lambda}, \qquad \tilde T^\alpha = \frac{dx^\alpha}{d\tilde\lambda}, \tag{8.328}$$

and we can now use our previous remark to set $\frac{\dot g}{g} = f$ and find a new parameter $\tilde\lambda$ such that

$$\frac{D\tilde T^\alpha}{D\tilde\lambda} = 0, \tag{8.329}$$

and we get back to our previous condition. Such a choice of parameter is called an *affine parameter* and it is unique up to an affine transformation

$$\tilde\lambda \to a\tilde\lambda + b, \qquad a,b \in \mathbb{R}. \tag{8.330}$$

8.8.1 Example: Auto-parallels on Lie Groups

The orbits and tangent vectors of of one parameter groups of left translations $g_a(t)$ and their tangent vectors $R_a = \frac{dg_a}{dt}$ are clearly invariant under right translations. Thus they must be invariant under what we called the right connection ∇_1. In other words they are autoparallels of the right connection. Similarly the orbits of one parameter families of right translations are auto=parallels of the left connection.

8.9 Examples of connections with torsion: Tele-parallelism

Given any basis $\mathbf{e}^a$ of vectors we can always define an associated connection by defining parallel transported vectors to have constant components in this frame. In other words in this basis

$$\Gamma_a{}^b{}_c = 0. \tag{8.331}$$

In general such a connection will have torsion given for example by Cartan's first structural equation as

$$T_a{}^b{}_c \, \mathbf{e}^a \wedge \mathbf{e}^c = d\mathbf{e}^b \,. \tag{8.332}$$

If the components of the metric in this frame

$$g_{ab} = g(\mathbf{e}_a, \mathbf{e}_b) \tag{8.333}$$

then this connection will be metric preserving with respect to that metric

Connections of this type,whether metric preserving or not provide a global notion of parallelism (as long as the frame is defined everwhere in M) called Tele-parallelism. A famliar example is given by the *Loxodromic connection* on the sphere S^2 envisged as the surface of the earth. For this connection a vector is parallely transported along a ship's course if it has constant components in the dyad

$$\mathbf{e}^1 = d\theta \,, \qquad \mathbf{e}^2 = \sin\theta d\phi \,. \tag{8.334}$$

Autoparallels of the loxodromic connection are called *loxodromic curves* by mathematicians or *rhumb lines* by sailors. They make a constant angle with the meridians and hence spiral into the north or south pole where the loxodromic connection has singukarities at which the reader may check that torsion diverges.

8.10 Projective Equivalence

This concept is similar to that of conformal equivalence except that we focus on auto-parallels. We say two linear affine connections $\Gamma_\beta{}^\gamma{}_\delta$ and $\tilde\Gamma_\beta{}^\gamma{}_\delta$ are *projectivley equivalent* if they share the same autoparallel paths[7], not necessarily with the same affine parameter. It follows that

$$\Gamma_{(\beta}{}^\gamma{}_{\delta)} = \tilde\Gamma_{(\beta}{}^\gamma{}_{\delta)} + \delta^\gamma_\beta A_\delta + \delta^\delta_\gamma A_\beta, \tag{8.335}$$

for some co-vector field A_β.

A *projective transformation* or *collineation* takes auto-parallel paths to auto-parallel paths.

The basic example is constructed from straight lines in $\mathbb{R}^n$.

Globally one should consider $\mathbb{RP}^n$. To describe this, introduce homogeneous coordinates X^α, $\alpha = 0, 1, \ldots, n$. We can think of $\mathbb{R}^n$ as the hyperplane Π given by $X^0 = 1$. Straight lines correspond to the intersections of 2-planes through the origin with the hyperplane Π. Acting with $SL(n+1, \mathbb{R})$ in the obvious way, will take straight lines to straight lines. However, some $SL(n+1, \mathbb{R})$ transformations will take straight lines to straight lines at 'infinity', i.e. to 2-planes which do not intersect the hyperplane Π. Moreover while almost all straight lines intersect once, some which are pararallel do not intersect at all. To obtain a more symmetrical picture, one adds extra points at infinity to Π. One defines $\mathbb{RP}^n$ as the set of lines through the origin in $\mathbb{R}^{n+1}$, i.e. $(n+1)$-tuples X^α identified such that $X^\alpha \equiv \lambda X^\alpha$, $\lambda \neq 0$. Clearly $GL(n+1, \mathbb{R})$ takes all straight lines to all straight lines but we need to factor by the action of $\mathbb{R} \setminus 0$ to get an effective action of $PSL(n+1, \mathbb{R})$. One may also check that every pair of distinct straight lines intesect once and only once.

Clearly, $S^n \subset \mathbb{R}^n$ may be mapped onto the set of directions through the origin. However, anti-podal points on S^n must be identifed since $X^\alpha \equiv -X^\alpha$. Under this $2-1$ mapping great circles, i.e. geodesics

[7] It is convenient to define a path as the image of a curve, i.e. to throw away the information about the parametrization.

on S^n map to straight lines in $\mathbb{RP}^n$. As we have seen, by considering Maxwell's lens, every distinct pair of geodesics on S^n intersect twice. On $\mathbb{RP}^n$ these two intersections are identified.

Of course these remarks also apply to de-Sitter or anti-de-Sitter spacetimes which are projectivley flat and which admit coordianate systems, called Beltrami coordinates in which the geodesics are straight lines.

8.10.1 Example: metric preserving connections having the same geodesics

A metric preserving connection with torsion $T_\alpha{}^\beta{}_\gamma$ is projectively equivalent to the Levi-Civita connection if and only if the torsion is totally antisymmetric

$$T_{\alpha\beta\gamma} = T_{[\alpha\beta\gamma]}. \tag{8.336}$$

Connections of this type frequently arise in supergravity and string theory. For example, $T_{\alpha\beta\gamma}$ may be a closed 3-form such as the Kalb-Ramond field strength. Another example is provided by left or right tanslation on a semi-simple Lie group.This provides two flat metric preserving connections with torsion.

9 The Curvature Tensor

Parallel transport is non-commutative. A measure of this is provided by the curvature tensor, called the Riemann tensor in the case of the Levi-Civita connection. We define it as a map $T(M)\otimes T(M)\wedge T(M) \rightarrow T(M)$ given by the formula

$$R(U,V)Z = \nabla_U\nabla_V Z - \nabla_V\nabla_U Z - \nabla_{[U,V]}Z\,. \tag{9.337}$$

That this expression defines a tensor of type $\binom{1}{3}$ it suffices to check that for functions f, g, h,

$$R(fU, gV)hZ = fghR(U,V)Z. \tag{9.338}$$

This is a routine, albeit tedious, caculation which will be left to te reader.

One may think of the curvature as a matrix valued two-form since

$$R(U,V)Z = -R(V,U)Z.. \tag{9.339}$$

Becaues

$$\nabla_{[U,V]}Z = (\nabla_U V - \nabla_V U - T(U,V))Z, \tag{9.340}$$

in components the definition reads

$$R^\alpha{}_{\sigma\mu\nu}Z^\sigma = (\nabla_\mu\nabla_\nu - \nabla_n\nabla_m)Z^\alpha + T_\mu{}^\sigma{}_\nu\nabla_\sigma Z^\alpha \tag{9.341}$$

which is sometimes referred to as the Ricci-Identity.

In a general basis, if, acting on functions,$\partial_a f = \mathbf{e}_a f$, one has

$$R^a{}_{bcd} = \partial_a\Gamma_d{}^a{}_b + \Gamma_c{}^a{}_e\Gamma_d{}^e{}_b - \frac{1}{2}D_c{}^e{}_d\Gamma_e{}^a{}_b - (c \leftrightarrow d)\,. \tag{9.342}$$

In a coordinate basis we have

$$R^\alpha{}_{\beta\gamma\delta} = \partial_\alpha\Gamma_\delta{}^\alpha{}_\beta + \Gamma_\gamma{}^\alpha{}_\epsilon\Gamma_\delta{}^\epsilon{}_\beta - (\gamma \leftrightarrow \delta)\,. \tag{9.343}$$

Equivalently one may adopt a differential form notation.

$$R^a{}_b = d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b \tag{9.344}$$

where the $\mathfrak{g}(n,\mathbb{R})$- valued two-form

$$R^a{}_b = \frac{1}{2}R^a{}_{bcd}\mathbf{e}^c \wedge \mathbf{e}^d\,. \tag{9.345}$$

This formula is called *Cartan's second structural equation.* The reader is invited to verify that under a change of basis

$$R \rightarrow \Lambda R\Lambda^{-1}\,. \tag{9.346}$$

9.1 Flat coordinates for flat torsion-free connections

If the curvature and torsion of an affine connection bot vanish, then there exist coordinates y^a in a local neighbourhood $U \subset M$ such that

(i) the connection cofficients vanish, $\Gamma_a{}^b{}_c = 0$.

(ii) there is a basis such that $\mathbf{e}_a = \frac{\partial}{\partial y^a}$, and a dual basis $\mathbf{e}e^a = dy^a$,

(iii) Autoparallels are straight lines of the form $y^a = T^a(0)\lambda + y^a(0)$ $T^a(0)$ and $y^a(0)$ are constants.

The first step is to construct a set of covariantly constant vector fields $\mathbf{e}_a$ such that $\nabla \mathbf{e}_a = 0$. This entails solving

$$\frac{\partial e^\beta_a}{\partial x^\alpha} + \Gamma_\alpha{}^\beta{}_\gamma e^\gamma_a = 0. \tag{9.347}$$

in some coordinate system x^α say. By taking mixed partials one sees that the integrability condition is

$$R^\beta{}_{\delta\gamma\alpha} = 0. \tag{9.348}$$

The vanishing torsion condition now implies. the vanishing of the Lie bracket

$$.[\mathbf{e}_a, \mathbf{e}_b] = 0\,, \tag{9.349}$$

and that the dual basis satisfies

$$d\mathbf{e}^a = 0. \tag{9.350}$$

We can now either use Froebenius or the Poincaré lemma to obtain the new coordianates y^a. In these coordinates, unlike in the original coordinates x^α, the components of the affine connection must vanish. It follows that the autoparallels are of the form stated.

9.1.1 Example: Beltrami coordinates for S^n and De-Sitter spacetimes

If we think of S^n embedded in $\mathbb{E}^{n+1}$ as

$$(X^0)^2 + (X^i)^2 = 1\,, \quad i = 1, 2, \ldots, n \tag{9.351}$$

we can project radially onto the plane $X^0 = 1$ by setting

$$X^i = \frac{x^i}{\sqrt{1+x^2}} \qquad X^0 = \frac{1}{\sqrt{1+x^2}}\,, \tag{9.352}$$

so that

$$x^i = X^i/X^0. \tag{9.353}$$

In this chart (we would need 8 to cover the entire sphere) the metric is

$$ds^2 = \frac{dx^i dx^i}{1+x^2} - \frac{(x_i dx^i)^2}{(1+x^2)^2}. \tag{9.354}$$

Since geodesics are the intersections with S^n of two-planes through the origin of $\mathbb{E}^{n+1}$, they will be straight lines in the Beltrami coordinates x^i. This may be verified directly from the metric. Now the Levi-connection for the metric is not flat, but S^n is projectively flat (we showed that the projective tensor vanishes earlier) and there is a projectively equivalent connection whose components vanish in Beltrami coordinates. In other words the connection takes the form

$$\left\{ {}_i{}^j{}_k \right\} = \delta^k_i A_j + \delta^k_j A_i \tag{9.355}$$

for some one-form A_i.

The adaption of this discussion to the de-Sitter metrics is straightfoward. The physical significance is as follows. Judged solely by the motion of freely falling particles, Minkowski spacetime and the De-Sitter spacetimes are locally indistinguishable. The same is true when judged by the motion of light rays, i.e. null geodesics, since both are conformally flat. It is only the additional information that comes from measuring lengths that allows one to distinguish them.

9.1.2 Bianchi Identities

For a general affine connection one may star with the expression for the curvature tensor in a natural basis we have of course.

$$R^{\alpha}{}_{\beta\mu\nu} = -R^{\alpha}{}_{\beta\nu\mu}\,, \tag{9.356}$$

but also the *first and second Bianchi identities* **check**

$$\boxed{\begin{array}{rcl} R^{\alpha}{}_{\beta\mu\nu} - \nabla_{\beta} T_{\mu}{}^{\alpha}{}_{,\nu} + \quad \text{cyclic} & = & 0 \\ \nabla_{\lambda} R^{\alpha}{}_{\beta\mu\nu} + T_{\mu}{}^{\sigma}{}_{\nu} R^{\alpha}{}_{\beta\sigma\lambda} + \quad \text{cyclic} & = & 0\,. \end{array}} \tag{9.357}$$

9.1.3 Ricci Tensor

This is may be defined without any metric by by contraction on the first and third

$$R_{\alpha\beta} = R^{\sigma}{}_{\alpha\sigma\beta} \tag{9.358}$$

9.1.4 The other contraction

In general, parallel transport does not preserve volume (the holonomy lies outside $SL(n,\mathbb{R}) \subset GL(n,\mathbb{R})$)and the other constraction is non-zero. Let

$$S_{\mu\nu} = R^{\sigma}{}_{\sigma\mu\nu}\,. \tag{9.359}$$

The first and second Bianchi identities become

$$R_{\alpha\beta} - R_{\beta\alpha} = S_{\alpha\beta} + \quad \text{torsion} \quad \text{terms} \tag{9.360}$$

$$\nabla_{\gamma} S_{ab} + \quad \text{cyclic} = \quad \text{torsion} \quad \text{terms.} \tag{9.361}$$

In particular if the connection is torsion free one obtains a closed two-form S_{ab} which may be interpreted as the piece of the curvature associated with parallel transport of the volume form.

9.1.5 Example: Weyl connections

In this case

$$S \propto dA, \tag{9.362}$$

where

$$\nabla g = A \otimes g,. \tag{9.363}$$

10 Integration on manifolds: Stokes' Theorem

The basic idea is that one can only integrate *p-forms* over *p-chains*. Roughly speaking, a p-chain is a p-dimensional submanifold, which may be regarded as the sum of a set of *p-cubes* (or alternatively of *p-simplices*). Each p-cube is the image in M under some map $\phi : \mathbb{R}^n \to M$ of a standard p-cube in $\mathbb{R}^n$. In other words, a p-cube C pushes foward to M to give a curvilinear p-cube $C_* = \psi_* C$. In order to integrate a p-form ω in M over the p-chain C_* in M, we pull ω back to $\mathbb{R}^n$ to give $\omega^* = \phi^* \omega$ and integrate it over C using the standard integration procedure, i.e by means of the definition

$$\int_{C_*} \omega = \int_C \omega^*\,. \tag{10.364}$$

On each p-cube $C \in \mathbb{R}^n$ we may introduce coordinates $0 \le x^1, x^2, \ldots, x^p \le 1$, and we define:

$$\int_C \omega^* = \frac{1}{p!} \int_C \omega^*_{\mu_1\mu_2\ldots\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} \tag{10.365}$$

$$= \int \cdots \int \omega_{12\ldots p} dx^1 dx^2 \cdots dx^p\,. \tag{10.366}$$

A general p-chain is a sum of p-cubes: $C = \sum_i C_i$. To each p-chain C we asscociate a *boundary* $\partial C = \sum_i \partial C_i$, where adjacent cubes contribute with the opposite sign and hence cancel. Thus the boundary of a boundary vanishes:

$$\partial^2 C = 0\,. \tag{10.367}$$

By linearity and because d commutes with pull-back, *Stokes' theorem* now reads

$$\boxed{\int_{C_*} d\omega = \int_{\partial C_*} \omega \ .} \tag{10.368}$$

In other words, we only need only check this formula on a standard p-cube in $\mathbb{R}^n$. To check that, recall that

$$d\omega^*_{\mu_1\mu_2\dots\mu_{p+1}} = \frac{(p+1)!}{p!}\partial_{[\mu_1}\omega^*{}_{\mu_2\dots\mu_{p+1}]} \ . \tag{10.369}$$

Thus

$$\int_C \frac{1}{(p+1)!}\, d\omega^*_{\mu_1\dots\mu_{p+1}}\, dx^{\mu_1}\wedge\dots\wedge dx^{\mu_{p+1}} \tag{10.370}$$

$$= \int_C \frac{(p+1)!}{(p+1)!}\frac{1}{p!}\,\partial_{[\mu_1}\omega^*{}_{\mu_2\dots\mu_{p+1}]}\, dx^{\mu_1}\wedge\dots\wedge dx^{\mu_{p+1}} \tag{10.371}$$

$$= \int_{\partial C}\frac{1}{p!}\,\omega^*_{\mu_1\mu_2\dots\mu_p} dx^{\mu_1}\wedge\dots\wedge dx^{\mu_p} \ , \tag{10.372}$$

where the last line follows from integration by parts.

As an example, let $F = \sin\theta\, d\theta\wedge d\phi$:

$$\int_{S^2} F = \int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta d\phi = \big[\phi\big]_0^{2\pi}\big[-\cos\theta\big]_0^{\pi} = 4\pi \ . \tag{10.373}$$

As a corollary we see that F is not exact, i.e. $F\neq dA$, because if it were, then

$$\int_{S^2} F = \int_{S^2} dA = \int_{\partial S^2} A = 0 \ , \tag{10.374}$$

because S^2 has no boundary: $\partial S^2 = \varnothing$.

More generally, the volume form of any compact manifold is closed but never exact.

10.0.6 The divergence operator

If we define an inner product on $\Omega^*(M)$ by

$$(\!(\alpha,\beta)\!) = \int_M (\alpha,\beta)\eta = \int_M *\alpha\wedge\beta = \frac{1}{p!}\int \alpha_{\mu_1\dots\mu_p}\alpha^{\mu_1\dots\mu_p}\sqrt{|g|}d^n x \ , \tag{10.375}$$

then the formal adjoint[8] δ of d is defined by

$$\boxed{(\!(\alpha, d\beta)\!) = (\!(\delta\alpha,\beta)\!) \ .} \tag{10.376}$$

In components,

$$\boxed{\delta\alpha_{\mu_1\dots\mu_p} = -\nabla^{\mu_0}\alpha_{\mu_0\mu_1\dots\mu_p} \ .} \tag{10.377}$$

Now if $\lambda\in\Omega^p(M)$, then $**\lambda = (-1)^t(-1)^{p(n-p)}\lambda$, and

$$\lambda\wedge\mu = (*\lambda,\mu)\eta \ , \qquad \Rightarrow \qquad *\lambda\wedge\mu = (-1)^t(-1)^{p(n-p)}(\lambda,\mu)\eta \ , \tag{10.378}$$

$$\Rightarrow \qquad (\alpha, d\beta)\eta = (-1)^t(-1)^{(p+1)(n-p-1)} *\alpha\wedge d\beta \ , \tag{10.379}$$

for $\alpha\in\Omega^{p+1}(M)$. Moreover,

$$d(*\alpha\wedge\beta) = d(*\alpha)\wedge\beta + (-1)^{n-p-1} *\alpha\wedge d\beta \ , \tag{10.380}$$

and hence up to a boundary term,

$$\int_M d(*\alpha)\wedge\beta = (-1)^{n-p}\int_M *\alpha\wedge d\beta \ . \tag{10.381}$$

Thus

$$\int_M (*d*\alpha,\beta)\eta = (-1)^{n-p}(-1)^t(-1)^{p(n-p)}\int_M(\alpha,d\beta)\eta \ , \tag{10.382}$$

and up to a sign,

$$\boxed{\begin{aligned} \delta &= \pm *d* \ , \\ \Rightarrow \delta^2 &= 0 \ . \end{aligned}} \tag{10.383}$$

[8]It is a *formal* adjoint because we are not worrying about boundary terms.

10.1 The Brouwer degree

Let $\phi : M \to N$ be a map between two manifolds of the *same* dimension. Let N be equipped with a volume form η_N. The *Brouwer degree* of the map ϕ is an integer given by

$$\boxed{\deg \phi = \frac{\int_M \phi^* \eta_N}{\int_N \eta_N} \ .} \tag{10.384}$$

Intuitively, $\deg \phi$ is the number of times that the map ϕ 'wraps' or 'winds' the manifold M over N. It is also the number of inverse images of the map at a generic point $q \in N$ counted with respect to orientation, that is subtracting those points for which ϕ reverses orientation from those for which ϕ preserves it.

10.1.1 The Gauss linking number

Suppose that γ_1 and γ_2 are two connected, closed curves in $\mathbb{E}^3$ which do not intersect each other, i.e. $\gamma_1(t) \neq \gamma_2(s)$ for all $t, s \in S^1$. Thus

$$\begin{aligned} \gamma_1 : S^1 \to \mathbb{E}^3 \ , \quad t \mapsto \gamma_1(t) = \mathbf{x}_1(t) \ , \qquad 0 \ \leq \ t < 2\pi, \text{ and} \qquad (10.385) \\ \gamma_2 : S^1 \to \mathbb{E}^3 \ , \quad s \mapsto \gamma_2(s) = \mathbf{x}_2(s) \ , \qquad 0 \ \leq \ s < 2\pi \ . \qquad (10.386) \end{aligned}$$

The unit vector

$$\mathbf{n}(t, s) = \frac{\mathbf{x}_2(s) - \mathbf{x}_1(t)}{|\mathbf{x}_2(s) - \mathbf{x}_1(t)|} \tag{10.387}$$

gives a map $\phi \colon S^1 \times S^1 \to S^2$, $(t, s) \mapsto \mathbf{n}(t, s)$ called the *Gauss Map.* The *Gauss linking number* $\mathrm{Link}\,(\gamma_1, \gamma_s)$ is defined by

$$\mathrm{Link}\,(\gamma_1, \gamma_2) = \deg \phi = \frac{1}{4\pi} \int_{S^1 \times S^1} \phi^* \eta \ , \tag{10.388}$$

where η is the volume form on S^2. If $\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1$, then

$$\eta = \frac{z\, dx \wedge dy + y\, dz \wedge dx + z\, dy \wedge dz}{(x^2 + y^2 + z^2)^3} \ , \tag{10.389}$$

and

$$\begin{aligned} \mathrm{Link}\,(\gamma_1, \gamma_2) \ &= \ \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \frac{\big((\mathbf{x}_2 - \mathbf{x}_1) \times \mathbf{dx}_1(t)\big) . \,\mathbf{dx}_2(s)}{|\mathbf{x}_2 - \mathbf{x}_1|^3} \qquad (10.390) \\ &= \ \oint_{\gamma_1} \mathbf{B}_2(\mathbf{x}_1) . \,\mathbf{dx}_1 \ , \qquad (10.391) \end{aligned}$$

where

$$\mathbf{B}_2(\mathbf{x}) = \frac{1}{4\pi} \oint_{\gamma_2} \frac{(\mathbf{x}_2 - \mathbf{x}) \times \mathbf{dx}_2}{|\mathbf{x}_2 - \mathbf{x}|^3} \tag{10.392}$$

is, by the *Biot-Savart Law*, the magnetic field due to a unit current along γ_2.

The reader should have no difficulty generalizing this example to the linking of a p-brane and a q-brane in $p + q + 1$ dimensions.

10.1.2 The Gauss-Bonnet Theorem

Suppose that S is a closed, connected, orientable 2-surface immersed into $\mathbb{E}^3$, given in any local coordinate neighbourhood on Σ by

$$\mathbf{x} = \mathbf{x}(u, v) \ , \tag{10.393}$$

where u, v are local coordinates on Σ and

$$\mathbf{n}(u, v) = \frac{\partial_u \mathbf{x} \times \partial_v \mathbf{x}}{|\partial_u \mathbf{x} \times \partial_v \mathbf{x}|} \tag{10.394}$$

is the unit normal. The normal provides the *Gauss map* $\phi \colon \Sigma \to S^2$. A calculation, which the reader is invited to do, shows that

$$\phi^* \eta_{S^2} = K \eta_\Sigma \ , \tag{10.395}$$

where K is the Gauss curvature of Σ, whose induced metric is

$$ds^2 = \partial_i \mathbf{x}\, \partial_j \mathbf{x}\, dx^i\, dx^j = g_{ij}\, dx^i\, dx^j = E\, du^2 + 2F\, du\, dv + G\, dv^2 \ . \tag{10.396}$$

That is, the Riemann tensor is

$$R_{ijmn} = K(g_{im} g_{jn} - g_{in} g_{jm}) \ . \tag{10.397}$$

Consideration of the Brouwer degree of ϕ then leads to an extrinsic version of the *Gauss-Bonnet Theorem*:

$$\frac{1}{2}\chi(\Sigma) = 1 - g(\Sigma) = \frac{1}{4\pi} \int_K \sqrt{EG - F^2}\, du\, dv \in \mathbb{Z} \ , \tag{10.398}$$

where χ is the Euler number and $g(\Sigma)$ is the *genus* or number of handles of the surface Σ.

The reader should have no difficulty generalizing this argument to a p-brane in $p+1$ dimensions.

10.2 A general framework for classical field and brane theories

Many physical theories can be cast in a — possibly procrustean — framework based on the space of maps $\mathrm{Map}(\Sigma, M) = \{\phi \colon \Sigma \to M\}$ from a manifold Σ to another manifold M, say.

10.2.1 Particles

The simplest example is when $\Sigma = \mathbb{R}$ and we obtain a point particle moving in a curved spacetime M, with $\phi \colon \mathbb{R} \to \mathbb{R}$ given by $x^\mu(\lambda)$, $\lambda \in \mathbb{R}$ being a parameter on the *world-line* of our particle. $\mathrm{Map}(\mathbb{R}, M)$ is then just *Feynman's space of histories* of our particle.

The action (for massless particles) is

$$S = \frac{1}{2} \int_{\mathbb{R}} N(\lambda)\, d\lambda\, g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \ . \tag{10.399}$$

Variation with respect to the Lagrange multiplier $N(\lambda)$ gives the *massless condition* or *constraint*:

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \ . \tag{10.400}$$

Mathematically, we may regard $N^{-2} d\lambda^2$ as a *metric* on Σ, and $N^{-1} d\lambda$ is the volume form or *einbein* on Σ, and

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = (\phi^* g)_{\lambda\lambda} \tag{10.401}$$

is the pull-back of the metric $g_{\mu\nu}$ on M to Σ.

10.2.2 Strings, membranes and p-branes

For *strings* we take Σ to be 2-dimensional. For *membranes* Σ is three-dimensional. If $\dim \Sigma = p+1$, we get a *p-brane*[9]. Now let λ^i, $i = 1, 2, \ldots, p+1$ be local coordinates on Σ and γ_{ij} a metric on Σ. The pull-back to Σ of the spacetime metric on M is given by

$$(\phi^* g)_{ij} = g^*_{ij} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda^i} \frac{\partial x^\nu}{\partial \lambda^j} \ , \tag{10.402}$$

and

$$\gamma^{ij} g^*_{ij} = \gamma^{ij} g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda^i} \frac{\partial x^\nu}{\partial \lambda^j} \tag{10.403}$$

is the *trace* with respect to γ of the pull-back of g. The volume form on Σ is

$$\eta_\gamma = \sqrt{\det(\gamma_{ij})}\, d^{p+1}\lambda \ , \tag{10.404}$$

and a suitable action is

$$S = \frac{1}{2} \int_\Sigma \sqrt{\gamma}\, d^{p+1}\lambda\, \gamma^{ij} g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda^i} \frac{\partial x^\nu}{\partial \lambda^j} \tag{10.405}$$

$$= \frac{1}{2} \int_\Sigma \big(\mathrm{Tr}_\gamma\, \phi^*\big) \eta_\gamma \ . \tag{10.406}$$

[9] p is the spatial dimension of the object and $p+1$ the dimension of the *world-sheet* or *world-volume*. Sometimes, when no notion of time is involved, a p-brane is taken to be the same as a p-dimensional immersed submanifold.

In string theory this is called the *Polyakov form* of the action. The independent fields to be varied are the embedding ϕ, i.e. one varies the world-volume scalar fields $x^\mu(\lambda^i)$, and the world-volume metric γ_{ij}. A different strategy is to set

$$\gamma_{ij} = g^*_{ij} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda^i} \frac{\partial x^\nu}{\partial \lambda^j} \ , \tag{10.407}$$

that is, one identifies the world-volume metric with the metric induced on the world-volume from the spacetime metric via pull-back. The *Dirac-Nambu-Goto action* is now taken to be

$$S = \frac{1}{2} \int_\Sigma \eta_{\phi^* g} = \frac{1}{2} \int_\Sigma \sqrt{\left| g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda^i} \frac{\partial x^\nu}{\partial \lambda^j} \right|} \, d^{p+1}\lambda \ . \tag{10.408}$$

This is to be varied with respect to the embedding (or immersion) ϕ, that is with resepct to the world-volume fields $x^\mu(\lambda^i)$. The reader is invited to carry out that exercise. In the mathematical literature solutions of the Euler-Lagrange equations are called *minimal submanifolds*, even though they may in fact only be stationary points of the Dirac-Nambu-Goto action functional.

10.2.3 Non-linear sigma models

So far we have been thinking of the target M of the map ϕ as our spacetime and the domain Σ as some particle or extended object moving in it. However, we can always change our minds and think of the domain Σ of the map as our spacetime and of the target M as some sort of field space or internal space. The simplest situation occurs when we think of the metric on Σ as fixed. For example, we could take Σ to be Minkowski spacetime $\mathbb{E}^{n-1,1}$, and we take the Polyakov form of action given above in (10.406). We then get the action of what in physics are called *non-linear sigma models*, the solutions of whose equations of motion are called in mathematics *harmonic maps* if the metric on Σ is positive definite and *wave maps* if it is Lorentzian. The terminology arises because if $(\Sigma, \gamma) = \mathbb{E}^n$ or $(\Sigma, \gamma) = \mathbb{E}^{n-1,1}$, then the equation of motion reduces to the linear Laplace equation or to the wave equation, respectively. If the target metric is not flat, the the equations of motion are non-linear. Historically, the name non-linear sigma model arose in pion physics. One has four fields π_i, σ which are subject (in suitable units) to a constraint

$$\pi_i \pi_i + \sigma^2 = 1 \ . \tag{10.409}$$

The target space may thus be identified with the round 3-sphere S^3 with the standard round metric. One may also think of S^3 as the group manifold of $SU(2)$. The high degree of symmetry, $SO(4) \equiv SU(2) \times SU(2)/\mathbb{Z}_2$, contains chiral symmetries, and so one also speaks of *chiral models* or *principal chiral models*.

Sigma models, and indeed all geometric theories based on $\mathrm{Map}(\Sigma, M)$ admit in general two types of reparametrization invariance, one of Σ and one of M. In the the metrics admit isometries these give rise, via Noether's theorems, to conserved currents on Σ. For example, the reader is invited to check that in the case of a sigma model, if M admits a Killing vector field K, we can convert it to a *Killing* 1*-form*

$$K_{\flat_g} = K_{\flat_g \mu} dx^\mu = g_{\mu\nu} K^\nu dx^\mu \ , \tag{10.410}$$

and pulling back $K_{\flat_g}$ to Σ gives a conserved current on Σ:

$$J_K = K^*_{\flat_g} \ , \quad d * J_K = 0 \ . \tag{10.411}$$

We can then define a conserved Noether charge

$$Q_K = \int_S * J_K \ , \tag{10.412}$$

where S is a suitable spacelike hypersurface in Σ. Because $d * J_K = 0$, by Stokes' theorem and assuming a suitable behaviour at infinity the charge Q_K will not depend on the choice of the spacelike hypersurface S.

10.2.4 Topological Conservation laws

The conservation of Noether charges depends upon the field equations holding, i.e. the fields must be *on shell*. Topological conservation laws hold independently of field equations and resemble the conservation of magnetic charges, which depend only on Bianchi identies.

Suppose that the spacetime that we are regarding as Σ has dimension n and topologically takes the form

$$\Sigma = S \times \mathbb{R} \ , \tag{10.413}$$

where S is a hypersurface and $t \in \mathbb{R}$ is a time coordinate. The field $\phi(\mathbf{x}, t)$ gives rise to a one-parameter family of maps $\phi_t \colon S \to M$. If M is equipped with a closed $(n-1)$-form ω, we can pull it back to Σ to give a closed $(n-1)$-form

$$\omega^* = \phi_t^* \omega \ , \qquad d\omega = 0 \Leftrightarrow d\omega^* = 0 \ . \tag{10.414}$$

The topological charge

$$Q_\omega = \int_S \omega^* \tag{10.415}$$

will, subject to suitable behaviour at infinity, be independent of t and thus conserved. If M is $(n-1)$-dimensional, we can choose for ω the volume form η_M, in which case the conserved charge is proportional to the Brouwer degree $\deg \phi_t$, which is clearly quantized, in other words, 'discrete, and hence by continuity constant.

Example Consider the *Skyrme model*. This is a non-linear theory of pions $\pi_i(\mathbf{x}, t)$ with higher derivatives. It is based on maps into $SU(2)$ given by $\mathbf{x} \mapsto U(\mathbf{x}) = \exp(i\pi_i \tau_i)$, with τ_i being Pauli matrices. Suppose $\pi_i(\mathbf{x})$ tends to some single value at infinity, independently of direction. Then we may extend the map to S^3, the *one-point compactification* of $\mathbb{E}^3$. Thus we get a map $S^3 \to SU(2) = S^3$, and its Brouwer degree is interpreted as the *Baryon number* in this model.

In fact, the one-point compactification is a *conformal compactification*. Consider stereographic projection $f \colon S^3 \setminus \{\text{north pole}\} \to \mathbb{E}^3$: Explicitly,

$$f \colon (X^1, X^2, X^3, X^4) \mapsto (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r\cos\theta) \ , \tag{10.416}$$
$$X^4 = \cos\chi \ , \qquad \sin\chi = \frac{2r}{4 + r^2} \ .$$

We can pull $U(\mathbf{x})$ back to get a map $f^* U(\mathbf{x}) \colon S^3 \setminus \{\text{north pole}\} \to S^3$, which we extend continuously over the north pole to give our map $\phi \colon S^3 \to S^3$ whose degree we must calculate. A *hedgehog* configuration is one for which $\pi_i = h(r)(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ with $h(0) = 0$ and $h(\infty) = \pi$. It has degree one.

Note that, since stereographic projection preserves angles, the compactification is conformal. In fact, if g_{round} is the standard metric on S^3 and g_{flat} the standard flat metric on $\mathbb{E}^3$, then

$$f^* g_{\text{flat}} = \Omega^2 g_{\text{round}} \ , \tag{10.417}$$

with $\Omega = \frac{1}{2}(4 + r^2)$.

An analogous construction works in Instanton Physics. The Yang-Mills equations in four dimensions are conformally invariant and solutions on $\mathbb{E}^4$ and can be pulled back to S^4 by stereographic projection.

10.2.5 Coupling of p-branes to $(p+2)$-forms

To begin with, consider a charged particle with charge e moving along a world-line Σ in a manifold M with electromagnetic field $F = dA \in \Omega^1(M)$. One adds to the usual action an interaction term

$$S_{\text{int}} = \int_\Sigma e A_\mu dx^\mu = \int_\Sigma A_\mu \frac{dx^\mu}{d\lambda} d\lambda = \int_\Sigma \phi^* A \ , \tag{10.418}$$

where

$$\phi^* A = A_\mu \frac{dx^\mu}{d\lambda} d\lambda \ . \tag{10.419}$$

In this notation we can carry over the same expression of a p-brane Σ with $\dim \Sigma = p+1$ moving in the background of a closed $(p+2)$-form $F + dA \in \Omega^p(M)$. Moreover, S_{int} is obviously invariant under abelian gauge transformations $A \to A + d\Lambda$, $\Lambda \in \Omega^p(M)$ up to a surface term, because under this transformation

$$S_{\text{int}} \ \to \ S_{\text{int}} + e \int_\Sigma \phi^* d\Lambda \tag{10.420}$$
$$= \ S_{\text{int}} + e \int_\Sigma d\phi \, d\Lambda \tag{10.421}$$
$$= \ S_{\text{int}} + e \int_\Sigma \phi^* \Lambda \ . \tag{10.422}$$

The brane equations of motion will thus be gauge-invariant.

In local coordinates, and using indices, we get the rather formidable expression

$$S_{\text{int}} = \frac{e}{(p+1)!} \int_\Sigma \sqrt{\gamma}\, A_{\mu_1 \ldots \mu_{p+1}} \frac{\partial x^{\mu_1}}{\partial \lambda^{i_1}} \cdots \frac{\partial x^{\mu_{p+1}}}{\partial \lambda^{i_{p+1}}}\, \eta^{i_1 \ldots i_{1+p}}\, d\lambda^1 \cdots d\lambda^{p+1} \ , \tag{10.423}$$

which is time-consumimg to type in LaTeX 2_ε, and showing that it is invariant up to a boundary term is rather painful, particularly if the details have to be written out.

The best known case is $p = 1$, which corresponds to string theory. One has a closed 3-form

$$H = dB \in \Omega^3(M) \ , \quad H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\lambda B_{\mu\nu} + \partial_\nu B_{\lambda\mu} \ , \tag{10.424}$$

and

$$S_{\text{string}} = \frac{1}{2} \int_\Sigma \left(g_{\mu\nu} \gamma^{ij} + B_{\mu\nu} \eta^{ij} \right) \frac{\partial x^\mu}{\partial \lambda^i} \frac{\partial x^\nu}{\partial \lambda^j} \sqrt{\gamma}\, d^2\lambda \ . \tag{10.425}$$

10.2.6 Topological defects

In so-called *mean field theory* approaches to condensed matter physics, or in cosmology, the space ϕ in which the *field* or *order parameter* ϕ takes it values is typically non-trivial. The simplest case to consider is the global vortex. For example, in theories involving a $\mathbb{C}$-valued field $\phi(x, y)$ in $2 + 1$ dimensions the energy is

$$\int_{\mathbb{E}^2} dx\, dy \left(\frac{1}{2} |\nabla \phi|^2 + V(\phi) \right) \ , \tag{10.426}$$

and the potential energy $V(\phi)$ may be $SO(2)$-invariant: $V(\phi) = V(|\phi|)$. Local minima of V define a *vacuum manifold* $N \subset M = \mathbb{C}$ which will be a collection of circles about the origin in $\mathbb{C}$.

Example Suppose there is only one such circle, of finite radius, on which $V(\phi)$ attains its absolute lower bound. For example, take $V(\phi) = |\phi|^4/4 - |\phi|^2/2$. The vacuum manifold is the unit circle $|\phi| = 1$. On a circle $S^1_\infty \subset \mathbb{E}^2$ near infinity, finite energy forces $|\phi| = 1$. Thus we get a map $S^1_\infty \to S^1$, whose degree may be interpreted as a conserved vortex number. Note that if the vortex number is non-zero, by an argument identical to the proof of the *fundamental theorem of algebra*, there must be at least one zero of ϕ inside S^1_∞. Such zeros are interpreted as vortex positions.

This example can be trivially generalized to three dimensions with $\mathbb{E}^2$, $\mathbb{C} \equiv \mathbb{R}^2$, $SO(2)$ and S^1 replaced by $\mathbb{E}^3$, $\mathbb{R}^3$, $SO(3)$ and S^2, respectively, and the words *global vortex* replaced by the words *global monopole*.

The defects can also be *local* if the usual gradient operator ∇ is replaced by a covariant derivative operator D with respect to a $U(1)$ or $SU(2)$ gauge group.

10.2.7 Self-interactions of p-forms

The free action for a p-form $F = dA$ is

$$S = -\frac{1}{2} \frac{1}{p!} \int_M (F, F) \eta = -\frac{1}{2} \frac{1}{p!} \int_M F \wedge *F \ . \tag{10.427}$$

The variation is

$$\delta S = \frac{(-1)^{p-1}}{p!} \int_M \delta A \wedge d * F \ , \tag{10.428}$$

which yields the linear equation of motion

$$d *F = 0 \ . \tag{10.429}$$

To get an interacting, i.e. nonlinear theory, we could try an action constructed out of higher exterior powers of F. It would have the interesting property of being 'topologically' independent of any metric on our manifold M. The problem is that such a term would either vanish identically, e.g $F \wedge F \wedge F$ for a 3-form in nine dimensions, or not contribute to the equations of motion because it is exact, for example $F \wedge F \wedge F$ in six dimensions. In fact, in all dimensions we have

$$F \wedge F \wedge F = d(A \wedge F \wedge F) \ . \tag{10.430}$$

However, this suggests that in five dimensions

$$S_{\text{Chern-Simons}} = c_{\text{C-S}} \int_M A \wedge F \wedge F \ , \quad F \in \Omega^2(M) \ , \tag{10.431}$$

where $c_{\text{C-S}}$ is a coupling constant, is a possibility. Under a gauge transformation $A \to A + d\Lambda$,

$$A \wedge F \wedge F \to A \wedge F \wedge F + d(\Lambda \wedge F \wedge F) \ , \tag{10.432}$$

and so the action $S_{\text{Chern-Simons}}$ changes by a surface term, which will not affect the equations of motion.

Now up to a surface term,

$$\delta S_{\text{Chern-Simons}} = 3c_{\text{C-S}} \int_M \delta A \wedge F \wedge F \ . \tag{10.433}$$

Thus the non-linear equations of motion are

$$d \ast F - 6c_{\text{C-S}} F \wedge F = 0 \ . \tag{10.434}$$

In three dimensions the Chern-Simons term gives a mass to the photon. In eleven dimensional supergravity, which has a 4-form, it gives

$$d \ast F \propto F \wedge F \ . \tag{10.435}$$

11 Actions of groups on manifolds

11.1 Groups and semi-groups

A group G is a set $\{g_i\}$ with a multiplication law, $G \times G \to G$ which we write multiplicatively and which satisfies

$$\boxed{\textbf{closure} \qquad (i) \quad g_1 g_2 \in G \quad \forall \quad g_1 \, , g_2 \in G} \tag{11.436}$$

$$\boxed{\textbf{associativity} \qquad (ii) \qquad (g_1 g_2) g_3 = g_1 (g_2 g_3) \quad \forall \quad g_1 \, , g_2 \, , g_3 \in G} \tag{11.437}$$

$$\boxed{\textbf{identity} \qquad (iii) \quad \exists e \in G \quad s.t. \qquad eg = ge = g \quad \forall \quad g \in G} \tag{11.438}$$

$$\boxed{\textbf{inverse} \qquad (iv) \quad \forall g \in G \quad \exists \quad g^{-1} \in G \quad s.t. \quad g^{-1} g = g g^{-1}.} \tag{11.439}$$

If one omits (iv) one gets a *semi-group*. These frequently arise in physics when irreversible processes, such as diffusion are involved.

11.2 Transformation groups: Left and Right Actions

Usually one is interested in *Transformation group* acting on some space X, typically a manifold, such that there is a map $\phi : G \times X \to X$ such that $X \ni x \to \phi_g(x)$ and

$$\boxed{\textbf{left} \quad \textbf{action} \qquad \phi_{g_1} \circ \phi_{g_2} = \phi_{g_1 g_2} , \qquad \phi_e(x) = x.} \tag{11.440}$$

Such an action is called a *left action* of the group G on X or sometimes in physics a *realization*[10]. The action of a map ψ would be called a *right action* if

$$\boxed{\textbf{right} \quad \textbf{action} \qquad \psi_{g_1} \circ \psi_{g_2} = \psi_{g_2 g_1} .} \tag{11.441}$$

[10] Often one speaks of a *non-linear realization* to distinguish the action from a linear action, *linear realization* or *representation* of G on X. We shall treat the special case of representations shortly.Of course the word 'action'has nothing to do with action functionals

In other words the order of group multiplication is reversed under composition. Our conventions for composition are that the second map is performed first, i.e

$$f \circ g(x) = f(g(x)) \tag{11.442}$$

which coincides with the standard order for matrix multiplication in the case of linear maps. In terms of arrows, and if we think of in the standard way of reading input to output from left to right, we have

$$X \to gX \to fX. \tag{11.443}$$

It would, perhaps, be more logical to read from right to left

$$X \leftarrow fX \leftarrow gX. \tag{11.444}$$

It sometimes happens that one needs to consider right actions. This gives rise to various sign changes and so one must be on one's guard. Of course, given a right action ψ , one may always replace it by a left action provided each group element is replaced by its inverse:

$$\psi_{g_1^{-1}} \circ \psi_{g_2^{-1}} = \psi_{g_2^{-1} g_1^{-1}} = \psi_{(g_1 g_2)^{-1}}. \tag{11.445}$$

11.3 Effectiveness and Transitivity

The action (either left or right) of a transformation group G on X is said to be *effective* if $\nexists g \in G$: $\phi_g(x) = x \,\forall,, x \in X$, i.e which moves no point. We shall mainly be interested in effective actions since if the action of $G$is not effective there is a subgroup which is.

The action is *transitive* if $\forall x_1, x_2 \in X, \quad \exists g \in G \quad : \quad \phi_g(x_1) = x_2$. In other words every pair of points in X may be obtained one form the other by means of a group transformation.

If the group element g is unique the action is said to be *simply transitive*. Otherwise the action is said to be *multiply transitive*.

11.4 Actions of groups on themselves

One has three important actions

$$\boxed{\textbf{Left action} \qquad L_h : g \to hg, \qquad h \text{ regarded as moving } g} \tag{11.446}$$

$$\boxed{\textbf{Right action} \qquad R_h : g \to gh, \qquad h \text{ regarded as moving } g} \tag{11.447}$$

$$\boxed{\textbf{Conjugation} \qquad C_h : g \to hgh^{-1}, \qquad h \text{ regarded as moving } g} \tag{11.448}$$

Note that the left and right actions are simply transitive and they commute with one another

$$\boxed{L_{g_1} \circ R_{g_2} = R_{g_2} \circ L_{g_1}\,.} \tag{11.449}$$

By contrast, , which is in fact a left action, commutes with neither and fixes the identity element or origin e of the group G

$$C_h(e) = e \qquad \forall h \in G. \tag{11.450}$$

Conjugation is a natural symmetry of a group. Roughly speaking, it is analogous to change of basis for a vector space. Another way to view it is a sort of rotation about the identity element e. Two subgroups $H_1, H_2 \subset G$ are said to be *conjugate* if there is a $g \in G$ such that for all $h_1 \in H_1$ and $h_2 \in H_2$, $gh_1g^{-1} = h_2$, and thus $gH_1g^{-1} = H_2$. Two subgroups related by conjugation are usually regarded as being essentially identical since one may be obtained from the other by means of a 'rotation'about the identity origin e. A subgroup H is said to *invariant* or *normal* if $gHg^{-1} = H$, $\forall g \in G$.

11.5 Cosets

If H is a subgroup of G it's set of *right cosets* $G/H = \{g \in G : g_1 \equiv g_2 \,\text{iff}\, \exists\, H \in H : g_1 = hg_2\}$. It's set of *left cosets* $H\backslash G = \{g \in G : g_1 \equiv g_2 \,\text{iff}\, \exists\, H \in H : g_1 = g_2 h\}$. Our notation, which may not be quite standard is such that right cosets are obtained by dividing on the right and left cosets by dividing on the left. Note that, because right and left actions commute, L_g the left action of G descends to give a well defined action on a right coset and the right action R_g descends to give a well defined action on a left coset.

11.6 Orbits and Stabilizers

The *orbit* $\mathrm{Orb}_G(x)$ of a transformation group G acting on X is the set of points attainable from x by a transformation, that is $\mathrm{Orb}_G(x) = \{y \in X : \exists\, g \in G : \phi_g(x) = y\}$. Clearly G acts transitively on each of its orbits. If G acts transitively on X, then $X = \mathrm{Orb}_G(x) \forall x \in X$. One may think of the set of right cosets G/H as the orbits in G of H under the left action of G on itself and the right cosets $H\backslash G$ as the orbits of H under the right action of G on itself.

If G acts multiply transitively on X then H is said to be a *homogeneous space* and $\forall\, x \in X$, there is a subgroup $H_x \subset G$ each of whose elements $h \in H_x$ fixes or stabilizes x, i.e. $\phi_h(x) = x \forall\, x\, \in H_x$. The subgroup H_x is called variously the *stabilizer subgroup*, *isotropy subgroup* or *little group* of x. For example, the group $SO(3)$ acts on $S^2 \subset \mathbb{E}^3$ in the standard way and an $SO(2) \equiv U(1)$ subgroup fixes the north pole and south poles. Another $SO(2)$ subgroup in fact a (different) $SO(2)$ subgroup fixes every pair of antipodal points. These subgroups are all related by conjugation. Thus if $y = \phi_g(x)$ so that $x = \phi_{g^{-1}}(y)$ then

$$\phi_g \circ \phi_h \circ \phi_{g^{-1}}(y) = y. \tag{11.451}$$

That is

$$\phi_{ghg^{-1}}(y) = y\,, \quad \Rightarrow H_y = gH_xg^{-1}. \tag{11.452}$$

Thus, up to conjugation, which corresponds to choosing an origin in X, there is a unique subgroup H and one may regard the homogeneous space X as a right coset G/H on which G acts on the left.

11.6.1 Example:De-Sitter and Anti-de-Sitter Spacetimes

De-Sitter spacetime, dS_4 and anti-De-Sitter spacetime AdS_4 are conveniently defined as quadrics in five dimensional flat spacetimes, with signature $(4,1)$ and $(3,2)$ respectively:

$$dS_4 : \qquad (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 - (X^5)^2 = 1. \tag{11.453}$$

$$AdS_4 : \qquad (X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2 - (X^5)^2 = -1. \tag{11.454}$$

They inherit a Lorentzian metric from the ambient flat spacetime which is preserved by $G = SO(4,1)$ or $SO(3,2)$ which acts transitivley with stablizer $H = S0(3,1)$ in both cases. Thus $dS_4 = SO(3,1)/S0(3,1)$ and $dS_4 = SO(3,1)/S0(3,1)$.

11.6.2 Example: Complex Projective space

A physical description of complex projective space, $\mathbb{CP}^n$ is as the space of pure states of an $n+1$ state quantum system with Hilbert space $\mathcal{H} \equiv \mathbb{C}^{n+1}$. Thus a general state is

$$\psi\rangle = \sum_{r=1}^{n+1} Z^r |r\rangle, \tag{11.455}$$

where $|r\rangle$ is a basis for $\mathcal{H}$. Now since $|\Psi\rangle$ and $\lambda|\psi\rangle$, with $\lambda \in \mathbb{C} \setminus 0$ define the same physical state the space of states is paramtrized by $Z^1, Z^2 \ldots Z^{n+1}$ but $(Z^1, Z^2 \ldots Z^{n+1}) \equiv (\lambda Z^1, \lambda Z^2, \ldots, \lambda Z^{n+1})$, with $\lambda \in \mathbb{C} \setminus 0$, which is the standard mathematical definition of $\mathbb{CP}^n$ which is the simplest non-trivial example of a complex manifold with complex charts and locally holomorphic transtion functions

Working in an orthonormal basis, we may attempt to reduce the redundancy of the state vector description by considering normalized states

$$\sum_{r=1}^{n+1} |Z^r|^2 = 1\,. \tag{11.456}$$

This restricts the vectors $\Psi\rangle$ to lie on the unit sphere

$S^{2n+1} = SO(2n+2)/SO(2n+1) = U(n+1)/U(n)$. However one must still take into account the redundant, and unphysical, phase of the state vector. The group $G = U(n+1)$ acts transitively on $\mathbb{CP}^n$ and the stabilizer of each point is $H = U(n) \times U(1)$. We may think of $\mathbb{CP}^n$ as the set of orbits in S^{2n+1} of the action of $U(1)$ given by

$$Z^r \to e^{i\theta} Z^r. \tag{11.457}$$

Each orbit is a circle S^1 and the assignment of each pooint to the orbit through it defines is called the Hopf map

$$\pi : S^{2n+1} \to \mathbb{CP}^n\,. \tag{11.458}$$

The structure we have just described provides perhaps the simplest example of a fibre bundle, in fact a circle bundle. We shall treat them in further detail later

The most basic case is the spin $\frac{1}{2}$ system which has $n = 1$. One has $\mathbb{CP}^1 \equiv S^2 \equiv SU(2)/U(1) \equiv U(2)/U(1) \times U(1)$. We may introduc two coordinate patches, $\zeta = Z^1/Z^2$ and $\tilde{\zeta} = Z^2/Z^1$, one covering the north pole and the other the south pole of S^2. On the overlap $\zeta = \frac{1}{\tilde{\zeta}}$. In fact these are stereographic coordinates.

11.7 Representations

If X is some vector space V and g acts linearly, i.e by endomorphism or linear maps we have a *representation* of G by matrices $D(g)$ such that

$$D(g_1)D(g_2) = D(g_1g_2) \quad \text{etc.} \tag{11.459}$$

The representation said to be *faithful* if the action is effective, i.e. no element of G other than the identity leaves every vector in V invariant. Another way to say this is that a representation is a homomorphism D of G into a subgroup of $GL(V)$ and a faithful representation has no kernel. Given an unfaithful representation we get a faithful representation by taking the quotient of G by the kernel.

11.7.1 Reducible and irreducible representations

The representation is said to be *irreducible* if no vector subspace of V is left-invariant, otherwise it is said to be *reducible*. The representation is said to be *fully reducible* if the complementary subspace is invariant.

A reducible but not fully reducible representation has

$$D(g) = \begin{pmatrix} \cdots & 0 \\ \cdots & \cdots \end{pmatrix} . \tag{11.460}$$

and vectors of the form $\begin{pmatrix} 0 \\ \cdots \end{pmatrix}$ are invarint but those of the form $\begin{pmatrix} \cdots \\ \cdots \end{pmatrix}$ are not. On the other hand for a fully reducible representation one has

$$D(g) = \begin{pmatrix} \cdots & 0 \\ 0 & \cdots \end{pmatrix} . \tag{11.461}$$

and both subspaces are invariant. If V admits a G -invariant *definite* inner product, B say, then any reducible representation is fully reducible.

Proof Use B to project any vector into W, the invariant subspace, and its orthogonal complement $W_\perp$ with respect to B Thus for any $v \in V$ we have $v = v_\parallel + v_\perp$. Now G takes $v_\parallel$ to $\tilde{v}_\parallel \in W$ and $v_\perp$ to $\tilde{v}_\perp \in W_\perp$,

11.7.2 The contragredient representation

If $D(g)$ acts on V then $(D^{-1}(g))^t$ acts naturally on $V^\star$ the dual space. In components

$$D(g) : V^a \to D^a{}_b(g)V^b = \tilde{V}^a . \tag{11.462}$$

$$D^{-1}(g) : \omega_a \to \omega_c(D^{-1})^c{}_a = \tilde{\omega}_a \tag{11.463}$$

$$(D^{-1})(g) : (\omega^t)^a \to (\omega^t)^a = ((D^{-1})^t)^a{}_b(\omega^t)^b . \tag{11.464}$$

Thus

$$\omega_a \; V^a = \tilde{\omega}_a\tilde{V}^a . \tag{11.465}$$

The reason we need to transpose and invert is that otherwise the matrix multiplication would be in reverse order. We would have an anti-homomorphism og G into $GL(V)$ or in our previous language a right action of G on V.

11.7.3 Equivalent Representations

Two representations D and $\tilde{D}$ acting on vector spaces V and W are said to be *equivalent* if there exist an invertible linear map $B : V \rightarrow B$ and $D = B^{-1}\tilde{D}B$.

Example Suppose that $W = V^{\star}$, the dual space of V, and B is a symmetric non-degnerate billinear form or metric invariant under the action of G, then D and the contragredient representation $(D^{-1})^t$ are equivalenet.

Proof Invariance of B under the action of G requires that

$$D^t B D = B. \tag{11.466}$$

Thus

$$D = B^{-1}(D^{-1})^t B. \tag{11.467}$$

In components

$$(D^{-1})^b{}_a = B_{ac} D^c{}_e B^{eb}. \tag{11.468}$$

Thus raising and lowering indices with the metric B, which need not be positive definite, gives equivalent represenions on V and $V^{\star}$. The maps B and B^{-1} are sometimes referred to as the *musical isomorphisms* and denoted by $\flat$ and $\sharp$ respectively.

In fact there is no need for B to be symmetric, $B = B^t$. It often happens that one have a non-degenerate bi-linear form which is anti-symmetric, $B = -B^t$. In that case V is said to be endowed with a *symplectic structure*, or *symplectic* for short.

11.7.4 Representations on functions

The following construction, and straightforward generalizations of it, frequently arise in physical applications. A group G acts on a manifold M (on the left), then the pull-back map induces an action on the functions $f \in V = C^{\infty}(M) = \Omega^0(M)$ which is linear and hence provides a representation of G Explicitly

$$D(g)f(x) = f(g^{-1}x) = f(\phi_{g^{-1}}(x)) = \phi_{g^{-1}} \circ f = \phi^{\star}_g f. \tag{11.469}$$

The inverse is required to ensure that

$$D(g)D(g') = D(gg'), \tag{11.470}$$

as the reader may easily verify. If M is equipped with a G-invariant measure μ then the represention respects the inner product

$$\int_M f f' \mu. \tag{11.471}$$

It is useful to examine this situation infinitesimally. We suppose that K_a are infinitesimal generators of the action of G on M so that $x^{\mu} \rightarrow x^{\mu} + K^{\mu}_a + \dots$. More over we write $D = 1 + T_a + \dots$, so that

$$T_a f = -K^{\mu}_a \partial_{\mu} f \,, \Rightarrow [T_a, T_b] f = -[K_a, K_b]^{\mu} \partial_{\mu} f \,. \tag{11.472}$$

If structure constants $C_a{}^b{}_c$

$$[T_a, T_b] = C_a{}^c{}_b T_b \,, \tag{11.473}$$

then

$$[K_a, K_b] = -C_a{}^c{}_b K_c \,. \tag{11.474}$$

This reversal of signs arises here essentially, because of the necessity of putting the inverse in (11.469).

Special cases of this construction arise when $M = G/H$ in which case we have the *quasi-regular representation* on $L^2(G/H, \mu)$. If $H = 1$ we get the *regular representation.* If G is a finite group of order $|G|$ then $L^2(G), \mu$ is a vector space V of dimension $|G|$, with each element of G corresponding to a basis vector $e_(g)$ in V. We then get what is usually called the *regular representation* of the finite group G by permutation matrices.

In the case of a manifold, the construction may also be generalized by replacing functions by sections of vector bundles over M.

11.8 Semi-direct products, group extensions and exact sequences

Given two groups G and H there is an obvious notion of product

$$G \times H : (g_1, h_1)(g_2, h_2) \to (g_1 g_2, h_1 h_2) \,. \tag{11.475}$$

Frequently however, one encounters a more sophisticated construction: the semi-direct product $G \ltimes H$. The Poincaré group for example, is the semi-direct product of the Lorentz group and the translation group, such that the translations form an invariant subgroup but the the Lorentz-transformstions do not. The construction also arises when one wants to extend a group G by another group H. It also arises in the theory of fibre bundles which we will outline later.

Suppose G acts on H preserving the group structure, i.e. there is a map $\rho : G \to H$ so that $\rho_{g_1}(h) \circ \rho_{g_2}(h) = \rho_{g_1 g_2}(h)$ and $\rho_g(h_1)\rho_g(h_2) = \rho_g(h_1 h_2)$. If, for example, H is abelian, thought of as additively it is a vector space, then then ρ_g would be a representation of G on H. In general ρ is homomorphism of G into $\mathrm{Aut}(H)$ the automrphism group of H.

We define the product law by

$$G \ltimes H : (g_1, h_1)(g_2, h_2) \to (g_1 g_2, h_1 \rho_{g_1}(h_2)) \,, \tag{11.476}$$

which is asociative with inverse

$$(g, h)^{-1} = (g^{-1}, \rho_{g^{-1}}(h^{-1}) \,. \tag{11.477}$$

A standard example is the Affine group $A(n) = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$, which acts on n-dimensional affine space $\mathbb{A}^n \equiv \mathbb{R}^n$ by translations and general linear transformations preserving the usual flat affine connection. A matrix representation is

$$\begin{pmatrix} \Lambda^a{}_b & a^a \\ 0 & 1 \end{pmatrix} , \qquad \Lambda \in GL(n, \mathbb{R}) \quad a^a \in \mathbb{R}^n \,. \tag{11.478}$$

we have

$$\begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda' & a' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \Lambda\Lambda' & a + \Lambda a' \\ 0 & 1 \end{pmatrix} . \tag{11.479}$$

Restricting to $O(n) \subset GL(n, \mathbb{R})$ we obtain the Euclidean group $E(n) = O(n) \ltimes \mathbb{R}^n$. The Poincaré group is obtained by restriction to the Lorentz group $SO(n-1, 1) \subset GL(n, \mathbb{R})$.

Glearly G may be regarded as a subgroup of $G \ltimes H$ because of the injection

$$i : G \to G \ltimes H \,, \qquad g \xrightarrow{i} (g, e). \tag{11.480}$$

One may also regard H as a subgroup of $G \ltimes H$ because

$$(e, h_1)(e, h_2) = (e, h_1 h_2) \,. \tag{11.481}$$

Moreover, H is an *invariant* subgroup of $G \times H$ because

$$(g, h)(e, h')(g^{-1}, h^{-1}\rho_{g^{-1}}(h^{-1})) = (e, h\rho_g(h')\rho_{g^{-1}}(h^{-1})) \in (e, H) \,. \tag{11.482}$$

By contrast, G is, in general (if $\rho \neq \mathrm{id}$), *not* an invariant subgroup because

$$(g, h)(g', h')(g^{-1}, h^{-1}\rho_{g^{-1}}(h^{-1})) \tag{11.483}$$
$$= (gg'g^{-1}, h\rho_g(h'\rho_{g'}(h^{-1}\rho_{g^{-1}}(h^{-1}))) \tag{11.484}$$
$$= (gg'g^{-1}, h\rho_g(h')\rho_{g'}(h)\rho_{g'g^{-1}}(h^{-1})) \tag{11.485}$$
$$= (gg'g^{-1}, h\rho_g(h')\rho_{g^{-1}}(h^{-1})) \,. \tag{11.486}$$

Thus even if $\eta' = e$, the second term is not necesarily the identity. In summary, we have what is called an *exact sequence*,

$$1 \longrightarrow G \xrightarrow{i} G \ltimes H \xrightarrow{\pi} H \longrightarrow, \tag{11.487}$$

that is the where the maps $\pi : (g, h) \to (e, h)$, $i \to (g, e)$ satisfy the relation that

$$\mathrm{Image}(i) = \mathrm{Kernel}(\pi) \,, \tag{11.488}$$

and the arrows at the ends are the obvious maps of identity elements. Note that in this language a map $\phi : G \rightarrow H$ is an isomorphism if

$$1 \longrightarrow G \xrightarrow{\phi} H \longrightarrow 1\,. \tag{11.489}$$

It is merely a homomorphism if

$$G \xrightarrow{\phi} H \longrightarrow 1\,. \tag{11.490}$$

and G is just a subgroup of H if

$$1 \longrightarrow G \xrightarrow{\phi} H\,. \tag{11.491}$$

If (11.487) holds, one says that $G \ltimes H$ is an *extension of G by H*. If $H \subset Z(G \ltimes H)$, the centre, of one speaks of a central extension. Thus Nil or the Heisenberg group is a central extension of the 2-dimensional abelian group of translations.

11.8.1 The five Lorentz groups

Other examples of semi-direct products arise when one considers the discrete symmetries parity P and time reversal T. These are two commuting involutions inside $O(3,1)$ acting on four-dimensional Minkowski spacetime $\mathbb{E}^{3,1}$ as

$$(x^0, x^i) \xrightarrow{P} (x^0, -x^i)\,, \qquad (x^0, x^i) \xrightarrow{T} (-x^0, x^i)\,. \tag{11.492}$$

We abuse notaton by calling P, T, PT, and (P,T) the $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ groups generated by ,P, T, PT and P, T respectively. Only PT is an invariant subgroup of $O(3,1)$, and indeed belongs to the centre of $O(3,1)$. We define the following invariant subsroups:

(i)$SO_0(3,1)$ is the connected component of the identity in $O(3,1)$,

(ii) $SO(3,1)$ is the *special Lorentz group* which is the subgroup with unit determinant.

(iii) $O\uparrow(3,1)$ is the *orthochronous Lorentz-group* which preserves time orientation,

(iv) $SO_+(3,1)$ as the subgroup preserving space-orientation.

One has

$$\boxed{\begin{array}{lcll} O(3,1)/SO_0(3,1) & = & \mathbb{Z}_2 \times \mathbb{Z}_2 & O(3,1) = (P,T) \ltimes SO_0(3,1) \\ O(3,1)/SO_0(3,1) & = & \mathbb{Z}_2 & O(3,1) = PT \ltimes SO(3,1) \\ O(3,1)/O\uparrow(3,1) & = & \mathbb{Z}_2 & O(3,1) = T \ltimes O\uparrow(3,1) \\ O(3,1)/O_+(3,1) & = & \mathbb{Z}_2 & O(3,1) = P \ltimes O_+(3,1)\,. \end{array}} \tag{11.493}$$

Thus for example

$$1 \longrightarrow (P,T) \xrightarrow{i} O(3,1) \xrightarrow{\pi} SO_0(3,1) \longrightarrow 1\,. \tag{11.494}$$

Geometrically, $O(3,1)$, for example, has four connected components and may be regarded as a principal bundle over a a conected base $B = SO_0(3,1)$ with fibres $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = (P,t)$ consisting of 4 disjoint points.

11.9 Geometry of Lie Groups

Lie groups are

(i) Differentiable Manifolds

(ii) groups

(iii) such that group multiplication and inversion are diffeomrphisms

The Lie algebra $\mathfrak{g}$ of G may be identified (as a vector space) with the tangent space of the origin or identity element e

$$\mathfrak{g} = T_e(G)\,. \tag{11.495}$$

Now if C_g is conjugation, then it fixes the origin and the derivative map $Ad_g = C_{g_\star}$ acts linearly on $T_e(g)$ via the what is called the *adjoint representation* and the contragredient representation acts on $\mathfrak{g}^\star$ the dual of the Lie algebra via what is called the *co-adjoint representation*. In general, these representations are not equivalent although given an invarint metric on G they will be. Now let $\mathbf{e}_a$ be a basis for $\mathfrak{g}$ and $\mathbf{e}^a$ a basis for $\mathfrak{g}^\star$ so that.

$$\langle \mathbf{e}^a | \mathbf{e}_b \rangle = \delta^a_b\,. \tag{11.496}$$

Since the right action R_g and left action L_g of G on itself are simply transitive we use the pull-forward and push-back maps $R_{g\star}, L_{g\star}$ and $R^\star_g, L^\star_g$ to obtain a set of *left-invariant vector fields* L_a and *right-invariant vector fields* R_a and *left–invariant one forms* λ^a and *right-invariant one forms* ρ^a .

Thus

$$R_{g\star}\mathbf{e}_a = R_a(g) \qquad R^\star_g \rho^a = \mathbf{e}^a\,, \tag{11.497}$$

$$L_{g\star}\mathbf{e}_a = L_a(g) \qquad R^\star_g \lambda^a = \mathbf{e}^a\,, \tag{11.498}$$

and

$$\langle \lambda^a | L_b \rangle = \delta^a_b \qquad \langle \rho^a | R_b \rangle = \delta^a_b\,. \tag{11.499}$$

Thus, for example,

$$L_{g_1\star} L_a(g_2) = L_{g_1\star} L_{g_2} \star \mathbf{e}_a = L_{g_1 g_2} \star \mathbf{e}_a = L_a(g_1 g_2)\,\text{etc...} \tag{11.500}$$

Thus every Lie Group is equipped with a global frame field (in two ways). We shall use these frame fields to do calculations.Note that, because left and right actions are global diffeomorphisms, the the frames are everywhere linearly independent. A manifold admitting such a global frame field is said to be *parallelizable*. Thus every Lie group is parallelizable. Among spheres, it is known that only S^1, S^3 and S^7 are parallelizable. The first two are groups manifolds, $U(1)$ and $SU(2)$ respectively but the third is not. In fact the parallelism of S^7 has some applications to supergravity theory and is closely related to the fact that S^7 may be regarded as the unit sphere in the octonions $\mathbb{O}$, just as S^1 and S^3 may be regarded as the unit sphere in the complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$ respectively.

11.9.1 Matrix Groups

The basic example of a Lie Group is $SU(2)$ which we may think of as complex valued matrices of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \qquad |a|^2 + |b|^2 = 1\,. \tag{11.501}$$

If $a = X^1 + iX^2$ and $b = X^3 + iX^4$ this defines a unit 3-sphere $S^3 \subset \mathbb{E}^4$.

Closely related is $SL(2,\mathbb{R})$ which we may think of as real valued matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad ad - bc = 1\,. \tag{11.502}$$

If $a = X^0 + X^3, d = X^0 - X^3, b = X^2 + X^4, c = X^2 - X^4$ this defines a a quadric in $\mathbb{E}^{2,1}$. In fact we may identify $SL(2,\mathbb{R})$ with three-dimensional Anti-de-Sitter spacetime AdS_3.

The group $SL(2,\mathbb{R})$ acting on $\mathbb{R}^2$ preserves the skew two by two matrix $\epsilon_{AB} = -\epsilon_{BA} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This is because for two by two matrices

$$\det X = X^A{}_1 X^B{}_2 \epsilon_{AB} = \epsilon_{12} = 1. \tag{11.503}$$

In fact $SL(2,\mathbb{R})$ is the first in an infinite series of Lie groups called the *symplectic groups* $Sp(2n,\mathbb{R})$ which preserve a skew-symmetric form $\omega_{ab} = -\omega_{ba}$. Just as the orthogonal group $SO(n)$ is the basis of *Riemanian geometry* involving asymmetric positive definite bi-linear form or metric $g_{ab} = g_{ba}$, the symplectic groups is the basis of *symplectic geometry* ,i.e. the geomery of phase space and involves an anti-symmetric tensor two-form $\omega_{ab} = -\omega_{ba}$.

11.10 Infinitesimal Generators of Right and Left Translations

If a group G acts freely on a manifold M on the left say we focus on the action of a one-parameter subgroup $g(t)$ such that $g(t_1)g(t_2) = g(t_1 + t_2)\,, g(-t) = g^{-1}(t)$ and $g(0) = e$. The orbits in M of of $g(t)$ are curves

$$x(t) = \phi_g(t)(x)\,. \tag{11.504}$$

There is at most one such curve through each point $p \in M$ and it has a tangent vector

$$K = \frac{dx}{dt}. \tag{11.505}$$

Thus we get a vector field $K(x)$ on M for each one parameter subgroup of G. A special case occurs when M is a Lie group G and the group action ϕ is by left translations. Each one parameter subgroup of G will have an orbit passing through the origin passing through the unit element or origin $e \in G$ and an initial tangent vector $\frac{dg}{dt}|_{t=0}$ which lies in the tangent space of the origin $T_e(G)$ which we identify with the Lie algebra $\mathfrak{g}$. If $\mathbf{e}_a$ is a basis for $\mathfrak{g}$, then get a map, called the *(left) exponential map* from $\mathbb{R} \times T_e(G) \equiv \mathbb{R} \times \mathfrak{g} : (t, V) \rightarrow g_a(t)$ by moving an amount t along the orbit through e with initial tangent V say.

11.10.1 Example Matrix groups

Near the origin we write

$$g = 1 + v^a M_a t + O(t^2) \,. \tag{11.506}$$

where M_a are a set of matrices spanning $\mathfrak{g}$. Then

$$g(t) = \exp tV^a M_a = 1 + tV^a M_a + \frac{t^2}{2!}(V^a M_a)^2 + \frac{t^3}{3!}(V^a M_a)^3 + \dots \,. \tag{11.507}$$

Thus the exponential map corresponds to the exponential of matrices.

11.11 Right invariant Vector Fields generate Left Translations and *vice versa*

By allowing the initial tangent vector V to range over a basis $\mathbf{e}_a$ of the initial tangent space $T_e(g) \equiv \mathfrak{g}$ we get family of vector fields K_a on G which are tangent to the curves $g(t)$ and are called the generators of left-translations.The give a global parallelization of G. Since left and right actions commute this parallelization is right-invariant. In fact it is clear that this parallelization must coincide with that defined earlier. That is we may set $K_a = R_a$.

It follows that

$$\boxed{\pounds_{L_a} \rho^b = 0 = \pounds_{R_a} \lambda^b \,.} \tag{11.508}$$

11.12 Lie Algebra

We can endow $\mathfrak{g}$ with a skew symmetric bracket satisfying the Jacobi identity by setting

$$[\mathbf{e}_a, \mathbf{e}_b] = -\pounds_{R_a} R_b|_{t=0} = -[R_a, R_b]_{t=0} \,. \tag{11.509}$$

We define the structure constants by

$$\boxed{[\mathbf{e}_a, \mathbf{e}_b] = C_a{}^c{}_b \mathbf{e}_c \,.} \tag{11.510}$$

Since the brackets of right-invariant vector fields are determined by their values at the origin we have

$$\boxed{[R_a, R_b] = -C_a{}^c{}_b R_c \,.} \tag{11.511}$$

Indeed this relation may also be taken as a definition of the Lie algebra. Note that the Jacobi Identity

$$\boxed{[\mathbf{e}_a, \mathbf{e}_b], \mathbf{e}_c] + [[\mathbf{e}_c, \mathbf{e}_a]\mathbf{e}_b] + [[\mathbf{e}_b, \mathbf{e}_c], \mathbf{e}_a] = 0 \,.} \tag{11.512}$$

follows from the Jacobi identity for vector fields

$$\boxed{[[X, Y], Z] + [[Z, X]Y] + [[Y, Z], X] = 0 \,.} \tag{11.513}$$

11.13 Origin of the minus sign

The sign conventions we are adoting are standard in mathematics but perhaps unfamilar to those whos knowledge of Lie algebras comes from some physics textbooks. Moreover it is illuminating to track them down.In order to understand a sign it is usually suficient to consider a particular example and so we shall consider the special case of matrix groups.

We move from e to x_1 along $g_a(t)$ and from x_1 tox_2 along $g_b(t)$ and compare with moving from e to x_3 along $g_b(t)$ followed by moving from x_3 to x_4 along $g_a(t)$. Now

$$g_a(t) g_b(t) = \exp tM_a \exp tM_b = 1 + t(M_a + M_b) + \frac{t^2}{2}(M_a^2 + M_b^2) + t^2 M_a M_b + \dots \,. \tag{11.514}$$

so that

$$x_4 - x_3 = g_a g_b - g_b g_a = t^2(M_a M_b - MBM_b) + \dots = t^2[M_a, M_b] + \dots . \tag{11.515}$$

From the theory of the Lie derivative

$$x_4 - x_3 = (\pounds_{R_b} R_a)t^2 + \dots = [R_b, R_a]T^2 + \dots \tag{11.516}$$

11.14 Brackets of left-invariant vector fields

If one repeats the execrcise for *right* actions one must the order of multiplication. This leads to a sign reversal and one has

$$\boxed{[L_a, L_b] = C_a{}^c{}_b L_c \, .} \tag{11.517}$$

11.14.1 Example:Right and Left vector fields commute

This is because the former generate left translations and the latter right translations and these commute. Thus infinitesimally they must commute, i.e.

$$\boxed{[R_a, L_b] = 0 \, .} \tag{11.518}$$

11.15 Maurer-Cartan Forms

Using the formula

$$d(\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) \tag{11.519}$$

and substituting $\rho^a(R_b) = \delta^a_b$ we get

$$\boxed{d\rho^a = \frac{1}{2} C_b{}^a{}_c \rho^b \wedge \rho^c \, .} \tag{11.520}$$

Similarly

$$\boxed{d\lambda^a = -\frac{1}{2} C_b{}^a{}_c \lambda^b \wedge \lambda^c \, .} \tag{11.521}$$

11.15.1 Example:Matrix groups

Clearly

$$\boxed{g^{-1} dg = \lambda^a M_a} \tag{11.522}$$

is invariant under $g \rightarrow hg$ with g constant. Now take d

$$d(g^{-1}) \wedge dg = d\lambda^a M_a \, . \tag{11.523}$$

But for matrices

$$d(g^{-1}) = -g^{-1} dg g^{-1} \, . \tag{11.524}$$

Thus

$$d\lambda^a M_a = -\lambda^b M_b \wedge \lambda^c M_c = -\frac{1}{2} \lambda^b \wedge \lambda^c [M_b, M_c] = -\frac{1}{2} C_b{}^a{}_c \lambda^b \wedge \lambda^c M_a \, . \tag{11.525}$$

One can similarly check the opposite sign occurs for the right-invariant forms

$$\boxed{dg g^{-1} = \rho^a M_a} \tag{11.526}$$

11.16 Metrics on Lie Groups

Given any symmetric invertible tensor $g_{ab} = g_{ba} \in \mathfrak{g}^\star \otimes_S \mathfrak{g}^\star$ we can construct a left-invariant metric on G by

$$ds^2 = g_{ab}\lambda^a \otimes \lambda^b \,, \tag{11.527}$$

because G_{ab} are constants and $\pounds_{R_b}\lambda^b = 0$. In general, however the metric so constructed will not be right-invariant. One calls a metric which is, *bi-invariant.*Since

$$L_h^\star g = g \,, \qquad \forall g \in G, \tag{11.528}$$

it suffices that g be invariant under conjugation

$$C_h : g \rightarrow hgh^{-1} \,. \tag{11.529}$$

Now acting at the origin $e \in G$, conjugation rotates the tangent space into itself via the Adjoint representation $Ad_h = C_{h\star} : \mathfrak{g} \rightarrow \mathfrak{g}$. This in turn acts on $\mathfrak{g}^\star$ by the contragredient or *co-adjoint representation.* A necessary and sufficent condition for the tensor g_{ab} to give rise to a bi-invariant metric is that it be invariant under the co-adjoint action. One such metric is the *Killing metric* which we will define shortly. We begin by discussing the infinitesimal version of the Adjoint action, called the *adjoint action* $\mathrm{ad}_X(Y)$, a linear map $\mathfrak{g} \rightarrow \mathfrak{g}$ (for fixed $X \in \mathfrak{g}$) defined by

$$Y \rightarrow [X, Y] \,. \tag{11.530}$$

Taking $X = \mathbf{e}_a$ we have

$$\mathrm{ad}_{\mathbf{e}_a}(Y) = [\mathbf{e}_a, Y] = C_a \,, \tag{11.531}$$

where C_a is some matrix or linear map on $\mathfrak{g}$ with components

$$(C_a)^b{}_c = C_a{}^b{}_c \,. \tag{11.532}$$

The Jocobi identity

$$[\mathbf{e}a, [\mathbf{e}_b, Y]] - [\mathbf{e}_b, [\mathbf{e}_a, Y]] = [[\mathbf{e}_a, \mathbf{e}_b], Y] \,, \tag{11.533}$$

becomes

$$C_a C_b - C_b C_a = C_a{}^c{}_b C_c \tag{11.534}$$

and therefore the matrices C_a provide a matrix representation of the Lie algbra called the *adjoint* or *regular representation.*

Multiplying the Jacobi identity by C_d and taking a trace gives

$$\mathrm{Tr}(C_d C_a C_b - C_d C_b C_a) = C_{adb} \,, \tag{11.535}$$

where

$$C_{adb} = B_{dc} C_a{}^c{}_b \,, \tag{11.536}$$

and

$$B_{dc} = \mathrm{Tr}(C_d C_c) = \mathrm{Tr}(C_c C_d) = B_{cd} \tag{11.537}$$

is a symmetric bi-linear form on $\mathfrak{g}$ called the *Killing form.*

Using the cyclic property of the trace we have

$$C_{abc} = -C_{bac} \,. \tag{11.538}$$

But because

$$C_a{}^d{}_b = -C_b{}^d{}_a \,, \tag{11.539}$$

we have

$$C_{adb} = -C_{bda}, \tag{11.540}$$

and hence

$$C_{abc} C_{[abc]} \,. \tag{11.541}$$

The tensor C_{abc} defines a 3-form on $\mathfrak{g}$ and hence by left or right translation a 3-form on G which by use of the Cartan-Maurer formulae may be shown to be closed.

11.16.1 Example: $SU(2)$

On $SU(2)$, or on $SL(2,\mathbb{R})$, we obtain a multiple of the volume form on S^3 or AdS_3 respectively.

11.17 Non-degenerate Killing forms

From now on in this subsection , unless otherwise stated, we assume that B_{ab} is non-degenerate, i.e. $\det B_{ab} \neq 0$. Thus we have an inverse B^{ab} such that $B^{ab}B_{bc} = \delta^a_c$. Therefore

$$C_a{}^c{}_b = B^{ce}C_{aeb} \tag{11.542}$$

and

$$C_a{}^c{}_c = C^{ce}C_{ace} = \operatorname{Tr} C_a = 0\,. \tag{11.543}$$

Groups for which $\operatorname{Tr} C_a = 0$ are said to be *unimodular.*

11.17.1 Necessary and sufficient condition that the Killing form is non-degenerate

This is that G is semi-simple, i.e. it contains no invariant abelian subgroups.We sahll establish the necessity and leave the sufficiency for the reader.

If G did contain an invariant abelian subgroup H , let $\mathbf{e}_\alpha$ span $\mathfrak{h}$ and $\mathbf{e}_i$ span the complement $\mathfrak{k}$ so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$. Since H is abelian

$$C_\alpha{}^\mu{}_\beta = 0 = C_\alpha{}^i{}_\beta. \tag{11.544}$$

Since H is invaraint

$$[\mathfrak{g},\mathfrak{h}] \subset \mathfrak{h}\,, \qquad \Leftarrow C_i{}^j{}_\alpha = 0\,. \tag{11.545}$$

One calculates and finds that

$$B_{\alpha\beta} = 0 = B_{\alpha i}\,. \tag{11.546}$$

Thus

$$B_{ab}\delta^a_\alpha = 0\,. \tag{11.547}$$

which implies that δ^a_α lies in the kernel of $B(\,,\,)$.

11.17.2 Invariance of the Killing metric under the co-adjoint action

let V^a ve the components of a vector $V \in \mathfrak{g}$ then under the adjoint action

$$(\delta_{\mathbf{e}_a} V)^b = [\mathbf{e}_a, V]^b = C_a{}^b{}_c V^c\,. \tag{11.548}$$

For example if $G = SO(3)$ $\mathfrak{g} = \mathbb{R}^3$ and

$$\delta\mathbf{v} = \mathbf{\Omega} \times \mathbf{v}\,, \qquad \mathbf{\Omega} \in \mathfrak{g}\,. \tag{11.549}$$

let ω_a be the components of a co-vector $\omega \in \mathfrak{g}^\star$. The *co-adjoint* action must leave invariant $\langle\omega|V\rangle = \omega_a V^a$. Therefore

$$\delta_{\mathbf{e}_a}\omega_b = -\omega_c C_a{}^c{}_b\,. \tag{11.550}$$

The induced action on $B \in \mathfrak{g}^\star \otimes_S \mathfrak{g}^\star$ is thus

$$\delta_{\mathbf{e}_a} B_{cb} = -C_a{}^e{}_b B_{ce} - C_a{}^e{}_c B_{eb} \tag{11.551}$$

$$= -C_{acb} - C_{abc} = 0\,. \tag{11.552}$$

It follows from our earlier work on representations that *if G is semi-simple, the Adjoint and the co-Adjoint representation are equivalent* since B is an invariant bi-linear form mapping $\mathfrak{g}$ to $\mathfrak{g}^\star$.

It also follows that *every semi-simple Lie group is an Einstein manifold with respect to the Killing metric.*

If we set $\lambda = \frac{1}{2}$ in the previous formulae for the $\nabla_{\frac{1}{2}}$ connection we get for the Ricci tensor

$$R_{eb} = -\frac{1}{4}C_d{}^f{}_b C_f{}^d{}_e = -\frac{1}{4}B_{de}\,. \tag{11.553}$$

Moreover, it is clear that since $\nabla_{\frac{1}{2}}$ preserves the Killing metric and is torsion free, then it is the Levi-Civita connection of the Killing metric.

The simplest examples are $G = SU(2) = S^3$ and $G = SL(2,\mathbb{R}) = AdS_3$ which are clearly Einstein since they are of constant curvature.

$$R_{abcd} = -\frac{1}{8}(B_{ac}B_{bd} - B_{ad}B_{bc})\,. \tag{11.554}$$

11.17.3 Signature of the Killing metric

The Killing form B is, in general, indefinite having one sign for compact directions and the opposite for non-compact directions. By a long standing mathematical convention, the sign is negative definite for compact directions and positive definite for non-compact directions. Thus in the Yang Mills Lagangian with $\mathfrak{g}$-valued curvatures $F^1_{\mu\nu}$,

$$-\frac{1}{2}g_{ab}F^a{}_{\mu}\nu F^b{}_{\mu\nu} \tag{11.555}$$

gauge invariance requires that the metric g_{ab} be invariant under the co-adjoint action and if g_{ab} is minus the Killing metric, then the group must be compact in order to have positive energy. In supergravity theories,scalars arise, on which which alllow g_{ab} is allowed to depend. That is g_{ab} becomes spacetime time-dependent. In that case it is posoible to arrage things to that the gauge group is non-compact while retaining positive energy.

We now illustrate the theory with some examples.

11.17.4 Example $SO(3)$ and $SU(2)$

These are clearly compact groups since $\mathbb{RP}^3 \equiv S^3/\mathbb{Z}_2$ is a compact manifold. In our conventions $C_i{}^j{}_k = -\epsilon_{ijk}$. Thus $B_{ij} = \epsilon_{irs}\epsilon_{jrs} = -2\delta_{ij}$. It is clear that since $SO(3)$ acts in the usual way on its Lie algebra $\mathbb{R}^3$, *any* bi-invariant metric must be a multiple of δ_{ij}. For example think of its double cover $SU(2)$ as 2×2 unimodular unitary matrices $U = (U^{-1})^t$, $\det U = 1$. We may set

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \qquad |a|^2 + |b|^2 = 1\,. \tag{11.556}$$

The left action is $U \to LU$, $L \in SU(2)$ The right action is $U \to UR$, $R \in SU(2)$.

The metric

$$\det dU = |da|^2 + |db|^2 \tag{11.557}$$

is invariant under $dU \to LdUR$, and hence bi-invariant.

11.17.5 Matrix Groups

In general, for a matrix group, the Killing metric is proportional to

$$\mathrm{Tr} g^{-1}dgg^{-1}dg = \lambda^a \otimes \lambda^b \mathrm{Tr}(M_a M_b). \tag{11.558}$$

For orthogonal groups the infinitesial generators are skew symmetric and $M_a = -M_a^t$ the Killing form is negative definite and these groups are compact.

11.17.6 Example: $SL(2,R)$

As before

$$g = \begin{pmatrix} X^0 + X^1 & X^2 + X^4 \\ X^2 - X^4 & X^0 - X^1 \end{pmatrix}, \qquad (X^0)^2 + (X^4)^2 - (X^1)^2 + (X^2) = 1.. \tag{11.559}$$

A calculation analogous to that above gives the signature $++1$. The group has topology $S^1 \times \mathbb{R}^2$ with the compact direction corresponding rotations in the 1-2 plane.

11.17.7 Example: $SO(2,1) = SL(2,\mathbb{R})/\mathbb{Z}_2$

If $SO(2,1)$ acts on $\mathbb{E}^{2,1}$ then

$$\mathbf{e}_3 = x\partial_y - y\partial_x \quad \text{generates rotations in x} - \text{y plane,} \tag{11.560}$$
$$\mathbf{e}_1 = t\partial_x + x\partial_t \quad \text{generates boosts in t} - \text{x plane,} \tag{11.561}$$
$$\mathbf{e}_2 = t\partial_y + y\partial_t \quad \text{generates boosts in t} - \text{y plane}. \tag{11.562}$$

Thus

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3\,, \qquad C_1{}^2{}_2 = -1\,. \tag{11.563}$$

$$[\mathbf{e}_3, \mathbf{e}_2] = \mathbf{e}_1\,, \qquad C_3{}^1{}_2 = -1\,. \tag{11.564}$$

$$[\mathbf{e}_3, \mathbf{e}_1] = -\mathbf{e}_2\,, \qquad C_1{}^2{}_1 = +1\,. \tag{11.565}$$

Therefore the Killing form is diagonal with $B_{33} = -2\,, B_{11} = B_{22} = +2$. This compatible with the fact that the group $SO(2)$ is compact and the group $SO(1,1)$ is non-compact, since for example as matrices they are

$$\begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}, \qquad 0 \le \psi < \psi, \tag{11.566}$$

but

$$\begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}, \qquad \infty < \phi < \infty\,. \tag{11.567}$$

11.17.8 Example: Nil or the Heisenberg Group

This is upper-triangular matrices

$$g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = 1 + xX + yY + zZ \tag{11.568}$$

with

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{11.569}$$

$$Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{11.570}$$

$$Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{11.571}$$

$$\tag{11.572}$$

and only non-vanishing commutator being

$$[X, Y] = Z\,. \tag{11.573}$$

Thus Z is in the *centre* $Z(\mathfrak{g})$ of the algebra, that is it commutes will all elements in $\mathfrak{g}$. The Killing form B is easily seen to vanish. Now

$$g^{-1} = \begin{pmatrix} 1 & -x & -z + xy \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}, \tag{11.574}$$

Thus

$$g^{-1}dg = Xdx + Ydy = Z(dz - xdy). \tag{11.575}$$

One finds that

$$\lambda^x = dx\,, \qquad L^x = \partial_x\,, \tag{11.576}$$

$$\lambda^y = dy\,, \qquad L_y = \partial_y + x\partial_z \tag{11.577}$$

$$\lambda^z = dz - xdy\,, \qquad L_z = \partial_z\,. \tag{11.578}$$

The non-trivial Cartan-Maurer relations and Lie brackets are

$$d\lambda^z = -\lambda^x \wedge \lambda^y\,, \qquad [L_x, L_y] = L_z\,. \tag{11.579}$$

Using the right-invariant basis one gets

$$\rho^x = dx\,, \qquad R^x = \partial_x + \partial_z\,, \tag{11.580}$$

$$\rho^y = dy\,, \qquad R_y = \partial_y \tag{11.581}$$

$$\rho^z = dz - ydx\,, \qquad R_z = \partial_z\,. \tag{11.582}$$

The non-trivial Cartan-Maurer relations and Lie brackets are now

$$d\rho^z = +\rho^x \wedge \rho^y\,, \qquad [R_x, R_y] = -R_z\,. \tag{11.583}$$

An example of a left-invariant metric on Nil is

$$(dz - xdy)^2 + dx^2 + dy^2\,. \tag{11.584}$$

The Lie algebra $\mathfrak{nil}$ is of course the same as Heisenberg algebra $[\hat{x}, \hat{p}] = i\frac{h}{2\pi}\mathrm{id}$.

11.17.9 Example: Two-dimensional Euclidean Group E(2)

This provides a managable example which illustrates more complicated cases such as the four-dimensional Poincaré group. A convenient matrix representation is

$$g = \begin{pmatrix} \cos\psi & -\sin\psi & x \\ \sin\psi & \cos\psi & y \\ 0 & 0 & 1 \end{pmatrix} . \qquad (11.585)$$

With $0 \le \psi < 2\pi$, $-\infty < x < \infty$, $-\infty < y < \infty$. The group is not semi-simple, it is in fact a semi-direct product of $\mathbb{R}^2 \bowtie SO(2)$. The translations form an invariant abelian sub-group The Killing form is degenerate, vansihing on the translations.

The matrix can be thought of as acting on the column vector

$$\begin{pmatrix} X^1 \\ X^2 \\ 1 \end{pmatrix} . \qquad (11.586)$$

The subspace

$$\begin{pmatrix} X^1 \\ X^2 \\ 0 \end{pmatrix} \qquad (11.587)$$

is invariant but the complementary subspace

$$\begin{pmatrix} 0 \\ 0 \\ X^3 \end{pmatrix} \qquad (11.588)$$

is not. The representation, which exhibits $E(2)$ as a subgroup of $SL(3, \mathbb{R}$, is thus *reducible but not fully reducible*. Geometrically, this is a projective construction.. Euclidean space $\mathbb{E}^2$ is identified with those directions through the origin of $\mathbb{R}^3$ which intersect the plane $X_3 = 1$. The full projective group of two dimensions is $SL(2, \mathbb{R}$ which takes straight lines to straight lines.

11.18 Rigid bodies as geodesic motion with respect to a left-invariant metric on $SO(3)$.

Consider a rigid body, a *lamina* constrained to slide on a plane $\mathbb{E}^2$. Every configuration may be obtained by acting with $E(2)$ on some standard configuration in a unique fashion,. Thus the *configuration space* Q may be regarded as $E(2)$. A dynamical motion corresponds to a curve $\gamma(t)$ in $E(2)$. If there are no frictional forces the motion is free and will be desribed by geodesic motion with repsct to the metric given by the kinetic energy

$$T = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu , \qquad (11.589)$$

where x^μ are local coordinates on Q.The symmetries of the situtation dictate that the metric is left-invariant and this gives rise to conservation of momentum and angular momentum.

This example obviously generalizes to a rigid body moving in *in vacuo* in $\mathbb{E}^3$ with $E(2)$ replaced by $E(3)$. The model would still apply if the rigid body moves in an incompressible frictionless fluid, an as such has been studied by fluid dynamicists in the nineteenth century. Note that in this case, the kinetic energy gets *renormalized* compared with its value in vaucum, because one must take into account the kinetic energy of the fluid. There is also an obvious generalization to a rigid body moving in certain curved spaces such as hyperbolic spaces $H^n = S(n,1)/SO(n)$ or a spheres $S^n = SO(n+1)/SO(n)$ and this idea formed the basis of Helmholtz's physical characterization of non-Euclidean geometry in terms of the axiom of the *free-mobility of a rigid body* which he thought of as some sort of measuring rod.

The simplest case to consider is when the mody is in moving in $\mathbb{E}^3$ but one point in the body is fixed. The configuration space Q now reduces to $SO(3)$. Each point in the body has coordinates $\mathbf{r}(t)$ in an inertial or *space-fixed* coordinate system with origin the fixed point , and where $\mathbf{r}(t) = O(t)\mathbf{x}$, where $\mathbf{x}$ *co-moving coordinates* fixed in the body with respect to some *body-fixed frame* and $O(t)$ is a curve in $SO(3)$ so that.

$$O^{-1}(t) = O^t(t) , \qquad (11.590)$$

that is,$O(t)$ is an orthogonal matrix for each time t. Now

Left actions correspond to rotations of with respect to the body fixed frame

$$\mathbf{r} \to L\mathbf{r}\,, \qquad \text{i.e.}\, O \to LO\,, \qquad L \in SO(3) \tag{11.591}$$

Left actions correspond to rotations of with respect to the body fixed frame

$$\mathbf{x} \to R^{-1}\mathbf{x}\,, \qquad \text{i.e.}\, O \to OR\,, \qquad R \in SO(3)\,. \tag{11.592}$$

The kinetic enery should be invariant under rotatations about inertial axes and this invriance gives rise to conserved angular momenta. Thus we anticipate that the kinetic energy T is left-invariant but not in general right-invariant. If $\rho(\mathbf{x})$ is the mass density, T is given by

$$T = \int_{\text{Body}} d^3\,x\,\frac{1}{2}\rho(\mathbf{x})\,\dot{\mathbf{r}}^2 \tag{11.593}$$

$$= \int_{\text{Body}} d^3\,x\,\frac{1}{2}\rho(\mathbf{x})\,\dot{\mathbf{r}}^t\,\dot{\mathbf{r}} \tag{11.594}$$

$$= \int_{\text{Body}} d^3\,x\,\frac{1}{2}\rho(\mathbf{x})\,(\dot{O}\mathbf{x})^t\,\dot{O}\mathbf{x} \tag{11.595}$$

$$= \int_{\text{Body}} d^3\,x\,\frac{1}{2}\rho(\mathbf{x})\,\mathbf{x}^t\,\dot{O}^t\,\dot{O}\,\mathbf{x} \tag{11.596}$$

$$= \int_{\text{Body}} d^3\,x\,\frac{1}{2}\rho(\mathbf{x})\,\mathbf{x}\,(O^{-1}\dot{O})^t(O^{-1}\dot{O})\,\mathbf{x} \tag{11.597}$$

$$= \int_{\text{Body}} d^3\,x\,\frac{1}{2}\rho(\mathbf{x})\,\mathbf{x}^t\,\omega^t\,\omega\,\mathbf{x} \tag{11.598}$$

$$\tag{11.599}$$

where

$$\omega_{ij} = (O^{-1}\dot{O})_{ij}\,. \tag{11.600}$$

In three dimensions we may dualize

$$\omega_{ik} = (O^{-1}\dot{O})_{ik}\epsilon_{ijk}\omega_j \tag{11.601}$$

where ω_i are the *instantantaneous angular velocities with reepect to a body fixed frame* and find

$$T = \frac{1}{2}\omega_i I_{ij}\omega_j \tag{11.602}$$

where

$$I_{ij} \in \mathfrak{so}(3)^\star \otimes_S \mathfrak{so}(3)^\star = \int_{\text{Body}} d^3\rho(\mathbf{x})\,(\delta_{ij}\mathbf{x}^2 - x_i x_j) = I_{ji} \tag{11.603}$$

is the (time-independent) *inertia quadric* of the body which serves as a left-invariant metric on $SO(3)$. Under right actions, i.e. under rotations of the body with respect to a space-fixed frame

$$I_{ij} \to R_{ik} I_{kl} R_{lj}\,, \tag{11.604}$$

where R_{ik} are the components of an orthogonal matrix. We may use this freedom to diagonalize the inertia quadric

$$T = \frac{1}{2}I_x(\omega^x)^2 + \frac{1}{2}I_y(\omega^y)^2 + \frac{1}{2}I_z(\omega^z)^2 \tag{11.605}$$

where I_x, I_y, I_z are called *principal moments of inertia.*

11.18.1 Euler angles

A geometrical argument shows that any triad may be taken into any other triad by an element of $SO(3)$ of the form may be written as the product of a rotation through an angle ψ about the third axis followed by a rotation through an angle θ about the new first axis followed by a rotation through an angle ϕ about the new third axis

$$O(\psi,\theta,\phi) = R_z(\phi)R_x(\theta)R_z(\psi) \tag{11.606}$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \tag{11.607}$$

with $0 \le \phi < 2\pi$, $0 \le \theta < \pi$, $0 \le \psi < 2\pi$. Thus

$$O^{-1}(\psi,\theta,\phi) = O(-\phi,-\theta,-\psi)\,. \qquad (11.608)$$

A covering element of $SU(2)$ is

$$S = \exp\frac{\phi\tau_z}{2i}\exp\frac{\theta\tau_x}{2i}\exp\frac{\psi\tau_z}{2i}\,, \qquad (11.609)$$

so that

$$S^{-1}dS = \frac{\lambda^x\tau_x}{2i} + \frac{\lambda^y\tau_y}{2i} + \frac{\lambda^z\tau_z}{2i}\,, \qquad (11.610)$$

with

$$\begin{aligned} \lambda^x &= \sin\theta\sin\psi d\phi + \cos\psi\delta\theta & (11.611)\\ \lambda^y &= \sin\theta\cos\psi d\phi - \sin\psi d\theta & (11.612)\\ \lambda^z &= d\psi + \cos\theta d\phi\,, & (11.613) \end{aligned}$$

and

$$\begin{aligned} L_x &= \cos\psi\,\partial_\theta - \sin\psi\,(\cot\theta\,\partial_\psi - \frac{1}{\sin\theta}\,\partial_\phi) & (11.614)\\ L_y &= -\sin\psi\,\partial_\theta - \cos\psi\,(\cot\theta\,\partial_\psi - \frac{1}{\sin\theta}\,\partial_\phi) & (11.615)\\ L_z &= \partial_\psi\,. & (11.616) \end{aligned}$$

The right-invariant basis satisfies

$$dSS^{-1} = \frac{\rho^x\tau_x}{2i} + \frac{\rho^y\tau_y}{2i} + \frac{\rho^z\tau_z}{2i}\,, \qquad (11.617)$$

and may be obtained form the observation that inversion $(\phi,\theta,\phi) \to (-\phi,-\theta,-\psi)$ takes $S^{-1}dS \to -dSS^{-1}$ and thus

with

$$\begin{aligned} -\rho^x &= -\sin\theta\sin\phi d\psi - \cos\phi\delta\theta & (11.618)\\ -\rho^y &= \sin\theta\cos\phi d\phi + \sin\phi d\theta & (11.619)\\ -\rho^z &= -d\phi - \cos\theta d\psi\,, & (11.620) \end{aligned}$$

and

$$\begin{aligned} -R_x &= -\cos\phi\,\partial_\theta + \sin\phi\,(\cot\theta\,\partial_\phi - \frac{1}{\sin\theta}\,\partial_\psi) & (11.621)\\ -R_y &= \sin\phi\,\partial_\theta - \cos\phi\,(\cot\theta\,\partial_\phi - \frac{1}{\sin\theta}\,\partial_\psi) & (11.622)\\ -R_z &= -\partial_\phi\,. & (11.623) \end{aligned}$$

We have

$$\begin{aligned} d\lambda^x &= -\lambda^y\wedge\lambda^z\,, \qquad [L_x\,L_y] = L_z\,\text{etc} & (11.624)\\ d\rho^x &= \rho^y\wedge\rho^z\,, \qquad [R_x,R_y] = -R_z\,\text{etc}\,. & (11.625) \end{aligned}$$

$\frac{\partial}{\partial\phi}$ generates left actions of $SO(2)$ (rotations about the 3rd space-fixed axis)

$\frac{\partial}{\partial\psi}$ generates right actions of $SO(2)$ (rotations about the 3rd body-fixed axis)

The kinetic energy metric may be written as

$$ds^2 = I_x(\sin\theta\sin\psi d\phi + \cos\psi d\theta)^2 + I_y(\sin\theta\cos\psi d\phi - \sin\psi d\theta)^2 + I_z(\ \psi + \cos\theta d\phi)^2\,. \qquad (11.626)$$

$\frac{\partial}{\partial\phi}$ is always a Killing vector corresponding to conservation of angular momentum about the third body fixed axis. In general, the full set of Killing vector fields is R_x, R_y, R_z

$$\pounds_{R_x}\,ds^2 = 0 \qquad \text{etc}\,. \qquad (11.627)$$

$\frac{\partial}{\partial\psi}$ is only a Killing vector if $I_x = I_y = I$ in which case

$$ds^2 = I_z(d\psi + \cos\theta)^2 + I_(d\theta^2 + \sin\theta^2 d\phi^2)\,.(\tag{11.628}$$

In mechanics this is *Lagrange's symmetric top*. In geometry $S^3 = SU(2)$ equipped with such a metric is such is sometimes called a *Berger-Sphere*. This requires our passing to the double cover by allowing the angle ψ, which is an angle along the S^1 fibres of the Hopf fibration, to run between 0 and π. In general,if we take $0 \le \frac{4\pi}{k}$ we could take a quotient to get the *lens spaces* $L(k,1) \equiv S^3/C_k$, where C_k is the cyclic group of order k. We shall later learn that these are the possible circle bundles over the 2-sphere S^2, with k the Chern number. They correspond to all possible Dirac mononopoles on S^2 and the metric provides their Kaluza-Klein description of these monopoles.

11.18.2 $SL(2,\mathbb{R})$ and the Goedel Universe

Using the 'cheap and dirty'trick of replacing $\theta \to i$, $\chi real$ results in

$$\begin{aligned} \lambda^x &\to i\lambda^x = \lambda^1, \qquad \text{non}-\text{compact} && (11.629)\\ \lambda^y &\to i\lambda^y = \lambda^2, \qquad \text{non}-\text{compact} && (11.630)\\ \lambda^z &\to i\lambda^z = \lambda^0 \qquad \text{compact}\,. && (11.631) \end{aligned}$$

Up to an overall scale, the family of left-invariant Berger metrics on $SU(2)$ with an additional $SO(2)$ right action are given by

$$ds^2 = \lambda(d\psi + \cos\theta d\phi)^2 + (d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{11.632}$$

become a family of metrics on $SL(2,\mathbb{R})$

$$\begin{aligned} ds^2 &= \lambda(d\psi + \cosh\chi\phi)^2 - (d\chi^2 + \sinh^2\chi d\phi^2\,. && (11.633)\\ &= \lambda(dt + 2\sinh^2(\frac{\chi}{2})d\phi)^2 - (d\chi^2 + \sinh^2\chi d\phi^2\,, && (11.634) \end{aligned}$$

with $t = \psi + \phi$. If $\lambda = 1$ we get the standard metric on Ad_3. If $\lambda > 1$ we get a family of Lorentzian metrics . For a certain value of λ we get a metric, which if multiplied by a line, gives Goedel's homogeneous solution of the Einstein equations containing rotating dust. If we chose $-\infty < t < \infty$ the manifold is simply connected and in fact homeomorphic to $\mathbb{R}^3$. It may be thought as an $\mathbb{R}$ bundle over the two-dimensional hyperbolic plane H^2. To anticipate a later section, the bundle is trivial and t provides a global section. However, if $\lambda > 1$, t is a *time function*, i.e. a function on a Lorentzian mnfold which increases along every timelike curve. We have

$$g_{\phi\phi} = -\sinh^2\chi + 4\lambda\sinh^2(\frac{\chi}{2}),\,. \tag{11.635}$$

Now ϕ must be periodic period 2π in order to avoid a conical singularity at $\chi = 0$. Thus the curves $t = \text{constant}, \theta = \text{constant}$ are the orbits of an $SO(2)$ action and hence circles. But if $\lambda > 1$ they are timelike, $g_{\phi\phi} > 0$, for sufficiently large χ. Thus the metric admits *closed timelike curves* or CTC's which is incompatible with the existence of a time function.

11.18.3 Example: Minkowski Spacetime and Hermitian Matrices

We can set up a one-one correspondence between $\mathbb{E}^{3,1} = (t,x,y,z)$ and two by two hermitian matrices $x = x^\dagger$ by

$$x = \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix}, \qquad \Rightarrow \det x = t^2 - x^2 - y^2 - z^2\,. \tag{11.636}$$

There is an action of $S \in SL(2,\mathbb{C})$ preserving hermiticity and the determinant

$$x \to SxS^\dagger\,, \qquad \Rightarrow \det x \to \det x\,. \tag{11.637}$$

Thus we get a homomomorphism $SL(2,\mathbb{C} \to SO_0(3,1)$, the identity component of the Lorentz group. The kernel, i.e. the pre-image of $1 \in SL(2,\mathbb{C})$ is $\pm 11 = \mathbb{Z}_2$ and thus

$$SO_0(3,1) = SL(2,\mathbb{C})/\mathbb{Z}_2\,. \tag{11.638}$$

Evidently $SL(2,\mathbb{C}$ acts faithfully on $\mathbb{C}^2$ preserving the simplectic form $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Elements of of $\mathbb{C}^2$ are *Weyl spinors*. One has $Spin(3,1) = SL(2,\mathbb{C})$. Specialization to $SU(2)$ yields $Spin(3) = SU(2)$, i.e. $SO(3) = SU(2)/\mathbb{Z}_2$.

Working over the reals

$$x = \begin{pmatrix} t+z & x+y \\ x-y & t-z \end{pmatrix}, \qquad \Rightarrow \det x = t^2 - x^2 + y^2 - z^2 \,. \tag{11.639}$$

There is an action of $S \times S' \in SL(2,\mathbb{R} \times SL(2,\mathbb{R})$ preserving the determinant

$$x \to SxS' \,, \qquad \Rightarrow \det x \to \det x \,. \tag{11.640}$$

Thus we get a homomomorphism $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \to SO_0(2,2)$. The kernel, i.e. the pre-image of $1 \in SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ is $\pm(1,1) = \mathbb{Z}_2$ and thus

$$SO_0(2,2) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\mathbb{Z}_2 \,. \tag{11.641}$$

We have $Spin(2,2) = SL(2\mathbb{R}) \times SL(2,\mathbb{R}$ and setting $y = 0$ and $S = S'$ yields $Spin(2,1) = SL(2,\mathbb{R})$, $SO_0(2,1) = SL(2,\mathbb{R})/\mathbb{Z}_2$. Elements of $\mathbb{R}^2$ or $\mathbb{R}^2 \oplus \mathbb{R}^2$ are called *Majorana* spinors for $Spin(2,1)$ or $Spin(2,2)$ respectively. Elements of $0 \oplus \mathbb{R}^2$ or $\mathbb{R}^2 \oplus 0$ are called *Majorana -Weyl* spinors for $Spin(2,2)$.

11.18.4 Example: $SO(4)$ and quaternions

If

$$i = \frac{\tau_1}{\sqrt{-1}} \qquad j = \frac{\tau_2}{\sqrt{-1}}, \qquad k = \frac{\tau_1}{\sqrt{-1}} \,, \tag{11.642}$$

then we have the algebra

$$ij + ji = jk + kj = ki + ik \quad = 0 \quad , \tag{11.643}$$

$$ij - k = jk - i = ki - j \quad = 0 \quad , \tag{11.644}$$

$$i^2 + 1 = j^2 + 1 = k^2 + 1 \quad = 0 \quad . \tag{11.645}$$

It may be verified that the quaternion algebra $\mathbb{H}$ coincides with the real Clifford algebra, Cliff$(0,2)$ of $\mathbb{E}^{0,2}$ generated by gamma matrices γ_1, γ_2 such that $\gamma_1^2 = \gamma_2^2 = -1$. One sets $\gamma_3 = \gamma_1\gamma_2 = -\gamma_2\gamma_1$ and identifies (i,j,k) with $(\gamma_1, \gamma_2, \gamma_3)$.

A general quaternion $q \in \mathbb{H}$ and its *conjugate* $\bar{q}$ are given by

$$q = \tau + xi + yj + zk, , \qquad \bar{q} = \tau - xi - yj - zk \,. \tag{11.646}$$

Thus

$$q\bar{q} = \bar{q}q = |q|^2 = \tau^2 + x^2 + y^2 + z^2 \,. \tag{11.647}$$

We may thus identify $\mathbb{H}$ with four-dimensional Euclidean space $\mathbb{E}^4$ and if

$$q \to lqr \tag{11.648}$$

with $|r| = |l| = 1$, we preserve the metric. We may identify unit quaternions with $SU(2) = Sp(1)$ and similar reasoning to that given earlier shows that $SO(4) = SU(2) \times SU(2))/\mathbb{Z}_2$, $Spin(4) = Sp(1) \times Sp(1)$, where $Sp(n)$ is the group of n times n quaternion valued matrices which are unitary with respect to the quaternion conjugate.

11.18.5 Example: Hopf-fibration and Toroidal Coordinates

The Hopf fibration of S^3 is the most basic example of a non-trivial fibre bundle. It is useful therefore to have a good intuitive understanding of its properties. One way to acquire this is by stereogrphic projection into ordinary flat 3-space .

We may think of S^3 as $(Z^1, Z^2) \in \mathbb{C}^2$ such that $|Z^1|^2 + |Z^2|^2 = 1$. We set

$$Z^1 = X^1 + iX^2 = e^{i\alpha} \tanh \sigma \,, \qquad X^2 = X^4 + iX^3 = e^{i\beta} \frac{1}{\cosh \sigma} \,. \tag{11.649}$$

The surfaces σ = constant or $|Z^1| = \sqrt{1-|Z^2|^2}$ = constant are called *Clifford tori* and α and β are coordinates on each torus. The Hopf fibration sends $(\alpha, \beta) \to (\alpha + c, \beta + c)$ and the anti-Hopf fibration sends $(\alpha, \beta) \to (\alpha + c, \beta - c)$, $c \in S^1$ and the fibres spiral around the torus.There are two degenerate tori, $\sigma = 0, |Z^1| = 0$, and $\sigma = \infty, |Z^2| = 0$, which collapse to circles. The induced round metric on S^3 is

$$\frac{d\sigma^2}{\cosh^2\sigma} + \tanh^2\sigma d\alpha^2 + \frac{1}{\cosh^2\sigma} d\beta^2 \,. \tag{11.650}$$

This looks more symmetrical if one sets $\sin\varpi = \tanh\sigma$,$0 \le \varpi \le \frac{\pi}{2}$

$$ds^2 = d\varpi^2 + \sin^2\omega d\alpha^2 + \cos^2\varpi d\beta^2 \,. \tag{11.651}$$

The torus $\varpi = \frac{\pi}{4}$ is square and a minimal surface. Stereographic projection may be achieved by setting

$$Z^1 = \sin\chi\sin\theta e^{i\phi} \,, \qquad Z^2 = \cos\chi + i\sin\chi\cos\theta e^{i\phi} \,, \tag{11.652}$$

where χ, θ, ϕ are polar coordinates on S^3 in which the metric is

$$ds^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2 d\phi^2) \tag{11.653}$$

$$= \frac{4}{(1+r^2)^2}(dr^2 + r^2(d\theta^2 + \sin^2 d\phi^2)) \tag{11.654}$$

with $r = \tan\frac{\chi}{2}$.

If $(x, y, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, \cos\theta)$ are Cartesian coordiantes for $\mathbb{E}^3$, on finds that

$$x + iy = e^{i\alpha}\frac{\sinh\sigma}{\cosh\sigma\cos\beta} \,, \qquad z = \frac{\sin\beta}{\cos\sigma + \cos\beta} \,, \tag{11.655}$$

and

$$(\sqrt{x^2+y^2} - \coth\sigma)^2 + z^2 = \frac{1}{\sinh^2\sigma} \,. \tag{11.656}$$

Thus we obtain the standard orthogonal coordinate system on $\mathbb{E}^3$ consisting of tori of revolution σ = constant obtained by rotating about the z-axis circles parameterized by the coordinate β lyinging in the planes ϕ = constant. Shifting α corresponds to rotating about the z-axis. The coordinate β. The two degenerate tori $\sigma = 0$ and $\sigma = \infty$ correspond to the z-axis and to a circle lying in the xy plane respectively. The flat metric turns out to be

$$ds^2_{\text{flat}} = \frac{1}{(\cosh\sigma + \cos\beta)^2}(d\sigma^2 + d\beta^2 + \sinh^2 d\alpha^2) \,. \tag{11.657}$$

The reader should verify that the *linking number* of any disjoint pairs of Hopf fibres is 1.

11.18.6 Example: Kaluza-Klein Theory

Consider the time-independent Schrödinger equation on on the Heisenberg group with left-invarinat metric

$$ds^2 = (dz - zdy)^2 + dx^2 + dy^2 \,. \tag{11.658}$$

One finds that

$$-\nabla^2\phi = E\phi \Rightarrow \phi_{xx} + \phi_{zz} + (\partial_+ x\partial_z z)^2\phi = -E\phi \,. \tag{11.659}$$

If one separates variables $\phi = \Psi(x, y)e^{iez}$ one gets

$$\Psi_{xx} - e^2\Psi + (\partial_y + iex)^2\Psi = -E\Psi \,. \tag{11.660}$$

This is of the form

$$(\partial_x - ieA_x)^2\Psi + (\partial_y - ieA_y)^2\Psi = -E\Psi \tag{11.661}$$

with

$$A = -xdy \Rightarrow F = -dx \wedge dy \,. \tag{11.662}$$

This is the Schrödinger equation for the Landau problem, i.e. that of a particle of charge e moving in the plane in a uniform magnetic field. *If the coordinate z is periodic,$0 \le z \le 2\pi L$ then the electric charge e is quantized $eL \in \mathbb{Z}$.*

A more general case works as follows. Let [11] WE have ametric on M

$$ds^2 = (\dot{z}) + A_\mu dx^\mu)^2 + g_{\mu\nu} dx^\mu dx^\nu \,, \tag{11.663}$$

with Killing vector field $\frac{\partial}{\partial z}$ generating an $S^1 = SO(2)$ isometry groups. The point particle Lagrangian on M

$$(\dot{z} + A_\mu \dot{x}^\mu)^2 = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \,. \tag{11.664}$$

The conserved charge is

$$\dot{z} + A_\mu \dot{x}^\mu = c \,. \tag{11.665}$$

The equation of motion for x^m turns out to be

$$\frac{d^2 x^\sigma}{dt^2} + \left\{ {}_\mu {}^\sigma {}_\nu \right\} \dot{x}^\mu \dot{x}^\nu = c g^{\sigma\tau} F_{\tau\mu} \dot{x}^\mu \tag{11.666}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This is a charged geodesic motion in the *quotient* or *base manifold* $B = M/SO(2)$.

As an example consider geodesics of the Berger spheres

$$L = (\dot{\psi} + \cos\theta \dot{\phi})^2 + \sin\theta^2 \dot{\phi} + \dot{\theta}^2 \,. \tag{11.667}$$

We have $A = \sin\theta d\phi \Rightarrow F = dA = -\sin\theta d\theta \wedge d\phi$. This is a magnetic monopole on the $B = S^2 = SU(2)/SO(2)$. *Electric charge quantization* becomes *angular momentum conservation* in the extra dimension, a view point with a curiously echo of the ideas of Lord Kelvin and other nineteenth century physicists based on the Corioli force, that magnetic forces were essentially gyroscpic in origin..

The reader should verify that the orbits are all little circles on S^2 with a size fixed by the magnitude of the charge, or equivalently the radius. One may repeat the problem on the Hyperbolic plane $H^2 = PSL(2, \mathbb{R} = SO_0(2,1)$. If this is mapped into the unit disc in the complex plane, one finds that all orbits are circles. For weak magnetic field the circles intersect the circle at infinity. For large enough magnetic field, the orbits are closed and lie inside the unit disc. If we pass to a compact quotient $\Sigma = H^2/\Gamma$, Γ a suitable discrete subgroup of $PSL(2, \mathbb{R}$, then it is known that the geodesic motion on an energy shell is ergodic. The charged particle motion remains ergodic as long as the orbits on the covering space are not closed circles.

12 Fibre Bundles

Fibre bundles are special types of manifolds which are locally products of a base manifold B with a fibre manifold F. Globally however they may be twisted, like a Moebius band for which $B = S^1$ and the fibre $F = I$, the interval. Fibre bundles and connections arise everywhere in physics whenever some idea of gauge-invariance is involved, the gauge group consisting roughly of reparameterizations of the fibres. They provide the natural language for Yang-Mills theory. During the 1970's and 80's the success of Yang-Mills theory stimulated a great deal of discussion of gravity as a gauge theory. In fact general relativity is very different from Yang-Mills theory but these two theories do share the common feature that their structure can be illuminated by using the language of fibre-bundles.

To begin with we define a *bundle E over a base manifold M* as a manifold e with smoooth projection map $\pi : E \rightarrow B$ onto a manifold B. The inverse image π_x^{-1} of a point $x \in B$ is called the *fibre* F_x above the point x. In what follows we shall assume that all fibres are diffeomorphic. Here are some examples.

The Moebius band This may be thought of as a ribbon with the ends ideentified by a twist through π. The base is the central circle, $B = S^1$ and the fibre is an interval.

The Tangent bundle $T(M) = \cup_x T_x(M)$. This is the space of positions and velocities $E = \{x, v; x \in M\, v \in T_x(M)\}$. Thus $B = M$ and $F_x = T_x(M)$, with

$$\pi : T(M) \rightarrow M \,, \qquad (x, v) \rightarrow x \,. \tag{12.668}$$

The Co-tangent bundle $T^\star(M)$. This is the space of positions and momenta $E = \{x, p; x \in M\, p \in T^\star_x(M)\}$.Thus $B = M$and $F_x = T^\star_x(M)$, with

$$\pi : T^\star(M) \rightarrow M \,, \qquad (x, p) \rightarrow x \,. \tag{12.669}$$

[11]Experts will realise that, for simplicity, we have set the 'gravi-scalar'to a constant value

A *Fibre Bundle* has more structure since the fibres F must lie inside E in a special way which is locally a product. We define it as a quintuple $\{E, \pi, B, F, G\}$ consisting of

(I) A manifold E , projection map π, base space B, fibre F together with a *structural group* G of diffomeomorphism of F acting on the left.

(II) An atlas of charts ,i.e a covering of B by open sets U_i, where i indexes the sets, and maps ϕ_i called *local trivializations* such that

$$\phi_{:}\pi^{-1}(U_i) \rightarrow U_i \times F \tag{12.670}$$

where

$$\phi_i(p) = \{\pi(p), g_i(p)\}\,, \quad p \in \pi^{-1}(U_i) \tag{12.671}$$

and

$$g_i : \pi^{-1}(U_i) \rightarrow F\,. \tag{12.672}$$

Moreover, if we define the restriction

$$g_i(x) = g_i|_{F_x}\,, \tag{12.673}$$

then $g_i(x) : F_x \rightarrow F$ is a left action of G on F.

(III) compatibility conditions such that $\forall U_i, U_j : U_i \cap U_j \neq \emptyset$ and if we define *transition functions* by

$$g_{ij}(x) = g_i(x) \circ g_j^{-1}(x) : F \rightarrow F\,, \tag{12.674}$$

then $\forall U_i, U_j, U_k : U_i \cup U_j \cup U_k \neq \emptyset$,

$$g_{ij}(x) g_{jk}(x) = g_{ik}(x) \forall x \in U_i \cup U_j \cup U_k\,. \tag{12.675}$$

The intuitive content is in each patch E looks like a product, in double intersections the fibres are twistd with the transition functions g_{ij} and these transition functions must satisfy the obvious compatibility codition, sometimes referred to as the *co-cycle condition* on triple intersections.

A *principal bundle* is one for which $F = G$ and G acts on G by left-translation.

A real *vector bundle* of rank n is one for which $F = \mathbb{R}^n$ and G acts on F via some representation.

Given a principal bundle one may thus construct an *associated vector bundle* with the same structural group using a suitable represensatation of G on F. In fact, given a *non-linear- realization*[12] of G on any manifold F we can construct an associated bundle with fibre F and structural group G.

12.0.7 Example: Frame bundle and associated Tangent bundle

Given an n-dimensional manifold M one may consider its *frame bundle* $F(M)$ consisting of all frames at at possible points $E = \{(x, \mathbf{e}_a)\}$ where $\mathbf{e}_a$ is a possible basis for $\mathbb{R}^n$. Since $GL(n, \mathbb{R})$ acts simply transitively on the set of frames for $\mathbb{R}^n$, the frame bundle is a principal bundle with structual group $GL(n, \mathbb{R})$. The tangent bundle $T(M)$ and co-tangent bundle $T^\star(M)$ are then associated vetor bundles via the defining representation and the contra-gredient representation respectively. The transition functions between two charts x^μ and $\tilde{x}^\nu$, are given by $\frac{\partial x^\mu}{\partial \tilde{x}^\nu}$ and the compatibility condition is satisfied by virtue of the chain rule.

One can go on to define further associated bundles of tensor fields since each fibre carries a representation of $GL(n, \mathbb{R})$.

It is possible to regard the frame bundle for $\mathbb{R}^n$ considered as an affine space, as the affine group $F(\mathbb{R}^n) = A(n) = \mathbb{R}^n \bowtie GL(n, \mathbb{R})$ and the base space as the coset $\mathbb{R}^n = GL(n, \mathbb{R}) \backslash A(n)$ The projection map π assigns each element of $A(n)$ to its coset with respect to $GL(n, \mathbb{R})$.

To make this more concrete, recall that $E = A(n) = \{S, x\}$ may be given a matrix representation as

$$\begin{pmatrix} S & x \\ 0 & 1 \end{pmatrix} \tag{12.676}$$

with $S \in GL(n\mathbb{R})$ and x column n vector acts on $\mathbb{R}^n$ considered as the column vector

$$\begin{pmatrix} X \\ 1 \end{pmatrix}\,. \tag{12.677}$$

[12]this is just an old-fashioned term, still used by some physicists for an group action which is not via a linear map on some vector space

The projection map π maps to

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \tag{12.678}$$

which may be affected by left multiplication by

$$\begin{pmatrix} S^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \tag{12.679}$$

and which commutes with right multilication by

$$\begin{pmatrix} S' & 0 \\ 0 & 1 \end{pmatrix}, \tag{12.680}$$

which takes the fibre $S \to SS'$ but leaves the base point x invariant.

This is a special case of a more general phenomenon which we describe in the next section.

12.0.8 Group extensions as Principal bundles

We take $E = G \ltimes H$. in (11.487), take G as the structural group and the base $B = H$. Thus the projection map $\pi : E \to B$ coincides with the map $\pi : G \ltimes H \to H$ in (11.487). Now, as we will prove shortly, any principal bundle (E, π, B, G) admits a global right action by G which commutes with π. To check this in the present example, we note that

$$(g,h)(g',e) = (gg', h\rho_g(e)) = (gg',h), \tag{12.681}$$

but

$$(g,h) \xrightarrow{\pi} h\,, \qquad (gg',h) \xrightarrow{\pi} h\,, \tag{12.682}$$

and we are done.

Note in addition that there is a left action of G on $E = G \ltimes H$ which does not, in general, commute with π since

$$(g,e)(g',h') = (gg', \rho_g(h')) \tag{12.683}$$

does not commute with

$$(g',h') \xrightarrow{\pi} h'\,, \qquad (gg',\rho_g(h')) \xrightarrow{\pi} \rho_g(h')\,. \tag{12.684}$$

12.1 Global right action on a Principal bundle

We shall now prove that *a principal bundle $\{E, \pi, B, G\}$ admits a right action by G which commutes with the projection $\pi : E \to B$.*

The action defined the action in each chart by

$$F_x = G \to G_x h\,, \qquad h \in G\,, \tag{12.685}$$

so that in the i'th chart we have

$$R^i_h = g_i^{-1} \circ R_h \circ g_i, \tag{12.686}$$

where R_h is right multiplication of G by h. Clearly R^i_h commutes with π and moreover defines a local right action

$$\begin{aligned} R^i_{h_2} \circ R^i_{h_1} &= g_i^{-1} \circ R_{h_2} \circ g_i \circ g_i^{-1} \circ R_{h_1} \circ g_i && (12.687)\\ &= g_i^{-1} \circ R_{h_2} \circ R_{h_1} \circ g_i && (12.688)\\ &= g^{-1} \circ R_{h_1 h_2} g_i && (12.689)\\ &= R^i_{h_1 h_s}\,. && (12.690) \end{aligned}$$

Now under change chart or trivialization

$$\begin{aligned} R^j_h &= g_j^{-1} \circ R_h \circ g_j && (12.691)\\ &= g_j^{-1} \circ g_i \circ g_i^{-1} \circ R_h \circ g_i \circ g_i^{-1} \circ g_j && (12.692)\\ &= g_{ji} \circ R^i_h \circ g_{ji}^{-1}\,. && (12.693) \end{aligned}$$

Now $g_{ji} = g_j^{-1} \circ g_i : G_x \to G_x$ acts on the *left* and hence commutes with R_h^i which acts on the *right* and thus

$$R_h^i = R_h^j \,, \tag{12.694}$$

and the action is independent of chart.

As a corollary we have that the base is the quotient of the bundle by the *right* action of G

$$B = E/G \,. \tag{12.695}$$

Thus for example,in Yang-Mills theory, local gauge transformations act on the left, while global gauge symmetries act on the right.

12.1.1 Examples: Coset spaces and monopoles

For example if E itself is a group, and H a subgroup and π the map assinging a group element to its coset G/H, then G is a principal bundle with structural group H and base $B = G/H$

Thus if $G = SU(2) = S^3$ and $H = SU(1)$ then $B = S^2$ and we get the simplest Hopf circle bundle. If $G = SO(3)$ and $H = SO(2)$ then again $B = S^2$ and we get another circle bundle over the 2-sphere. These bundles arise in Dirac's theory of magnetic monopoles. The former has the least possible magnetic charge, the latter twice the minimum value.

Another example was given in the previously: the frame bundle $E = F(\mathbb{R}^n) = A(n)$, $B = \mathbb{R}^n = A(n)/GL(n,\mathbb{R}$. Thus local frame rotations should be thought of as acting on the left, but there remains a global right action. We shall return to this example when we consider the bundle of pseudo-orthonormal frames.

12.2 Reductions of Bundles:

A bundle $\{E, \pi, B, F, G\}$ is said to be *reducible* if one may restric the transition functions g_{ij} to lie in a subgroup H of the structural group G to obtain a new bundle $\{E', \pi', B, F', H\}$ with the same base B.

12.2.1 Reduction of the frame bundle

The frame bundle $F(M)$ is a principal bundle with structural group $GL(n,\mathbb{R}$ because $GL(n,\mathbb{R})$ acts simply transitivley on the set of all frames at a point. Introduction of a metric $g(\,,\,)$ with signature (s,t) allows one to reduce the frame bundle to $OF_g(M)$ of pseudo-orthonormal frames. whose structural group is $0(s,t) \subset GL(n,\mathbb{R})$.

Such a reduction may not always be possible. Every manifold admits a riemannian metric [13] and so reduction t$O(n)$ is always possible. It is not hard to see that a further reduction to $SO(n)$ is possible if and only if the manifold is orientable .

In the case of Euclidean space $\mathbb{E}^n$ the ortho-normal frame bundle bundle may be identified with the Euclidean group, since it acts simply transitively on all possible orthonormal frames at all possible points. Thus

$$OF(\mathbb{E}^n) = E(n) \,, \tag{12.696}$$

$$\mathbb{E}^n = OF(\mathbb{E}^n)/O(n) = E(n)/O(n) \,. \tag{12.697}$$

A similar statement holds for spheres

$$OF(S^n) = SO(n+1) \,, \tag{12.698}$$

$$S^n = OF(S^n)/SO(n) = SO(n+1)/SO(n) \,. \tag{12.699}$$

12.2.2 Existence of Lorentzian metrics

A manifold may or may not admit a Lorentzian metric for which $(s-1)(t-1) = 0$. The necessary and sufficent condition is that the manifold M admits a direction field, i.e. an everywhere non-vanishing vector field V determined only up to a sign $\pm V$. To see why, let $g_R(\,,\,)$ be some Riemannnian metric on M and $g_L(\,,\,)$ the desired Lorentzian metric. To see the necessity we can diagonalize $g_L(\,,\,)$ with respect to $g_R(\,,\,)$ at every point. The single timelike eigen vector, which is determined only up to a scalar

[13] This may be proved using a partition of unity argument or by emebdding M in $\mathbb{R}^m$ for some m and using the induced metric. It is known that one may take $m = 2n + 1$

multiple, will define V. Conversely given V we can normalize it with respect to $g_R(\, , \,)$, and call it T say. Then we may put

$$g_L(\, , \,) = g_R(\, , \,) - 2T_\flat \otimes T_\flat \, , \tag{12.700}$$

where $T_\flat = g_R(T, \,)$.

Thus,perhaps remarkably, the odd dimensional spheres S^{2m+1} all admit Lorentzian metrics. This is because the Hopf fibrations provide everywhere non-vanishing vector fields and we can take as our riemanian metric the round or canonical metric on S^{2m+1}.. For a compact manifold M a necessary and sufficient condition that such a direction field exists is that the Euler number vanishes. Thus

$$\mathcal{H}\left(S^{2m+1}\right) = 0 = \chi(\mathbb{RP}^{2m+1}) \, . \tag{12.701}$$

For a non-compact manifold the same is true, subject to suitable conditions at infinity.

The Hopf vector field on S^{2m+1} has closed S^1 orbits. Thus these Lorentzian metrics have *closed timelike curves* often abbreviated to *CTC*s. In fact it is a general theorem that any compact Lorentzain manifold must have at least one CTC.

In the case of Minkowski spacetime $\mathbb{E}^{n-1,1}$ the orthonormal frame bundle is known as *the space of all possible reference frames* and it may be identified with the Poincaré group $E(n-1,1)$. Thus

$$\begin{aligned} OF(\mathbb{E}^{n-1,1}) &= E(n-1,1) \, , & (12.702) \\ \mathbb{E}^{n-1,1} &= OF(\mathbb{E}^{n-1,1})/O(n-1,1) = E(n-1,)/O(n-1,1) \, . & (12.703) \end{aligned}$$

Similarly for the de-Sitter spacetimes

$$\begin{aligned} OF(dS_n) &= SO(n,1) \, , & (12.704) \\ dS_n &= OF(dS_n)/SO(n-1,1) = SO(n,1)/SO(n-1,1) \, . & (12.705) \\ OF(AdS_n) &= SO(n-1,2) \, , & (12.706) \\ AdS_n &= OF(AdS_n)/SO(n-1,1) = SO(n-1,2)/SO(n-1,1) & (12.707) \end{aligned}$$

12.2.3 Time orientation

Clearly the direction field $\pm T$ defines a local direction of time. The spacetime $\{M, g_L(\, , \,)\}$ will be *time-orientable*, if and only the $\pm$ ambiguity can be removed. This will certainly be true if M is simply connected, $\pi_1(M) = \emptyset$. For $n > 3$, Lorentz group $O(n-1,1$ has four connected components, two of which $O^\uparrow(n-1,1)$ preserve the direction of time. Evidently a spacetime is time-orientable if and only if we can reduce the structual group of the frame bundle to $O^\uparrow(n-1,1)$. The Lorentzian metrics on S^{2m+1} constructed above may have closed timelike curves but they are at least time-orientable. Even though one may reach one's past, one may do so only in the forward direction of time.

Similar remarks apply to space orientation and total orientation.

The fact that the standard model of particle physics has interactions which break P,T, and PT may be used to argue, at least in all regions of our spacetime in which the standard model is valid, that the universe is time space and spacetime orientable. In other words the frame bundle may be reduced to to the identity component $SO_0(3,1)$.

12.3 Global and Local Sections of Bundle

This generalizes the idea of function or field. A *smooth section s* of a bundle (E, π, B) is a smooth map $s : B \to E$ such that $\pi \circ s = \mathrm{id}_B$, where id_B is the identity map on B.

Thus a vector field $V^\mu(x)$ is a *section of the tangent bundle* $T(M)$ and a co-vector field $\omega_\nu(x)$ is a *section of the co-tangent bundle* $T^\star(M)$.

For a general bundle, smooth global sections may not exist globally, i.e. they may not be well defined for all B. For a *vector bundle* a smooth section always exists, for example the *zero section* for which $s(x) = 0 \, \forall x \in B$. However *everywhere non-vanishing sections* may not exist. For example, by the Hairy Ball Theorem, there are no every where non-vanising vector fields on the two-sphere S^2. Since we may think of $SO(3)$ as the bundle of unit tangent vectors to S^2, $ST(S^2)$, or equivalently as $OF(S^2)$, the ortho-normal frame bundle of S^2,it follows that this bundle has no global sections. The same is true $SU(2)$ considered as the Hopf bundle over S^2.

12.3.1 Globally Hyperbolic four-dimensional spacetimes are pararlellizable

It is known that the frame bundle of any three-manifold is trivial, that is any three-manifold Σ_3 is parallelizable and thus admits a global triad field or drei-bein $\mathbf{e}_i$, $i = 1, 2, 3$. Now a globally hyperbolic n-dimensional spacetime M_4 is of the form

$$\mathbb{R} \times \Sigma \,, \tag{12.708}$$

where Σ is an $n-$ manifold, and $t \in \mathbb{R}$ is a time function on M_4 which increases along every timelike curve. Thus if $n = 4$, the triad or drei-bein $\frac{\partial}{\partial t}, \mathbf{e}_i$ is a global frame field.

12.3.2 Euler number of a vector bundle

For a vector bundle whose rank equals the dimension n of the base,such as the tangent bundle for example, any section S is an n-dimensional submanifold of a $2n$-dimensional manifold. If it is everywhere non-vanishing it will never interesect the zero-section $B \subset E$. In general however, it will intersect B in a finite number number of points. If they are counted taking into account orientation ('intersections and anti-intersections ') we get a topological invariant of the vector bundle called the Euler number $\chi(E)$. A necessary, and in fact sufficient condition that E admit an everywhere non-vanishing section is $\chi(E) = 0$. One defines the Euler number of the manifold as the Euler number of the tangent bundle $\chi(M) = \chi(T(M))$.

12.3.3 Global sections of principal bundles

In fact *a principal bundle E is trivial, i.e. $E = B \times G$ if it admits a global section.* To see why, suppose that $s(x)$ is the section. Then any point $p \in G_x$ may be written uniquely as $s(x)R_g(p)$, where $R_g(p)$ is the right action of G on G_x required to move s(x) to p. This gives a map taking $E \to (x = \pi(p), g(p)) \in B \times G$ which is a global trvialization. Conversely,if E is trvial, then *any* map from $B \to G$ provdes a global section.

The tangent bundles and frame bundles of Lie groups are trvial
$T(G) = G \times \mathbb{R}^n, F(G) = G \times GL(n, \mathbb{R})$.

12.4 Connections on Vector Bundles

Let $\Gamma(E, \pi, B, V, G)$ be the space of smooth sectioons of of a vector bundle. A connection ∇ provides a map from $T(M) \times \Gamma(E, \pi, B, V, G) \to \Gamma(E, \pi, B, V, G)$ written $\nabla_X s$, with $X \in T(M)$ and $s \in \Gamma(E, \pi, B, V, G)$ such that

(i) $\nabla_X(s + s') = \nabla_X s + \nabla_X s'$.

(ii) $\nabla_{X+X'} s = \nabla_X s + \nabla_{X'} s$.

(iii) $\nabla_X(fs) = f\nabla_X s + (Xf)s \,, \qquad \forall f \in C^\infty(B)$.

(iv) $\nabla_{fX} s = f\nabla_X s \,, \qquad \forall f \in C^\infty(B)$.

The curvature 2-form F is defined by

$$F(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \,. \tag{12.709}$$

In a natural basis

$$F_{\mu\nu} s = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)s. \tag{12.710}$$

Suppose that $\mathbf{s}_a$, $a = 1, 2, \ldots, \mathrm{dimV}$ is a local basis of section such that $s = \mathbf{s}_a s^a$ let

$$\mathbf{s}_a A_\mu{}^a{}_b = \nabla_\mu \mathbf{s}_b \,, \tag{12.711}$$

then

$$(\nabla_\mu s)^a = \partial_\mu s^a + A_\mu{}^a{}_b s^b \,. \tag{12.712}$$

The connection 1-forms transform under change of basis in the obvious way

$$s^a \to \Lambda^a{}_b s^b \,, \qquad \mathbf{s}_a \to \mathbf{s}_b (\Lambda^{-1})^b{}_a \,, \tag{12.713}$$

$$A_\mu{}^a{}_b \to \Lambda^a{}_e A_\mu{}^e{}_c (\Lambda^{-1})^c{}_b - (\Lambda^{-1})^a{}_c \partial_\mu \Lambda^c{}_b \,. \tag{12.714}$$

The curvature two-form is given by

$$F^a{}_{b\,\mu\nu} = \partial_\mu A_\nu{}^a{}_b - \partial_\nu A_\mu{}^a{}_b + [A_\mu, A_n]^a{}_b \,, \tag{12.715}$$

and transforms homogeneously

$$F^a{}_{b\,\mu\nu} \to \Lambda^a{}_e F^e{}_c (\Lambda^{-1})^c{}_b \,. \tag{12.716}$$

12.4.1 Form Notation

Set

$$A^a{}_b = (A)^a{}_b = dx^\mu A_\mu{}^a{}_b \tag{12.717}$$

and one has

$$F = dA + A \wedge A\,, \tag{12.718}$$

where A is a matrix valued 1-form and matrix multiplication is understood.

12.4.2 Metric preserving connections

We take ∇ to be a metric preserving connection on the tangent bundle $T(M)$. Consistency with our previous notation means that $\mathbf{s}_a \to \mathbf{e}_e$ the vielbein and the one-forms $A^a{}_b$ become the one-forms $\Gamma^a{}_b$ and the curvature $F^a{}_b$ is replaced by $R^a{}_b$

$$R^a{}_b = d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b\,. \tag{12.719}$$

The metric preserving condition becomes

$$dg_{ab} = \Gamma_{ab} + \Gamma_{ba}\,. \tag{12.720}$$

In adition, because we are dealling with the tangent bundle $T(M)$ we need to impose a torsion condition

$$d\mathbf{e}^a + \Gamma^a{}_b \wedge \mathbf{e}^b = T^a\,. \tag{12.721}$$

13 Symplectic Geometry

Symplectic geometry is a generalization of Hamiltonian mechanics to manifolds more general than the cotangent bundle $T(Q)$ of some n-dimensional configuration space Q. In that case one might start with a Lagrangian $L(q^i, v^j)$ where q^i are local coordinates for Q and $v^i = \frac{dq^i}{dt} = \dot{q}^i$ is the velocity of a typical dynamical trajectory, curve $q(t)$ in Q. Thus the lagrangian is a function on the tangent bundle $L :\to T(M) \to \mathbb{R}$. For example for *geodesic or free motion* on Q we take $L = \frac{1}{2} g_{ij} v^i v^j$, where g_{ij} is some metric on Q. Given an Lagrangian $L(q^i, v^j)$ on we get a map, called a Legendre transform, to the cotangent bundle $T^\star(Q)$ via

$$p_i = \frac{\partial L}{\partial v^i}\,. \tag{13.722}$$

The Euler-Lagrange equations now read

$$\frac{dp_i}{dt} = -\frac{\partial L}{\partial q^i}\,. \tag{13.723}$$

Note that the motion may be lifted to give a vector field $v^i \frac{\partial}{\partial q^i} + \dot{v}^j \frac{\partial}{\partial v^j}$ and flow on $T(M)$.

Now if the Lagrangian $L(q^i, v^j)$ is a convex function of the fibre coordinate v^i then the Legendre map may be inverted and the velocity v^i may be regarded as a single valued function of the momentum p_i. In other words for a given Lagrangian, we can set up an isomorphism between the tangent bundle $T(Q)$ and the co-tangent bundle called *Legendre duality.* For the Lagrangian associated to geodesic motion Legendre duality is just the musical isomorphism, i.e index raising and lowering. Physically one may think of Legendre duality as *Wave-Particle duality* because given any wave vector we can obtain particle velocity. We may now define the Hamiltonian $H(q^i, p_j) : T^\star(Q) \to \mathbb{R}$ as the Legendre transform, of the Lagrangian with respect to the fibre coordinate

$$H(q^i, p_j) = p_i v^j(p_k) - L(q^i, v^j(p_k))\,. \tag{13.724}$$

The flow on $T(M)$ may be pushed forward $T^\star(Q)$ and is given by *Hamilton's equations*

$$\boxed{\begin{array}{ccc} \frac{dp_i}{dt} & = & -\frac{\partial H}{\partial q^i}\,, \\ \frac{dq^i}{dt} & = & \frac{\partial H}{\partial p_i} \end{array}}\,. \tag{13.725}$$

This flow in $T^\star(Q)$ is tangent to the *Hamiltonian vector field*

$$X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^j} \frac{\partial}{\partial_j}\,. \tag{13.726}$$

More generally, given any function (or *classical oberservable*) $f = f(q^i, p_j) : T^\star(Q)$ we can define a vector field by

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^j}\frac{\partial}{\partial_j} \,. \tag{13.727}$$

We now try to 'covariantize'our treatment by introducing coordinates $x^\mu = (q^i, p_j)$, with $\mu = 1, 2, \ldots, 2n$ so that Hamilton's equations become

$$\frac{dx^\mu}{dt} = X^\mu_H = \omega^{\mu\nu}\partial_\nu H \,, \tag{13.728}$$

where

$$H^\mu = (\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q^j}), \tag{13.729}$$

the bi-vector [14] $\omega^{\mu nu}$ is given by and

$$\omega^{\mu\nu} = -\omega^{\nu\mu} = \begin{pmatrix} 0 & \delta^i_j \\ -\delta^i_j & 0 \end{pmatrix} \,. \tag{13.730}$$

The inverse symplectic 2-form $\omega_{\mu\nu}$ such that $\omega^{\mu\nu}\omega_{\nu\sigma} = \delta^\mu_\sigma$ is

$$\omega = \frac{1}{2}\omega_{\mu\nu} dx^\mu \wedge dx^\nu = dp_i \wedge dq^i \,. \tag{13.731}$$

has the properties that

(i) it is non-dgenerate $\det \omega \neq 0$

(ii) It is closed $d\omega = 0$.

In fact ω is exact, $\omega = d\theta$, where $\theta = p_i dq^i$ is the canonical 1-form on $T^\star(Q)$. One may now define a general *symplectic manifold* or *generalized phase space* $\{P, \omega\}$ as a manifold P equipped with a two-form ω satisfying conditions (i) and (ii) Obviously P must be even dimensional dimP $= n = 2m$ say. We don't insist that ω is exact. Some non-standard examples are which are not co-tangent bundles are

The two-sphere $P = S^2$ and $\omega = \sin\theta d\theta \wedge d\phi = \eta$ the volume form. This is certainly closed but it is not exact. Note that in general if P is compact then $\eta = \frac{(-1)^m}{m!}\omega^m$ is a volume form. If ω were exact $\omega = d\theta$ then η would be exact $\eta = \frac{(-1)^m}{m!} d(\theta \wedge \omega^{m-1})$. Therefore $\int_P \; eta = 0$, which is a contradiction. Thus if P is compact, then ω cannot be closed.

The weather In the geostrophic approximation the wind velocity $\mathbf{v}$ is governed by the balance between the Corioli force and presure gradient,

$$2\mathbf{\Omega} \times \mathbf{v} = -\frac{1}{\rho}\nabla P, \tag{13.732}$$

where $\mathbf{\Omega}$ is the angular velocity of the earth, ρ is the density fo the atmosphere and P the pressure. This a Hamiltonian flow with the role of the Hamiltonian being played by the pressure. The symplectic form is not proportional to the volume form.

Vortex motion Consider k parallel Kirchoff line vortices moving perpendicular to the the $x-y$plane with coordinates $\mathbf{r}_a(x_a, y_a)$, $a = 1, 2, \ldots k$. If $r_{ab} = |\mathbf{r}_a - \mathbf{r}_b|$, the equations of motion are motion are

$$\frac{dx_a}{dt} = \sum_{b\neq a} \gamma_b \frac{y_a - y_b}{r^2_{ab}} \,, \qquad \frac{dy_a}{dt} = -\sum_{b\neq a} \gamma_b \frac{x_a - x_b}{r^2_{ab}} \,, \tag{13.733}$$

where γ_a is the strength of the a'th vortex. This is a Hamiltonian flow on $\mathbb{R}^2 \setminus \Delta$, where Δ is the diagonal on which more than one vortex position coincides. The symplectic form is

$$\omega = \sum \gamma_a dx_a \wedge dy_a \,. \tag{13.734}$$

and the Hamiltonian is

$$H = -\sum\sum \gamma_a \gamma_b \gamma \ln r_{ab} \tag{13.735}$$

[14] A p-vector is totally anti-symmetric contravariant tensor, i.e. an element of $\Lambda^*(V^\star)$

13.0.3 Poisson Brackets

Let X_g be a Hamiltonian vector field and define

$$\{f,h\} = X_g f \quad (13.736)$$
$$= \omega^{\mu\nu}\partial_\mu f \partial_\nu g \quad (13.737)$$
$$= \omega^{-1}(df,dg)\,. \quad (13.738)$$

Now $X^\mu_g = \omega^{\mu\nu}\partial_\nu g$ implies that

$$\omega_{\sigma\mu}X^\mu_g = \omega_{\sigma\mu}\omega^{\mu\nu}\partial_\nu g = \partial_\sigma g\,,. \quad (13.739)$$

that is

$$dg = \omega(\,,X_g)\,. \quad (13.740)$$

Thus

$$Yg = \omega(Y,X_f)\,, \quad (13.741)$$

$$X_g = \{f,g\} = \omega(X_g,X_f)\,. \quad (13.742)$$

13.0.4 A generalization: Poisson Manifolds

These area manifolds P equipped with a bi-vector field $\omega^{\mu\nu} = -\omega^{\nu\mu}$. We then define a skew-symmetic map $\mathfrak{F}(P)\times\mathfrak{F}(P)\to\mathfrak{F}(P)$ by

$$f,g \to \{f,g\} = \omega^{\mu\nu}\partial_\mu f \partial_\nu g\,. \quad (13.743)$$

The standard case is $P = \mathbb{R}^{2n}$ with

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \qquad \{f,g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\,. \quad (13.744)$$

Clearly,

(i) $\{f,g\} = -\{g,f\}\,,$ skew symmetry.

(ii) $\{f_1+f_2,g\} = \{f_1,g\}+\{f_2,g\}\,,$ linearity.

(iii) $\{cf,g\} = c\{f,g\}\,,$ $c\in\mathbb{R}$.

(iv) $\{f,gh\} = \{f,g\}h + g\{f,h\}\,,$ Leibniz.

(v) $\{f,\{g,h\}\}+\{h,\{g,f\}\}+\{g,\{f,h\}\} = J^{\mu\nu\sigma}\,\partial_\mu f\,\partial_\nu g\,\partial_\sigma h\,,$ where

$$J^{\mu\nu\sigma} = J^{[\mu\nu\sigma]} = \frac{1}{3}(\omega^{\mu\lambda}\partial_\lambda\omega^{\nu\sigma} + \omega^{\sigma\lambda}\partial_\lambda\omega^{\mu\nu} + \omega^{\nu\lambda}\partial_\lambda\,\omega^{\sigma\mu})\,. \quad (13.745)$$

Despite the fact that only partial derivatives appear in $J^{\mu\nu\sigma}$ it is a tensor, called the *Jacobi tri-vector* , as may be seen by noting that if ∇ is a torsion free connection

$$J^{\mu\nu\sigma} = \frac{1}{3}(\omega^{\mu\lambda}\nabla_\lambda\omega^{\nu\sigma} + \omega^{\sigma\lambda}\nabla_\lambda\omega^{\mu\nu} + \omega^{\nu\lambda}\nabla_\lambda\,\omega^{\sigma\mu})\,. \quad (13.746)$$

defines a *Poisson Manifold* is such that the Jacobi tri-vector vanishes

$$J^{\mu\nu\sigma} = 0\,. \quad (13.747)$$

It follows that the ring of functions $\mathfrak{F}(P)$ is endowed with a Lie algebra structure, called the *Poisson Algebra*.

The basic example is symplectic manifold for which

$$\partial_\sigma\omega^{\mu\nu} = -\omega^{\mu\lambda}\,\partial_\sigma\,\omega_{\lambda\rho}\,\omega^{\rho\nu}\,. \quad (13.748)$$

Thus

$$\omega^{[\mu|\lambda|}\,\partial_\lambda\,\omega^{\nu\sigma]} = 0 \quad\Leftrightarrow\quad \omega^{\mu|\lambda|}\omega^{\nu\rho}\,\partial_\lambda\,\omega_{\rho\tau}\omega^{|\tau|\sigma]} = 0 \quad (13.749)$$
$$\Leftrightarrow\quad \omega^{\mu\lambda}\omega^{\nu\rho}\partial_{[\lambda}\omega_{\rho\tau]}\omega^{\lambda\sigma} = 0\,. \quad (13.750)$$

The last line follows because

$$d\omega = 0 \Leftrightarrow \partial_{[\lambda}\omega_{\rho\sigma]} = 0\,. \quad (13.751)$$

13.0.5 Poisson manifolds which are not Symplectic

Let $P = \mathbb{R}^3 = \{s_1, s_2, s_3\}$ be a *spin* with magnetic moment μ undergoing precesion in a constant magnetic field $\mathbf{B}$. The equations of motion are

$$\frac{d\mathbf{s}}{dt} = \mu \mathbf{B} \times \mathbf{s} \,. \tag{13.752}$$

or

$$\frac{ds_i}{dt} = \epsilon_{ijk} \mu B_j \times s_k \,. \tag{13.753}$$

or

$$\frac{ds_i}{dt} = s_k \epsilon_{ijk} p_j H \,, \tag{13.754}$$

with

$$H = \mu \mathbf{B}.\mathbf{s} \,. \tag{13.755}$$

This has the form of Hamilton's equations

$$\frac{dx^\mu}{dt} = \{x^\mu, H\} = \omega^{\mu\tau} \partial_\tau H, \tag{13.756}$$

with

$$\omega^{ij} = s_k \epsilon_{kij} \,, \tag{13.757}$$

and hence

$$\{f, g\} = \mathbf{s}.(\nabla f \times \nabla g) \,. \tag{13.758}$$

Since P is odd dimensional it cannot be symplectic but it is a Poisson since the Jacobi identity holds. This example can be trivially generalized to n spins on the n-fold product manifold, perhaps with non-trivial interactions between spins encoded in a non-trivial Hamiltonian function H. We shall not pursue that generalization here but rather notice that $\mathfrak{so}(3) = \mathbb{R}^3$ and consider $P = \mathfrak{g}^\star$, the dual of the Lie algebra $\mathfrak{g}$ of some Lie group G. let coordinates on P be μ_a, $a = 1, 2, \ldots \mathrm{dim} G$. (the choice of lowered indices and the letter μ is deliberate). We consider the Hamilton equations

$$\frac{d\mu_a}{dt} = \mu_c C_a{}^c{}_b \frac{\partial H}{\partial \mu_b} \,. \tag{13.759}$$

The Poisson structure is given by

$$\omega_{ab} = \mu_c C_a{}^c{}_b \,, \qquad \{f, g\} = \mu_a C_b{}^a{}_c \frac{\partial f}{\partial \mu_b} \frac{\partial g}{\partial \mu_c} \,. \tag{13.760}$$

One may check that the Jacobi identity for the Possion structure is satisfied by virtue of the Jacobi identity for the Lie algebra

$$C_{[a}{}^s{}_{|d|} C_b{}^d{}_{c]} = 0 \,. \tag{13.761}$$

The choice $\mathfrak{g} = \mathfrak{so}(3)$ gives our first example above. The case $G =$ Nil or the Heisenberg group of upper triangular matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \tag{13.762}$$

works as follows. We set $\mu = (p, q, r)$ and find

$$\{f, g\} = r\left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\right) . \tag{13.763}$$

Note that an arbitrary function of r Poisson commutes with any function of p, q, r and hence lies in the centre of the algebra. Such functions are called *Casimir* functions.

13.0.6 Co-adjoint orbits

The Lie algebras may or may not be odd dimensional but the bi-vector $\omega_{ab} = \mu_c C_a{}^c{}_b$ does not have maximal rank. However the orbits in $\mathfrak{g}^\star$ under the co-adjoint action are even dimensional and if G is compact then, by a construction due to Kostant and Kirilov, the bi-vector induces a G-invariant symplectic sructure on them. In fact, the co-adjoint orbits are complex manifolds, and in fact Kähler manifolds. The case of $SO(3)$ gives the familar 2-sphere. The case $SU(3)$ has 6 dimensional orbits and four dimensional orbits, $SU(3)/U(1) \times U(1)$ and $SU(3)/U(2)$ respectively. The latter is complex projective space $\mathbb{CP}^2$.

In Yang-Mills theory, these manifolds arise as *vacuum manifolds* when an adjoint valued Higgs field breaks a compact gauge group G down to a subsgroup H.

13.0.7 The canonical one-form

We give here a formal proof that $T^\star(Q)$ is a symplectic manifold that will illustrate some of the ideas introduced earlier. We need to show that $T^\star(Q)$ admits a global canonical 1-form θ. We could give it in local coordinates but we wish to give a global definition. We must assign to very point in $T^\star(Q)$, a 1-form, that is an element of $T^\star(T^\star(Q)$. the projection map

$$\pi : T^\star(Q) \to Q\,, \tag{13.764}$$

and its pull-back goes in the oposite direction:

$$\pi^\star : T^\star(q) \to T^\star(T^\star(Q)) \tag{13.765}$$

Thus, define,

$$\theta = \pi^\star(p), \qquad p \in T^\star(Q)\,. \qquad \omega = d\theta\,. \tag{13.766}$$

In local coordinates $\theta = p_i dq^i \Rightarrow \omega = dp_i \wedge dq^i$ and ω has maximal rank $2\dim Q = \dim T^\star(Q)$. But the rank, being an integer, is clearly constant throughout $T^\star(Q)$.

13.0.8 Darboux's Theorem and quantization

Asserts the existence, for any symplectic manifold, not just one which is a co-tangent bundle, of local charts,$(p_i, q^)$, clearly not unique, in which

$$\omega = dp_i \wedge dq^i\,. \tag{13.767}$$

On S^2 such charts are known to geographers as *equal area map projections*. Acrhimedes's theorem on the equality of the areas of infinitesimal annuli on the sphere and on its tangent cylinder, which was inscribed on his grave stone, is equivalant to the existence of the Darboux chart with $q = \cos\theta$ and $p = \phi$,

$$\omega = \sin\theta d\theta \wedge d\phi = dp \wedge dq\,. \tag{13.768}$$

Note that S^2 is mapped just into the square $-1 \le q \le 1, 0 \le p \le 2\pi$. *Lambert's equal area projection* is given by

$$p = 2\sin\frac{\theta}{2}\cos\phi\,, \qquad q = 2\sin\frac{\theta}{2}\sin\phi\,. \tag{13.769}$$

It maps S^2 into the disc of radius 2.

A frequently adopted strategem when quantizing a physical system is to pass to a local Darboux chart and promote p, q to operators $\hat{p}, \hat{q}$ satisfying the Heisenberg commtutation relations.

$$[\hat{p}, \hat{q}] = i\frac{h}{2\pi}\,. \tag{13.770}$$

For a general symplectic manifold, this is not obviously a valid procedure, since as we have seen the Darboux charts may extend over a much larger region than the original phase space. Moreover, it is rather unclear that the quantization obtained in this way is independent of the charts used. The stdy of problems like this is called *Geometric Quantization* and will be dealt with later.

13.0.9 Lie Bracket and Poisson Bracket

Every function $f \in C^\infty(M)$ determines a vector field Hamiltonian vector field $X_f \in \mathfrak{X}(M)$ which acts on a function ϕ as

$$X_f\phi = \{\phi, f\} \Rightarrow (X_g X_f - X_f X_g)\phi \tag{13.771}$$

$$= \{\{\phi, f\}, g\} + \{\{g, \phi\}, f\} \tag{13.772}$$

$$= -\{\{f, g\}, \phi\} \tag{13.773}$$

$$= +\{, \phi, \{f, g\}\}\,, \tag{13.774}$$

which means that

$$\boxed{[X_f, X_g] = -X_{\{f,g\}}\,.} \tag{13.775}$$

That is there is n algebra *anti-homorphism* from the Poisson Algebra of functions to the Lie algbra of vector fields.

13.0.10 Canonical transformations as symplecto-morphisms

A *canonical transfornation* in mechanics is one which preserves the form of Hamilton's equations. We therefore take it to be a diffeomorphism

$$f : P \to P\,, \qquad \text{s.t.} \quad f^\star \omega \,=\, \omega. \tag{13.776}$$

Thus if $\dim P = 2n$, $n = 2$, these are area-preserving maps, and in higher dimensions they preserve the volume form

$$\eta = \frac{(-1)^n}{n!}\omega^n, . \tag{13.777}$$

In fact we can raise the symplectic form to any intermediate exterior power to get a closed $2r$ form

$$\omega^r, \tag{13.778}$$

which is preserved by a symplectomorphism.

An *anti-canonical transformation* or *anti-symplecto-morphism* is one which reverses the sign of ω. On the S^2 complex conjugation of the stereographic coordinates is an

An example on a cotangent bundle $T^\star(Q)$ would be *time reversal* $(p_i, q^i) \to (-p_i, q^j)$. Symplecto-morphisms which arise by pulling back, or lifting a diffeomorphism of Q its cotangent bundle, $T^\star(Q)$

$$(q^i\,,\, p_j) \to \tilde{(}q^i\,,\, \frac{\partial q^k}{\partial \tilde{q}^i} p_k) \tag{13.779}$$

are called in mechanics *point transformations.*

13.0.11 Infinitesimal canonical transformations : Poincaré's Integral Invariants:

If these are generated by a vector field X then

$$\begin{aligned} \pounds_X \omega &= 0 && (13.780)\\ &= i_X d\omega + d(i_X \omega) && (13.781)\\ \Rightarrow d(i_X) &= 0 && (13.782)\\ \Rightarrow i_X &= -dH && (13.783)\\ \Rightarrow \omega(X, \,) &= -dH && (13.784)\\ \Rightarrow X &= X_H\,, && (13.785) \end{aligned}$$

in other words *any infinitesimal canonical transformation is generated locally, by a Hamiltonian vector field for some function H and conversely.* Moreover the Hamiltonian is constant, $H = E$ along the flow

$$X_H H = \{H, H\} = 0, \tag{13.786}$$

which means that locally the flow lies in an $(2n-1)$-dimensional in the level set of H, i.e. in the submanifold Γ_E defined by

$$\Gamma_E = H^{-1}(E). \tag{13.787}$$

The Hamiltonian function H is only determined up to a constant and if P is not simply connected so that

$$\int_\gamma \omega(X, \,) \neq 0, \tag{13.788}$$

for some non-contractible 1-cycle (i.e. closed curve) γ, then no H will exist globally. It follows that if we have a Hamiltonian flow, generated by a vector field X_f, then all exterior powers of the symplectic form are Lie dragged along the flow

$$\pounds_{X_H} \omega^r = 0\,. \tag{13.789}$$

Thus if C_r is some p-chain which is also dragged along the flow $\pounds_{X_H} C_r = 0$, then *Poincaré's r-th integral invariant*

$$\int_{C_r} \omega^r \tag{13.790}$$

is constant along the flow.

Intuitively the integral invariants count the number of flow lines passing through the r-chain C_r.

In *classical kinetic theory* one thinks of an *ensemble* of many identical replicas of a Hamiltonian systems e.g. of a particle, and one introdces a *distribution function* or non-negative function $f(p,q,t)$ describing the *classical state* of the ensemble such that

$$f\frac{(-1)^n}{n!}\omega^n = f d^n\, p\, d^n\, q \tag{13.791}$$

is the number of systems in the interval $d^n\, p\, d^n\, q$, or if normalized

$$\int_P f\frac{(-1)^n}{n!}\omega^n = 1, \tag{13.792}$$

their probability density. If $P = T^\star(Q)$, then the number density on Q is obtained by integrating over the fibres is an $n-$form $nd^n q$ on Q given by

$$nd^n\, q = \int_{T^\star_q(Q)} f\eta\,, \qquad n(q) = \int_{\mathbb{R}^n} d^n\, pf(p,q)\,. \tag{13.793}$$

The expectation value of a classical observable, i.e. of a function $o(p,q) : P \to \mathbb{R}$ is

$$\bar{o} = \int_P of\eta\,. \tag{13.794}$$

Conservation of systems implies

$$\frac{\partial(f\eta)}{\partial t} + \pounds_{X_H}(f\eta) = 0\,, \tag{13.795}$$

That is

$$\boxed{\frac{\partial f}{\partial t} + \{f,H\} = 0\,, \qquad \text{Collisionless Boltzmann equation.}} \tag{13.796}$$

In the time-independent case, *Jean's Theorem* states that if $H, K_1, K_2, \dots K_k$ are $k+1$ mutually commuting quantities, then an *arbitrary* function of the $k+1$ variables

$$f = f(H, K_1, K_2 \dots K_k) \tag{13.797}$$

provides a solution. The *Maxwell-Boltzmann distribution* is one such solution

$$f = \frac{1}{Z}\exp -\beta(H - \mu^i K_i)\,, \tag{13.798}$$

where $\beta = \frac{1}{T}$ is the *inverse temperature*, μ^i is the i'th *chemical potential* and $Z = Z(\beta, \mu^i)$ the partition function given by

$$Z(\beta,\mu_i) = \int_P \exp -\beta(H - \mu^i K_i)\eta\,. \tag{13.799}$$

From it we can can calculate the mean energy

$$\bar{H} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta}\,. \tag{13.800}$$

13.1 Hamiltonian Symmetries and moment maps

Suppose that a group G acts on a symplectic manifold (P,ω) on the left preserving ω. The will be associated Hamiltonian vector fields X_a, $a = 1, 2, \dots \dim G$ such that

$$[X_a, X_b] = -C_a{}^c{}_b X_c\,, \tag{13.801}$$

where $C_a{}^c{}_{,b}$ are the structure constants of $\mathfrak{g}$. Now

$$\pounds_{X_a}\omega = 0 \Rightarrow X_a = X_{\mu_a}\,, \tag{13.802}$$

where

$$\mu(x^\mu) : P \to \mathfrak{g}^\star \tag{13.803}$$

are called *moment maps* or *momentum maps*. Clearly

$$X_{\{\mu_a,\mu_b\}} = C_a{}^c{}_b X_c \Rightarrow \{\mu_a,\mu_b\} = C_a{}^c{}_b \mu_c + \nu_{ab}\,, \tag{13.804}$$

where $\nu_{ab} = -\nu_{ba}$ are constants subject to the Jacobi identity

$$C_{[a}{}^d{}_b \nu_{|d|c]} = 0\,. \tag{13.805}$$

Now the moment maps μ_a are determined at best up to constants

$$\mu_a \to \mu_a + a_a\,, \Rightarrow \nu_{ab} \to \nu_{ab} - C_a{}^c{}_b a_c\,. \tag{13.806}$$

We now ask whether one may choose a_a such that $\nu_{ab} = 0$? This requires that one may solve

$$i) \quad \nu_{ab} = C_a{}^c{}_b a_c \tag{13.807}$$

$$\text{subject to} \quad (ii) \quad C_{[a}{}^d{}_b \nu_{|d|c]} = 0\,. \tag{13.808}$$

We shall now prove that if G is semi-simple then we may with no loss of generality set $\nu_{ab} = 0$. We make use of the non-degenerate Killing form:

$$C_a{}^c{}_b = B^{ce} C_{aeb}\,, \qquad C_{acb} = C_{[acb]}\,. \tag{13.809}$$

Now

$$C_{abc} C_e{}^{bc} = -B_{ae} \Rightarrow C_{bac} C^{ebc} = -\delta^c_e\,. \tag{13.810}$$

Now contract (i) with $C^a{}_e{}^b$ and find

$$a_e = -\nu_{ab} C^a{}_e{}^b\,. \tag{13.811}$$

Note that $\nu_{ab} = -\nu_{ba}$ determines a two form on the Lie algebra $\nu_a \in \Lambda^2(\mathfrak{g})$ which may be converted to a bi-invariant 2-form on the group $\nu \in \Omega^2(G)$. The reader may verify that condition (ii) says that the two-form is closed. Condition (i) is that it is exact. This is an example of Lie algebra co-homology. In effect we have proved that, if $\mathfrak{g}$ is semi-simple, then $H^2(\mathfrak{g}) = \emptyset$. If G is compact and semi- simple we get a more global result $H^2(G,\mathbb{R}) = \emptyset$.

13.1.1 Geodesics and Killing tensors

We consider the geodesic flow on a co-tangent bundle $T^\star(M)$ with Hamiltonian

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \tag{13.812}$$

and suppose K^μ is a Killing vector field then we get a conserved quantity $K^\alpha p_\alpha$ linear in the fibre coordinate

$$\begin{aligned} \{K^\alpha p_\alpha, H\} &= \{K^\alpha p_\alpha, \frac{1}{2} g^{\mu\nu} p_\mu p_\nu\} &(13.813)\\ &= \frac{\partial K^\alpha}{\partial x^\lambda} p_\alpha g^{\lambda\nu} P_\nu - K^\lambda \frac{1}{2} p_\lambda g^{\mu\nu} p_\mu p_\nu &(13.814)\\ &= -\frac{1}{2} (\pounds_K g^{\mu\nu}) p_\mu p_\nu = 0\,. &(13.815)\end{aligned}$$

The associated Hamitonian vector field is

$$X_{p_\alpha K^\alpha} = K^\lambda \frac{\partial}{\partial x^\lambda} - \frac{\partial K^\alpha}{\partial x^\lambda} p_\alpha \frac{\partial}{\partial p_\lambda}\,. \tag{13.816}$$

Under the projection map π this pushes down to the original Killing vector field on M.

$$\pi \star X_{p_\alpha K^\alpha} = K^\lambda \frac{\partial}{\partial x^\lambda}\,. \tag{13.817}$$

This should be contrasted with the behaviour of *Killing-Stäckel tensor fields* which are totally symmetric contravariant tensor fields of rank k say, $K^{\alpha_1\alpha_2\ldots\alpha_k} = K^{(\alpha_1\alpha_2\ldots\alpha_k)}$ such that

$$K = K^{\alpha_1\alpha_2\ldots\alpha_k} p_{\alpha_1} p_{\alpha_2} \ldots p_{\alpha_k}\,, \tag{13.818}$$

commutes with the geodesic Hamitonian

$$\{H, K\} = 0\,. \tag{13.819}$$

The reader may check that this true if and only if the following generalization of Killing's equations holds:

$$K^{(\alpha_1\alpha_2\ldots\alpha_k;\alpha_{k+1})} = 0\,, \tag{13.820}$$

where ; denotes covariant differentiation with respect to the Levi-Civita connection of the metric $g_{\mu\nu}$. The associated Hamiltonian vector field

$$X_K = k\,K^{\alpha_1\ldots\alpha_k} p_{\alpha_1}\ldots p_{\alpha_{k-1}}\frac{\partial}{\partial x^{\alpha_k}} - \frac{\partial K^{\alpha_1\ldots\alpha_k}}{\partial x^{\alpha_{k+1}}} p_{\alpha_a}\ldots p_{\alpha_k}\frac{\partial}{\partial p_{\alpha_k+1}}\,. \tag{13.821}$$

The push forward vector field now vanishes

$$\pi_\star X_K = 0\,. \tag{13.822}$$

Thus the Killing-Stäckel tensor does not generate an the action of a geometric symmetry on the the configuration space M. Nevertheless, it is a symmetry of the Hamiltonian system. Such symmetries are called *hidden* or *dynamical* symmetries. Important non-trivial examples are provided by the Runge-Lenz vector of the Hydrogen atom or of the motion of a planet around the sun and the geodesics of the Kerr rotating black hole metric.

13.1.2 Lie Bracket and Poisson algebra's of Killing fields

Suppose we have a group G acting by isometries on M. Thus the Killing fields $K_a^\lambda\partial_\lambda$ have moment maps $\mu_a = K_a^\alpha p_\alpha$ and have Lie brackets

$$\boxed{[K_a, K_b]^\mu = -C_a{}^c{}_b\,K_c^\mu\,.} \tag{13.823}$$

where $C_a{}^c{}_b$ are the structure constants of the Lie algebra $\mathfrak{g}$. The Poisson brackets of the moment maps are

$$\begin{aligned}
\{K_a^\alpha p_\alpha, K_b^\beta p_\beta\} &= \frac{\partial K_a^\alpha p_\alpha}{\partial x^\lambda}\frac{\partial K_b^\beta p_\beta}{\partial p_\lambda} - (a \leftrightarrow b) &(13.824)\\
&= K_{a\;,\lambda}^{\alpha} K_p^\lambda p_\alpha - (a \leftrightarrow b) &(13.825)\\
&= -[K_a, K_b]^\lambda p_\lambda &(13.826)\\
&= C_a{}^c{}_b K_c^\lambda p_\lambda\,. &(13.827)
\end{aligned}$$

In other words

$$\boxed{\{\mu_a, \mu_b\} = C_a{}^c{}_b\,\mu_c\,.} \tag{13.828}$$

13.1.3 Left-invariant metrics and Euler equations

Consider a left-invariant metric on a Lie group G with left-invariant one -forms λ^a.

$$ds^2 = g_{ab}\lambda^a\lambda^b\,. \tag{13.829}$$

The Killing fields are the right-invariant vector fields R_a^α. which generate left actions

$$\pounds_{R_a} ds^2 = 0\,. \tag{13.830}$$

and the associated conserved momenta, or moment maps are

$$N_a = R_a^\alpha p_\alpha\,, \qquad R_a = X_{N_a}\,. \tag{13.831}$$

We may also define *in general non-conserved* moment maps which generate right actions

$$Ma = L_a^\alpha p_\alpha\,, \qquad L_a = X_{M_a}\,. \tag{13.832}$$

The Possion algebra is

$$\begin{aligned}
\{N_a, N_b\} &= C_a{}^c{}_b N_c &(13.833)\\
\{M_a, M_b\} &= -C_a{}^c{}_b M_c &(13.834)\\
\{N_a, M_b\} &= 0\,. &(13.835)
\end{aligned}$$

The Hamiltonian is

$$
\begin{aligned}
H &= \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta & (13.836)\\
&= \frac{1}{2} L_a^\alpha L_b^\beta g^{ab} p_\alpha p_\beta & (13.837)\\
&= \frac{1}{2} g^{ab} M_a M_b \Rightarrow \dot{N}_a = \{N_a, H\} = 0\,. & (13.838)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\dot{M}_a &= \{M_a, H\} & (13.839)\\
&= g^{bc}\{M_a, M_b\} M_c & (13.840)\\
&= -g^{bc} C_a{}^e{}_b M_e M_c\,. & (13.841)
\end{aligned}
$$

Thus

$$
\boxed{\dot{M}_a = -g^{bc} C_a{}^e{}_b M_e M_c \qquad \text{Euler's equations.}} \qquad (13.842)
$$

For example if $G = SO(3)$ we have

$$
\mathbf{M} = (I_x, \omega^x, I_y \omega^y, I_z \omega^z), \qquad (13.843)
$$

where ω^i are the angular velocities with respect to the moving a body fixed frame and M_1 the angular momenta with respect to the moving body fixed frame, which, of course, are not conserved. The inverse metric $g^{ab} = \text{diag}(I_x^{-1}, I_y^{-1}, I_z^{-1})$ and the equations become the familar

$$
I_x \dot{\omega}^x = (I_y - I_z)\omega^y \omega^z\,, \text{etc.} \qquad (13.844)
$$

13.1.4 Liouville Integrable Systems

If $\dim P = 2n$, we need n mutually commuting constants of the motion, i.e. mutually Possion commuting functions including the Hamiltonian $H, K_1 K_s, \dots K_{n-1}$ in order to integrate the system completely. By Froebenius's theorem, these will generate n commuting flows on a set of *invariant tori* in P of the form

$$
H^{-1}(E) \cap K_1^{-1}(c_1) \cdots \cap K_{n-1}^{-1}(c_{n-1}) \qquad (13.845)
$$

where the constants E, c_i label the tori.

In the case of the top $n = 3$ and we have

$$
H = \frac{M_x^2}{2I_x} + \frac{M_y^2}{2I_y} + \frac{M_z^2}{2I_x} = \frac{1}{2}(I_x(\omega^x)^2 + I_y(\omega^y)^2 + I_z(\omega^z)^2) \qquad (13.846)
$$

$$
N_3 \qquad (13.847)
$$

and the Casimir

$$
\begin{aligned}
\mathbb{M}^2 &= M_x^2 + m_y^2 + M_z^2 & (13.848)\\
&= I_x^2(\omega^x)^2 + I_y^2(\omega^x)^2 + I_z^2(\omega^x)^2 & (13.849)\\
&= -B^{ab} M_a M_b & (13.850)
\end{aligned}
$$

where B^{ab} is the Killing form.

13.2 Marsden-Weinstein Reduction

This formalizes the idea of eliminating constants of the motion and their asscociated conjugate 'ignorable'coordinates to obtain a Hamitonian system with two fewer canonical coordinates. It arises, for example, in Dirac's theory of constrained Hamiltonian systems.

Formally, we suppose that $SO(2)$ (or $\mathbb{R}$), with Lie alegbra $\mathfrak{so}(2) = \mathbb{R}$, generated by the vector field X acts on a $2n$ dimensional symplectic manifold $\{P, \omega\}$ preserving ω. Infinitesimally $\pounds_X \omega = 0$, which means that there exists a moment map $\mu :\rightarrow \mathbb{R}$. Consider now the $(2n-1)$-dimensional level sets

$$
P \supset \Gamma = \mu^{-1}(c) \qquad c \in \mathbb{R}\,. \qquad (13.851)
$$

Since $X\mu = 0$, the orbits of $SO(2)$ lie in Γ and we may take the quotient $\Gamma/SO(2)$ to get, if the action of $SO(2)$ is suitably 'nice 'a smooth $(2n-2)$-dimensional manifold P' say. In fact if we restrict ω to $T(\Gamma)$, i,e. to vectors which are tangent to Γ, call it ω_G will have a one-dimensional kernel generated by, i.e. consisting of multiples of X. Locally Γ is a circle bundle over P' with the projection map π being that which assigns a point in Γ to the $SO(2)$ orbit through that point. The tangent space $T(P)$ may thus be identified with the vector space quotient $T(\Gamma)/\text{Kernel}\,\omega$, and on this quotient the restriction of the sympletic form ω_Γ, call it ω' is non-degenerate and moreover it is closed $d\omega' = 0.$.

In this way we get a new symplectic manifold

$$\{P', \omega'\} = \mu^{-1}(c)/SO(2) = P//SO(2) \tag{13.852}$$

called the *Marsden-Weinstein symplectic quotient*.

13.2.1 Example: $\mathbb{CP}^n$ and the isotropic harmonic oscillator

We are in $\mathbb{E}^n$ and thus $P = T^\star(\mathbb{E}^n) = \mathbb{R}^{2n} \equiv \mathbb{C}^n$ with canonical coordinates $q^r + ip_r = z^r$. Hamiltonian

$$H = \frac{1}{2}\sum(p_r)^2 + (q_r)^2 = \frac{1}{2}\sum|z^r|^2\,, \tag{13.853}$$

The Hamiltonian generates the $S^1 = SO(2) = U(1)$ action

$$z^r \to e^{-it}z^r \tag{13.854}$$

which gives the Hopf-fibration of the S^{2n-1} level sets of the Hamiltonian

$$H = \frac{1}{2}\sum|Z^r|^2 = E = \text{constant}. \tag{13.855}$$

The Marsden-Weinstein quotient $\mathbb{C}^n//U(1) = S^{2n-1}/U(1)$ is just complex projective space $\mathbb{CP}^n$, which is indeed a symplectic manifold and, as we shall see shortly, a Kähler manifold.

13.2.2 Example: The relativistic particle

In the 8-dimensinal co-tangent space of flat Minkowski spacetime $T^\star(\mathbb{E}^{3,1}), \omega = dp_\mu dx^\mu$ this has the *super-Hamiltonian*

$$\mathcal{H} = \frac{1}{2}p_\mu p^\nu, \tag{13.856}$$

The *mass-shell constraint* reduces the motion to the 7 -dimensional level set

$$\mathcal{H}^{-1}(-\frac{1}{2}m^2) \subset T^\star(\mathbb{E}^{3,1})\,, \tag{13.857}$$

where m is the rest mass. The physical, non-covariant or *reduced phase space* is $T^s tar(\mathbb{E}^3) = T^\star(\mathbb{E}^{3,1})//\mathbb{R}, \omega = d\mathbf{p} \wedge d\mathbf{x}$ and the reduced Hamitonian is just

$$H = \sqrt{\mathbf{p}^2 + m^2}\,. \tag{13.858}$$

13.2.3 Reduction by G

The construction may be generalised to the case when a k dimesional Lie group acts with Lie algebra $\mathfrak{g}$. If $\mu : P \to \mathfrak{g}^\star$ is the moment map we take $\Gamma = \mu^{-1}(p)$, $p \in \mathfrak{g}^\star$, we want G to act on Γ and commute with the quotient and therefore we take

$$P' = \mu^{-1}(p)/G\,, \qquad p \in \mathfrak{g}^\star \tag{13.859}$$

in which case p should be invariant under the co-adjoint action. If G is semi-simple,$\mathfrak{g} \equiv \mathfrak{g}^\star$ and this means that $p \in Z(\mathfrak{g})$, the centre of the Lie algebra. The result now is an $(2n-2k)$-dimensional symplectic manifold

$$\{P', \omega'\} = \{P, \omega\}//G\,. \tag{13.860}$$

13.3 Geometric Quantization

The basic aim of *Geometric Quantization* is given a classical symplectic manifold $\{P, \omega\}$, and its *classical algebra of observables* or functions $f, g, \dots \in C^\infty(P) = \Omega^0(P)$ equipped with with its Poisson bracket $\{f, g\}$ to find a Hilbert space $\mathcal{H}$ and Hermitian operators $\hat{f}, \hat{g} \dots$ such that

$$[\hat{f}, \hat{g}] = i\hbar \widehat{\{f, g\}} + \mathcal{O}(\eta^2) . \tag{13.861}$$

By a result of Van-Hove, it is known that one cannot arrange

$$[\hat{f}, \hat{g}] = i\hbar \widehat{\{f, g\}} \qquad \forall f, g \in C^\infty(M) , \tag{13.862}$$

and so either one tolerates an $\mathcal{O}(h^2)$ correction or chooses to 'quantize'only a subset of the clasical variables.

The Geometric Quantization procedure falls into two parts

(i) *Pre-quantization*

(ii) Finding a *polarization*

Prequantization. One demands that the symplectic form ω is such that $\frac{2\pi}{h}\omega$ is the curvature of some connection ∇ of some vector bundle E over P with fibre $F = \mathbb{C}$ and structural group $GU(1)$. Thus locally, acting on a section $s \in \Gamma(E, \pi, P, \mathbb{C}, U(1))$ as

$$\nabla_\mu s = \partial_\mu s - \frac{2\pi i}{h} \theta_\mu s , \tag{13.863}$$

with

$$\omega_{\mu\nu} = \partial_\mu \theta_\nu - \partial_\nu \theta \mu . \tag{13.864}$$

Note that θ may not be defined globally, However if $p^\star_T(Q)$ it will and

$$\nabla_\mu = (\frac{\partial}{\partial q^j} - \frac{2\pi i}{h} p_j, \frac{\partial}{\partial p_k}) . \tag{13.865}$$

We define as *pre-quantization Hilbert space* $\Gamma(E, \pi, P, \mathbb{C}, U(1))$ with norm

$$||s||^2 = \int_P |s|^2 \frac{(-1)^n}{\omega^n} , \tag{13.866}$$

where $|s|^2 = \overline{s} s$ and $\overline{s}$ is the complex conjugate of the $\mathbb{C}$-valued section s. It is important to realise that $\Gamma(E, \pi, P, \mathbb{C}, U(1))$ is *not* the physical Hilbert space of Quantum Mechnics, and neither is the section s a quantum mechanical wave function. This is because it depends upon twice as many varibles, i.e. p_j, q^k as expected. Nevertheless, given a classical observable $f \in C^\infty(P)$, one define a self-adjoint operator $\check{f}$ acting on $\Gamma(E, \pi, P, \mathbb{C}, U(1))$ as

$$\check{f} s = \frac{h}{2\pi i} \nabla_{X_f} s + f s = f s \tag{13.867}$$

where X_f is the Hamitonian vector field assicited to the function f. In local Darboux coordinates

$$\check{f} s = \frac{h}{2\pi i} \Big(\frac{\partial f}{\partial_j} \frac{\partial s}{\partial q^j} - \frac{\partial f}{\partial q^j} \frac{\partial s}{\partial_j} - \frac{2\pi i}{h} p_j \frac{\partial s}{\partial q^j} \Big) + s f . \tag{13.868}$$

Not only is $\check{f}$ self adjoint with respect to the inner product coming form $||s||^2$, but

$$[\check{f}, \check{g}] s = \frac{ih}{2\pi} \widecheck{\{f, g\}} s . \tag{13.869}$$

Polarizations We must cut the number of independent variables upon which the sections depend by half. One approach is to look for a *real polarization*, that is an *integrable distribution with Lagrangian leaves* .

A *k-dimensional distribution* W in a manifold M is a or continuous assignment to each $p \in M$ of a k-dimemensional vector subspace $W_p \in T_p(M)$ of the tangent space. In other words it is a k-dimensional sub-vector bundle of the tangent bundle $T(M)$. For example, if $k = 1$ it would be a non-vanishing section of the tangent bundle.

A distribution W is *integrable*

$$\forall X, Y \in \mathfrak{X}(M)\, X, Y \in W \Rightarrow [X, Y] \in W. \tag{13.870}$$

By Froebenius's theorem, an integrable distribution or *foliation*, W, of M is tangent at every point $p \in M$ to a submanifold Leaf(W_p) of M called its *leaf*, $W_p = T_p(\text{Leaf}(W_p)$. One says that M is *foliated by the leaves*. The set of leaves constitute the $n-k$-dimensional *leaf space* N.

A submanifold Σ of a $2n$-dimensional symplectic manifold $\{P, \omega\}$ is called *isotropic* if the restriction of the symplectic form ω to Σ, $\omega|_\Sigma$ vanishes, that is $\omega(X, Y) = 0 \forall X, Y. \in T(\Sigma) \subset T(M)$. It is called Lagrangian if it is maximally isotropic, i.e it has the largest possible dimension, which is n. The leaf space N has the same dimension

For example, if $P = T^\star(Q)$ has an obvious *vertical polarization* whose leaves are the fibres $T^\star_q$ If $Q = \mathbb{R}^n$, and whose leaf space $N = Q$. *horizontal polarization* has leaves $p_i = constant$. For a general configuration space Q it is not clear that a horizontal polarization exists, but if did it might correspond to certain (local) sections of $T^\star(Q)$, with leaf space N some particular fibre $N = F_p = T^\star_p(Q)$.

Given a real polarization one may restrict attention to sections or *wave-functions* which are constant along the leaves and hence depend only on the n-dimensional leaf space N

$$\nabla_X s = 0\,, \qquad \forall X \in W. \tag{13.871}$$

because

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X)a - \nabla_{[X,y]} s = \frac{2\pi i}{h} \omega(X, Y) s\,, \tag{13.872}$$

the Lagrangian condition

$$\omega(X, Y) = 0 \forall\, X, Y\, \in W \tag{13.873}$$

is a necessary condition for consistency.

The obvious example is the vertical polarization for $P = T^\star(Q)$,

$$\nabla_{\frac{\partial}{\partial p_j}} s = \frac{\partial s}{\partial p_j} = 0\,. \tag{13.874}$$

Thus $s = \Psi(q^j)$, i.e. the wave functions depend only on the configuration space variables q^i. There remains the issue of an appropriate inner product. If Q carries a metric it is natural to use the associated riemannian measure or volume form

$$\langle s|s \rangle = \int_Q |\Psi|^2 \eta\,. \tag{13.875}$$

If $Q = \mathbb{E}^n$, the horizontal polarization gives

$$\nabla_{\frac{\partial}{\partial q^j}} s = \frac{\partial s}{\partial q^j} - \frac{2\pi i}{h} p_j s = 0 \Rightarrow s(p, q) = e^{\frac{2\pi i}{h} p_j q^j} \Psi(p_k)\,. \tag{13.876}$$

This is the familiar momentum represention.

An alternative procedure is to introduce a *complex polarization*, that is, to introduce complex coordinates $z^r = q^i + i p_r$ and consider *holomorphic waves functions* $s = s(z^k)$.