1 Maps Between Manifolds

In addition to the exterior derivative \( d \), there is another type of derivative one may define on a manifold possessing no further structure, it is called the Lie Derivative. To define it we need to a smooth map \( \psi : M \to N \) from a smooth manifold \( M \) to another smooth manifold \( N \), which need not have the same dimensions, \( m \) and \( n \) respectively. Of course an interesting special case will correspond to taking \( M = N \) but for clarity and also with other applications in mind, we keep \( M \) and \( N \) distinct. By smooth map, we mean one given by smooth functions in every smooth coordinate chart. Thus if \( x^\alpha \) are local coordinates for \( M \) and \( y^\beta \) for \( N \), then \( \psi \) takes a point \( p \) to \( p' = \psi(p) \) and if \( p \) has coordinates \( x^\alpha, \alpha = 1, 2, \ldots m \) and \( p' \) has coordinates \( y^\beta, \beta = 1, 2, \ldots n \), then \( y^\beta = y^\beta(x^\alpha) \).

Associated with \( \psi \) are two maps

Push Forward \( \psi_* : T_p(M) \to T_{p'}(N) \) (1)

Pull Back \( \psi^* : T^*_p(M) \leftarrow T^*_{p'}(N) \) (2)

The push forward acts on curves vectors and contra-variant tensor, the pull back acts on functions, co-vectors, p-forms and co-variant tensor fields.

1.0.1 The Push Forward Map

Consider a curve in \( M \), i.e a map \( c : \mathbb{R} \to M \) given by \( x^\alpha(\lambda) \). The pushed forward a curve \( c_* \) is just obtained by composition, in other words \( c_* = \psi \circ c \), or \( y^\beta(\lambda) = y^\beta(x^\alpha(\lambda)) \). The chain rule now allows us to push forward the tangent vector \( T \)

\[
\frac{dy^\beta}{d\lambda} = \frac{\partial y^\beta}{\partial x^\alpha} \frac{dx^\alpha}{d\lambda}
\] (3)
or if $\psi_* T = \ast T$

$$\ast T^\beta(p') = \frac{\partial y^\alpha}{\partial x^\beta} T^\alpha(p).$$  \hspace{1cm} (4)

Clearly we can extend $\psi_*$ to arbitrary contra-variant tensor fields.

### 1.0.2 The Pull Back Map

Consider a function on $N$, i.e a map $N \rightarrow N$ given by $f(y^\beta)$. The pulled back function $f^*$ is just obtained by composition, in other words $f^* = f \circ \psi$, or $f^*(x^\alpha) = f(y^\beta(x^\alpha))$. The chain rule now allows us to pull back the gradient covector $\omega = df$

$$\frac{\partial f^*}{\partial x^\alpha} = \frac{\partial y^\beta}{\partial x^\alpha} \frac{\partial f}{\partial y^\beta}$$  \hspace{1cm} (5)

or if $\psi^* \omega = \ast \omega$,

$$\ast \omega_\alpha(p) = \frac{\partial y^\beta}{\partial x^\alpha} \omega_\beta(p').$$  \hspace{1cm} (6)

Clearly we can extend $\psi_*$ to arbitrary co-variant tensor fields.

Clearly push forward and pull backs are related. One could define one in terms of the other by

$$V f^*_{p'} = V^* f_p$$  \hspace{1cm} (7)

or

$$\langle \ast \omega | V \rangle_p = \langle \omega | \ast V \rangle_{p'}.$$  \hspace{1cm} (8)

### 1.0.3 Exterior Derivative commutes with pullback

The formula given above is identical to that used earlier to show that the exterior derivative takes $p$-form fields to $p+1$-form fields. In the present context that calculation shows that $d$ commutes with pull back, i.e.

$$d(\psi^* \omega) = \psi^* (d\omega).$$  \hspace{1cm} (9)

For example consider the map

$$\psi : \mathbb{R}^3 \setminus z - axis \rightarrow S^1, \quad (x, y, z) \rightarrow \theta = \tan^{-1}(\frac{y}{x}).$$  \hspace{1cm} (10)

We can pull-back the volume form on $S^1$

$$\psi^* d\theta = \frac{-y dx + x dy}{x^2 + y^2} = B \quad dB = 0.$$  \hspace{1cm} (11)

Physically, $B$ is the magnetic field $\mathbf{B}$ due to a current flowing along the $z$-axis and $dB = 0 \Leftrightarrow \text{curl} \mathbf{B} = 0$. Note that

$$\oint_{\gamma} \mathbf{B} \cdot dx = 2\pi = \oint_{S^1} d\theta$$  \hspace{1cm} (12)

where $\gamma$ is any closed curve encircling the $z$-axis once.
Another example is provided by the map

$$\psi : S^2 \to \mathbb{R}^3 \text{s.t.} (x, yz) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad (13)$$

which embeds the 2-sphere into three-dimensional space. Let

$$\omega = zdx \wedge dy + xdy \wedge dz + ydz \wedge dx \Rightarrow d\omega = 3dx \wedge dy \wedge dz. \quad (14)$$

We have

$$\psi^* \omega = \sin \theta \wedge d\phi \Rightarrow d\psi^* \omega = 0 = \psi^* d\omega. \quad (15)$$

The last equality follows because $\psi^* d\omega$ is a 3-form in 2-dimensions.

Our last example is related to the Dirac monopole in the same way that the first was related to the vortex.

$$\psi : \mathbb{R}^3 \setminus 0 \to S^2 \quad (x, y, z) \to \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) \quad (16)$$

with $r = \sqrt{x^2 + y^2 + z^2}$. If

$$F = \frac{zdx \wedge dy + xdy \wedge dz + ydz \wedge dx}{r^3} = \frac{1}{2} \epsilon_{ijk} \frac{x_k}{r^3} dx^i \wedge dx^j \quad (17)$$

One has $F = \psi^* \eta$, with $\eta = \sin \theta d\theta \wedge d\phi$ the volume form on $S^2$ and so $F$ is closed, $d\eta = 0 \Rightarrow dF = 0$. Thus if $B$ is the magnetic field then curl $B = 0$ and

$$\int_S B \cdot dS = 4\pi, \quad (18)$$

for any closed surface $S$ which encloses the origin once.

This calculation easily generalizes to arbitrary dimensions and is applied to the Ramond-Ramond and Neveu-Schwarz charges of various $p$-brane solutions in supergravity and string theory.

1.0.4 Diffeomorphisms: Active versus Passive Viewpoint

In terms of diagrams we are just following the arrows. Analytically we only need the Jacobian matrix $\frac{\partial y^\beta}{\partial x^\alpha}$, we never need what may not exist, i.e. $\frac{\partial x^\beta}{\partial y^\alpha}$. However if $M = N$, and $\psi$ is, at least locally invertible, so that $\psi$ is a global diffeomorphism, then $\frac{\partial x^\alpha}{\partial y^\beta}$ will exist and what we have been doing is identical to what we earlier thought of as a local coordinate transformation or change of chart from $\phi$ say to $\phi' = \phi \circ \psi$. This is just a change of viewpoint, initially one may have thought of $\psi$ actively as moving the points of the manifold $M$ while we usually think of coordinate transformations passively as relabeling the same points. The two concepts are essentially interchangeable.
1.0.5 Invariant tensor fields

We can now give a condition that a contra-variant tensor field $S$ of rank $k$ is invariant under the action of a diffeomorphism $\psi$, it must equal its push forward $\psi^* S$ under $\psi$, i.e.

$$\psi^* S(p) = S(p'),$$  \hspace{1cm} (19)

or in components

$$S^{\beta_1 \beta_2 \ldots \beta_k}(y) = \frac{\partial y^{\beta_1}}{\partial x^{\alpha_1}} \frac{\partial y^{\beta_2}}{\partial x^{\alpha_2}} \cdots \frac{\partial y^{\beta_k}}{\partial x^{\alpha_k}} S^{\alpha_1 \alpha_2 \ldots \alpha_k}(x).$$ \hspace{1cm} (20)

Similarly a co-variant second rank tensor field $g$, for example the metric tensor, is invariant under the action of $\psi$ is that $g$ equals its pull-back,

$$g(p) = \psi^* g(p'),$$ \hspace{1cm} (21)

i.e.

$$g_{\alpha \beta}(x) = g_{\mu \nu}(y) \frac{\partial y^{\mu}}{\partial x^{\alpha}} \frac{\partial y^{\nu}}{\partial x^{\beta}}, \quad ds^2(x) = ds^2(y).$$ \hspace{1cm} (22)

Thus infinitesimal lengths are preserved by such diffeomorphisms which are called isometries. Later we will discuss infinitesimal isometries and Killing vector fields.

1.1 Immersions and Imbeddings

The derivative map or Jacobian

$$\psi_*: T_p(M) \to T_{p'}(M)$$ \hspace{1cm} (23)

is a linear map, explicitly the matrix $\frac{\partial y^\mu}{\partial x^{\alpha}}$. Its rank is the number of linearly independent rows. Then $\psi_*$ is injective if rank $\psi_* = \dim M$, i.e. no vector in $T_p(M)$ is mapped to zero $\psi_*$ is surjective if rank $\psi_* = \dim N$, i.e. every vector in $T_{p'}(N)$ is the image of some vector in $T_p(M)$.

The map $\psi$ is called a smooth immersion if locally the inverse $\psi^{-1}$ exists. The Implicit Function theorem states that $\psi$ is an immersion iff $\psi_*$ is injective. We say that $\psi(M)$ is an immersed submanifold $A$ smooth imbedding\textsuperscript{1} is a smooth immersion which is 1-1 on its image and the inverse image of any compact set is compact.

For example an imbedding of $S^1$ into $\mathbb{R}^2$ is a circle or an ellipse or any simple smooth curve which does not intersect itself. An example of an immersion would be a figure of eight curve. For each point on $S^1$ there is a unique tangent vector but at two points $S^1$ with the same image in $\mathbb{R}^2$ there are two distinct tangent vectors.

\textsuperscript{1}sometimes called an embedding
A hypersurface, for example a t = constant surface in spacetime is an imbedded submanifold of co-dimension one, i.e. \( \dim N = \dim M - 1 \).

In this language, a smooth diffeomorphism is such that \( \psi \) is 1-1 and \( \psi^{-1} \) is smooth. The implicit function theorem then asserts that this is true if \( \psi_* \) is both surjective and injective. Thus

\[
\text{rank } \psi_* = \dim M = \dim N.
\]  

(24)

It follows that \( (\psi_*)^{-1} = \psi_*^{-1} \) are isomorphisms of \( T_p(M) \) and \( T_{\psi(p)}(N) \) respectively and that the two tensor algebras are also isomorphic.

Thus, in the case of a diffeomorphism the distinction between push forward and pull back is to some extent lost. We can, by convention, define the push forward of a co-variant tensor field \( \Omega \) by the formula

\[
\psi_* \Omega(y) = \psi_*^{-1} \Omega(x)
\]  

(25)

Thus on functions, for example,

\[
(f^\ast)_* = f.
\]  

(26)

2 One-parameter families of diffeomorphisms: Lie Derivatives

Suppose \( \psi_t : \mathbb{R} \times M \to M \) be one parameter family of smooth maps such that

\[
\psi_{t+t'} = \psi_t \circ \psi_{t'}
\]  

(27)

and \( \psi_0 \) is the identity map. Thus

\[
\psi_{-t} = \psi^{-1}.
\]  

(28)

and we have an action of the additive group \( \mathbb{R} \) on \( M \). One should have in mind the example of time translations. The orbit of \( \psi_t \) through a point \( p \in M \) with coordinates \( X^\alpha \), \( \psi_t(p) \) is a curve \( y^\alpha(t) \) with \( y^\alpha(0) = x^\alpha \), the coordinates of the point \( p \). The curves \( \psi_t(p) \) have tangent vectors \( T(t) \) and define a vector field \( K(x) \) on \( M \). Conversely given a vector field \( K \) we obtain a one parameter family of diffeomorphisms \( \psi_t \) by moving the points of \( M \) up the integral curves of \( K \) and amount \( t \). Thus in local charts, \( \psi_t \) is given by

\[
\psi_t : x^\alpha \to y^\alpha(t),
\]  

(29)

where \( y^\alpha(t) \) is a solution of

\[
\frac{dy^\alpha}{dt} = K^\alpha(y^\beta(t))
\]  

(30)

with \( y^\alpha(0) = x^\alpha \). Infinitesimally, for small \( t \) we have

\[
y^\alpha(t) = x^\alpha + V^\alpha(x)t + \mathcal{O}(t^2).
\]  

(31)
Given $\psi_t$ we can Lie-Drag curves, and tensor fields along the integral curves, using the push-forward map $\psi_t$. Thus, for a vector $V$, $\delta V = \psi_t V$ is given by

$$\delta V^\alpha(y^\alpha(t)) = \frac{\partial y^\alpha(t)}{\partial x^\beta} V^\beta(x^\gamma).$$

(32)

We now define the Lie Derivative $\mathcal{L}_K S$ of any contra-variant tensor field $S$ say by

$$\mathcal{L}_K S(x) = -\frac{d}{dt}\psi_t S(x)\bigg|_{t=0} = \lim_{t \to 0} \frac{(S(x) - \psi_t S(x))}{t}.$$  

(33)

The origin of the minus sign is as follows. We are evaluating the Lie derive of $S$ at the point $x$. To do so we compare $S(x)$ with the value obtained by dragging $S$ from the point $\psi_t(x) = \psi_t^{-1}(x)$ to the point $x$.

As mentioned above, for a (local) diffeomorphism we can extend the push forward map to co-variant tensor fields and the definition of the Lie Derivative extends as well. Alternatively one could define it by using the pull back map but one now gets a plus sign:

$$\mathcal{L}_K \Omega(x) = \frac{d}{dt}\psi_t^* \Omega(x)\bigg|_{t=0} = \lim_{t \to 0} \frac{(\psi_t^* \Omega(x) - \Omega(x))}{t}.$$  

(34)

2.0.1 Example: Functions

The simplest case to consider is that of functions

$$\mathcal{L}_K f = \lim_{t \to 0} \frac{f(x) - f(y^\alpha(-t))}{t}.$$  

(35)

Thus

$$\mathcal{L}_K f = \lim_{t \to 0} \frac{f(x) - f(x^\alpha - tK^\alpha(x)))}{t} = K^\alpha \frac{\partial f}{\partial x^\alpha} = K f.$$  

(36)

Doing it the other way we get the same answer

$$\mathcal{L}_K f = \lim_{t \to 0} \frac{f(x^\alpha + tK^\alpha(x^\alpha)) - f(x^\alpha)}{t} = K^\alpha \frac{\partial f}{\partial x^\alpha} = K f.$$  

(37)

A, slightly symbolic, but nevertheless illuminating notation makes use of the exponential map $e^{tK}$ of a vector field. Geometrically this takes points an amount $t$ up the integral curves of the vector field $K$

$$e^{tK} x^\alpha = y^\alpha(t).$$  

(38)

. On functions therefore we have

$$e^{tK} f = *f = \psi^* f.$$  

(39)

The push-forward is given by

$$e^{-tK} f = \delta f = \psi_* f.$$  

(40)
2.0.2 Example: Vector Fields

We need

\[(\mathcal{L}_K V)^\alpha(x) = -\frac{d}{dt}\left(V^\beta(y^\gamma(-t))\frac{\partial y^\alpha(-t)}{\partial x^\beta}\right).\] (41)

Since

\[\frac{\partial y^\alpha(-t)}{\partial x^\beta}(x) = \delta^\alpha_\beta - tK^\alpha_\beta(x) + O(t^2),\] (42)

we obtain

\[(\mathcal{L}_K V)^\alpha = V^\alpha_\beta K^\beta - K^\alpha_\beta V^\beta = [K, V]^\alpha.\] (43)

This calculation can also be done using the exponential notation.

\[(\star V f)(p') = (V^* f)(p) \Rightarrow (\star V f)(p) = (V^* f)(p),\] (44)

thus

\[\psi_{t*} V = e^{-tK} V e^{tK} \Rightarrow \frac{d}{dt}\psi_{t*} V \bigg|_{t=0} = [K, V].\] (45)

It is straightforward to see how to take the Lie derivative of higher rank contravariant tensor fields. One gets terms with a partial derivative of \(K\) for each upper index. For example of a second rank contravariant tensor we have

\[(\mathcal{L}_K S)^\alpha_\beta = K^\gamma \partial_\gamma S^\alpha_\beta - \partial_\alpha K^\beta S^\gamma_\beta - \partial_\beta K^\gamma S^\alpha_\gamma.\] (46)

2.0.3 Lie Derivative on covariant tensors

We have

\[(L\omega)_a = -\frac{d}{dt}\left(\omega_\beta(y^\gamma(-t))\frac{\partial y^\alpha(-t)}{\partial x^\alpha}\right)_{t=0}.\] (47)

Using the fact that

\[\frac{\partial y^\alpha(-t)}{\partial x^\alpha} = \delta^\alpha_\beta - t\partial_\alpha K^\beta + O(t^2),\] (48)

one gets

\[(\mathcal{L}_K \omega)_a = K^\beta \partial_\beta \omega_a + \partial_\alpha K^\beta \omega_\beta.\] (49)

For a second rank co-variant tensor, such as the metric tensor one has

\[(\mathcal{L}_K g)_{\alpha\beta} = K^\gamma \partial_\gamma g_{\alpha\beta} + \partial_\alpha K^\beta g_{\gamma\beta} + \partial_\beta K^\gamma g_{\alpha\gamma}.\] (50)

Note that we have not assumed a particular symmetry for \(g_{\alpha\beta}\), but if we had, that symmetry would be inherited by the Lie derivative.
2.0.4 Example: Adapted Coordinates, Stationary and Static metrics

From its definition, it is clear that the Lie derivative takes tensor fields to tensor fields. The presence of the terms involving $\beta K^\alpha$ guarantees that. However, it is always possible locally to introduce adapted coordinates $t, x^i$ such that

$$ K^\alpha = \delta^\alpha_t, \quad \iff K'' = \frac{\partial}{\partial t}. \quad (51) $$

In these coordinates for any tensor field

$$ \mathcal{L}_K = \frac{\partial}{\partial t}. \quad (52) $$

Thus if $K$ is a Killing vector field, that is

$$ \mathcal{L}_K g = 0 \quad (53) $$

, where $g$ is the metric tensor, then in adapted coordinates the metric is independent of the coordinate $t$

$$ ds^2 = g_{00}(x^k)dt^2 + 2g_{0i}(x^k)dx^i + g_{ij}(x^k)dx^i dx^j. \quad (54) $$

The existence of adapted coordinates may be shown as follows. We introduce an initial hypersurface nowhere tangent to $\Sigma$ so that (at least in a local neighbourhood $U$) the integral curves of $K$ intersect $\Sigma$ once and only once. The coordinates $x^i$ are chosen on $\Sigma \cap U$ and then Lie dragged along the integral curves, in other words they are defined to be constant along the integral curves. Points on $\Sigma$ are assigned the coordinate $t = 0$. We then assign to each point $p$ in $U \subset M$ the coordinates $t, x^i$, where $x^i$ labels the integral curve passing through $p$ and $t$ the parameter necessary to reach $p$ starting from $\Sigma$.

There is thus clearly some "gauge freedom" in choosing the hypersurface $\Sigma$. Changing $\Sigma$ to some other hypersurface $\Sigma'$ will alter the origin of the $t$ parameter along each integral curve. If $\Sigma'$ intersects the integral curve labelled by $x^i$ at $t = \tau(x^i)$ then the new coordinate $t'$ will be given by

$$ t' = t - \tau(x^i). \quad (55) $$

In physical applications, if the Killing vector is timelike, one calls the metric stationary unless there is an additional time reversal symmetry, in which one may put $g_{00} = 0$. The metric is then said to be static. For example a non-rotating Schwarzschild black hole has gives rise to a static metric. A rotating Kerr balck hole has a stationary metric.

2.1 Lie Derivative and Lie Bracket

So far we have not verified explicitly that the Lie derivative $\mathcal{L}_V W$ is a vector field. This may be seen directly by introducing the Bracket or Commutator

$$ [V, W] = -[W, V] \quad (56) $$

8
of two vector fields $V$ and $W$ by its action on a function $f \in \mathfrak{X}(M)$
\[ [V,W]f = V(Wf) - W(Vf). \] (57)

The bracket of two vector fields is itself a vector field. To see this one needs to check that the Leibniz property
\[ [V,W]fg = g[V,W]f + f[V,W]g. \] (58)
This is a simple, albeit tedious, calculation.

Moreover, a short calculation in local coordinates reveals that that
\[ [V,W]\alpha = (\mathcal{L}_V W)\alpha. \] (59)

2.1.1 Jacobi Identity

In the same vein, one easily verifies that the Lie bracket on functions satisfies the Jacobi Identity
\[ [[U,[V,W]] + [V,[W,U]] + [W,[U,V]] = 0. \] (60)

It follows from (??) and (??) that the set of vector fields $\mathfrak{X}(M)$ forms an infinite dimensional Lie Algebra This is sometimes denoted $\text{diff}(M)$.

2.1.2 Example: $\text{diff}(S^1)$ is called the Virasoro Algebra

It crops up in string theory. If $\theta \in (0, 2\pi]$ is a coordinate on $S^1$, one defines a basis index by $n \in \mathbb{Z}$ by
\[ D_n = i e^{in\theta} \partial_\theta. \] (61)
The brackets are
\[ [D_n, D_m] = (n - m)D_{m+n}. \] (62)
There is a finite sub-algebra spanned by $D_{-1}, D_0, D_{-1}$ whose Lie-algebra is that of the projective group $\text{sl}(2, \mathbb{R})$.

2.2 Lie Bracket and Closure of infinitesimal rectangles

Suppose that we move from $O$ to $A$ an amount $t$ along the integral curves of $U$ and then from $A$ to $B$ an amount $s$ along the integral curves of $V$. We may compare our final position $B$ with what would have happened if we had first moved from $O$ to $C$ an amount $s$ along the integral curves of $V$ and then from $C$ to $D$ an amount $t$ along the integral curves of $U$. A short calculation in local coordinates using Taylor’s theorem shows that to lowest no trivial order:
\[ x_B ^\mu - x_D ^\mu = [U,V]^\alpha st + \ldots. \] (63)

Thus, if the bracket closes the infinitesimal rectangle $DCOAB$ will close. More generally it may be shown that if the bracket vanishes then the two vector
fields lie in a 2-surface and conversely. It is not necessary that the bracket vanish for the vector fields to lie in a 2-surface. It suffices that the bracket $[U, V]$ is a linear combination of $U$ and $V$, with coefficients $a(x)$ and $b(x)$ which may depend on $x$. This is a special case

2.3 Frobenius’s Theorem

Suppose that $k$ non-vanishing vector fields $U_a$, $a = 1, 2, \ldots k$ satisfy

$$[U_a, U_b] = -D_a{}^c {}_b U_c$$

where the functions $D_a{}^c {}_b = D(x)_a{}^c {}_b$ are called structure functions, then the vector fields are tangent to some $l$-dimensional sub-manifold with $0 < l \leq k$.

Note that if the structure functions are independent of position they become the structure constants of some Lie-algebra.

2.3.1 Example: The simplest non-trivial Lie algebra

$$x \frac{\partial}{\partial x} \text{ and } \frac{\partial}{\partial x}.$$  \hspace{1cm} (65)

Here $k = 2, l = 1$. Changing the $x \frac{\partial}{\partial x}$ to $a(x) \frac{\partial}{\partial x}$ would lead to a structure function rather than a structure constant, but the two vector fields would still be tangent to a one-dimensional manifold.

2.3.2 Example: Two commuting translations

$$\frac{\partial}{\partial x} \text{ and } \frac{\partial}{\partial y}.$$ \hspace{1cm} (66)

Here $k = 2$ and $l = 2$

2.3.3 Example Kaluza or Torus Reductions

Following the initial idea of Kaluza one could imagine a $4 + m$ dimensional space-time invariant under the action of the abelian torus group $T^m = (S^1)^m$. There will be $m$ mutually commuting Killing fields $K_a$, $a = 1, \ldots \mu$

$$[K_a, K_b] = 0.$$ \hspace{1cm} (67)

By a combination of the Frobenius theorem and the discussion above about adapted coordinates, we may introduce coordinates $y^a$ such that

$$K_a = \frac{\partial}{\partial y^a},$$ \hspace{1cm} (68)

and the metric takes the form

$$ds^2 = g_{\lambda \mu}(x^\lambda)(dy^a + A^a_\mu(x^\lambda)dx^\mu)(dy^b + A^b_\nu(x^\lambda)dx^\nu) + g_{\mu \nu}(x^\lambda)dx^\mu dx^\nu.$$ \hspace{1cm} (69)
One may interpret $g_{\mu\nu}(x^\lambda)$ as the metric on our lower dimensional spacetime with coordinates $x^\mu$. The co-vector fields $A_\mu(x^\lambda)$ may be interpreted as $n$ abelian gauge fields which are called gravi-photons. This works because the coordinates $y^m$ are determined only up to a coordinate transformation of the form

$$y^a \rightarrow \tilde{y}^a = y^a + \Lambda^a(x^\lambda),$$  \hspace{1cm} (70)

under which the one-forms $A^a_\mu(x^\lambda)$ undergo a gauge transformation

$$A^a_\mu(x^\lambda) \rightarrow A^a_\mu(x^\lambda) - \partial_\mu \Lambda^a(x^\lambda).$$  \hspace{1cm} (71)

The fields $g_{ab}(x^\lambda)$ are invariant under both gauge transformations, and $(u(1)^m$ gauge-transformations. They are scalar fields, called gravi-scalars, which take their values in the space of symmetric matrices or $m$-dimensional metrics which as we shall see may be identified with the symmetric space $GL(m,R)/SO(m)$. It is sometimes convenient to split of the determinant $\det(g_{ab})$ and to consider unimodular matrices in which case the remaining scalars take their values in $SL(m,R)/SO(m)$. Some multiple of the logarithm of $\det(g_{ab})$ is often called the dilaton.

What has been achieved here is to embed the local $(U(1))^m$ gauge group as a subgroup of the the diffeomorphism group of the higher dimensional spacetime. The analogue of time reversal is charge conjugation symmetry under which $y^a \rightarrow -y^a$. If it holds for one of the $A^a_\mu$’s then we may set that $A^a_\mu = 0$ and the $U(1) \equiv SO(2)$ contribution to the isometry group is augmented to $O(2)$.

2.3.4 Example: $so(3)$

\[ L_z = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \hspace{1cm} \text{and cyclically.} \]  \hspace{1cm} (72)

Here

$$[L_x, L_y] = -L_z \hspace{1cm} \text{and cyclically,}$$  \hspace{1cm} (73)

and we get the Lie algebra of the rotation group $so(3)$. If $r^2 = x^2 + y^2 + z^2$, one verifies that

$$L_z r^2 = 0, \hspace{1cm} \text{and cyclically,}$$  \hspace{1cm} (74)

Thus $k = 3$ but $l = 2$, all three vector fields are tangent to 2-spheres centred on the origin of three-dimensional Euclidean space $E^3$. Note that while the $L_x, L_y, L_z$ generate rotations in the positive sense in the three orthogonal coordinate planes, the structure constants are the negative of the standard structure constants for $so(3)$, in other words

$$[L_i, L_j] = -\epsilon_{ijk} L_k.$$  \hspace{1cm} (75)

This turns out to be universal sign reversal, as will be explained later.
2.4 Lie Derivative and Exterior Derivative

It is important that $d$ commutes with Lie, i.e.

$$L_U(d\omega) = d(L_U\omega),$$

(76)

for any $p$-form field $\omega$ and vector field $U$. This is no more than the fact that the exterior derivative commutes with pull-back, but it may be proved more concretely by writing out the relevant expressions in local coordinates. Another important fact is that one may swap the exterior derivative for the bracket. Thus, for a 1-form $\omega$

$$\langle d\omega|U,V \rangle = U\langle \omega|V \rangle - V\langle \omega|U \rangle - \langle \omega|[U,V] \rangle.$$  

(77)

2.5 Cartan’s formula: Lie derivative and interior product

In the same spirit one has the extremely useful formula

$$L_U \omega = i_U d\omega + d(i_U \omega),$$

(78)

where $\omega$ is a two-form and $U$ a vector field.

2.5.1 Example: electrostatic potentials

Suppose that $\omega = F$, a Maxwell 2-form which by Faraday’s law is closed, $dF = 0$, and suppose $F$ is invariant under time-translation or some other symmetry, generated by a vector field $U$, which could be a Killing vector field. We have:

$$\mathcal{L}_U F = 0 \Rightarrow i_U F = d\Phi,$$

(79)

for some function $\Phi$ which plays the role of an an electrostatic or magneto static potential. Thus in the Schwarzschild solution, for example, if

$$F = Q\frac{dt \wedge dr}{r^2} \quad \text{and} \quad U = \frac{\partial}{\partial t},$$

(80)

then

$$\Phi = -\frac{Q}{r}.$$  

(81)

This example can readily be generalized to higher rank forms which appear in higher dimensions in supergravity and superstring theory.

3 Integration on Manifolds: Stokes’ Theorem

The basic idea is that we can only integrate $p$-forms over $p$-chains. Roughly speaking, a $p$-chain is a $p$-dimensional sub-manifold which may be regarded as the sum of a set of $p$-cubes (or alternatively of $p$-simplices). Each $p$-cube is the image in $M$ under some map $\phi: R^n \to M$ of a standard $p$-cube in $R^n$. In other
words, a p-cube $C$ pushes forward to $M$ to give a curvilinear p-cube $C = \psi \ast C$.
In order to integrate a p-form $\omega$ in $M$ over the p-chain $C$ may we pull $\omega$ back to $R^n$ to give $\omega = \phi \ast \omega$ and integrate it over $C$ using the standard integration procedure, i.e by means of the definition

$$\int_C \omega = \int_C \omega^*.$$  \hspace{1cm} (82)

On each p-cube $C \in R^n$ we may introduce coordinates $0 \leq x^1 \leq 1, 0 \leq x^2 \leq 1, \ldots, 0 \leq x^p \leq 1$.

$$\int_C \omega^* = \frac{1}{p!} \int_C \omega^*_{\mu_1 \mu_2 \ldots \mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \ldots \wedge dx^{\mu_p} = \int \ldots \int \omega_{12 \ldots p} dx^1 dx^2 \ldots dx^p.$$  \hspace{1cm} (83)

A general p-chain is a sum of p-cubes $C = \sum_i C_i$. To each p-chain $C$ we associate a boundary $\partial C = \sum_i \partial C_i$, where adjacent cubes contribute with the opposite sign and hence cancel. Thus the boundary of a boundary vanishes:

$$\partial^2 C = 0.$$  \hspace{1cm} (85)

By linearity and Because $d$ commutes with pull-back, Stokes’s theorem now reads

$$\int_C d\omega = \int_{\partial C} \omega.$$  \hspace{1cm} (86)

In other words we only need only check this formula on a standard p-cube in $R^n$. To check that, recall that

$$d\omega^*_{\mu_1 \mu_2 \ldots \mu_{p+1}} = \frac{(p+1)!}{p!} \partial_{[\mu_1} \omega^*_{\mu_2 \ldots \mu_{p+1}]}.$$  \hspace{1cm} (87)

Thus

$$\int_C \frac{1}{(p+1)!} d\omega^*_{\mu_1 \ldots \mu_{p+1}} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{p+1}} = \int_C \frac{(p+1)!}{(p+1)!} \partial_{[\mu_1} \omega^*_{\mu_2 \ldots \mu_{p+1}]} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{p+1}} = \int_{\partial C} \omega^*_{\mu_1 \mu_2 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p},$$  \hspace{1cm} (88)

where the last line follows from integration by parts.

As an example let $F = \theta d\theta \wedge d\phi$.

$$\int_{S^2} = \int \int \sin \theta d\theta d\phi = [\cdot \cos \theta \frac{\pi}{2}]^\theta_0 = 4\pi.$$  \hspace{1cm} (91)
As a corollary, we see that $F$ is not exact $F \neq dA$ because so

$$\int F_{S^2} = \int_{S^2} dA = \int_{\partial S^2} = 0,$$

(92)

because $S^2$ has no boundary $\partial S^2 = 0$.

More generally the volume form of any compact manifold is closed but never exact.

### 3.0.2 The divergence operator

If we define an inner product on $\Omega^p(M)$ by

$$(\alpha, \beta) = \int_M (\alpha, \beta) \eta = \int_M \star \alpha \wedge \beta = \frac{1}{p!} \int \alpha_{\mu_1 \ldots \mu_p} \alpha^{\mu_2 \ldots \mu_p} \sqrt{|g|} d^n x,$$

(93)

then the formal adjoint $^2$ of $\delta$ of $d$ is defined by

$$(\delta \alpha, \beta) = (\delta \alpha, \beta).$$

(94)

In components

$$\delta \alpha_{\mu_1 \ldots \mu_p} = -\nabla^{\mu_0} \alpha_{\mu_0 \mu_1 \ldots \mu_p}.$$  

(95)

Now if $\lambda \in \Lambda^p$, $\star \star \lambda = (-1)^l (-1)^{p(n-p)} \lambda$ and

$$\lambda \wedge \mu = (\star, \mu) \eta, \Rightarrow \star \lambda \wedge \mu = (-1)^l (-1)^{p(n-p)} (\lambda, \mu) \eta,$$

(96)

$$\Rightarrow (\alpha, d\beta) \eta = (-1)^l (-1)^{(p+1)(n-p)} \star \alpha \wedge d\beta,$$

(97)

for $\alpha \in \Lambda^{p+1}$. Moreover

$$d(\star \alpha \wedge \beta) = d \star \alpha \wedge \beta + (-1)^{n-1-p} \star \alpha \wedge d\beta,$$

(98)

hence up to a boundary term

$$\int_M (d \star \alpha) \wedge \beta = (-1)^{(n-p)} \int_M \star \alpha \wedge d\beta.$$

(99)

Thus

$$\int_M (\star d\alpha \wedge \beta) \eta = (-1)^{(n-p)} (-1)^l (-1)^{p(n-p)} \int_M (\alpha, d\beta) \eta.$$

(100)

Thus up to a sign

$$\delta = \pm \star d\star \Rightarrow \delta^2 = 0.$$

(101)

$^2$It is a formal adjoint because we are not worrying about boundary terms
3.1 Brouwer degree

Let $\phi : M \to N$ be a map between two manifolds of the same dimension. Let $N$ be equipped with a volume form $\eta_N$. The Brouwer degree of the map $\phi$ is an integer given by

$$\text{deg} \, \phi = \frac{\int_M \phi^* \eta_N}{\int_M \eta_M}. \quad (102)$$

Intuitively, $\text{deg} \, \phi$ is the number of times the map $\phi$ "wraps" or "winds" the manifold $M$ over $N$.

It is also the number of inverse images of the map at a generic point $x \in N$ counted with respect to orientation. That is subtracts the those for which $\phi$ preserve orientation from those which reverse it.

3.1.1 Gauss Linking Number

Suppose that $\gamma_1$ and $\gamma_2$ are two connected closed curves immersed into $E^3$ and which do not intersect each other $\gamma_1 \cap \gamma_2 = \emptyset$. Thus

$$\gamma_1 : S^1 \to E^3 \quad t \to \mathbf{x}_1(t) \quad 0 \leq t < 2\pi, \quad (103)$$

$$\gamma_2 : S^1 \to E^3 \quad s \to \mathbf{x}_2(s) \quad 0 \leq s < 2\pi. \quad (104)$$

The unit vector

$$\mathbf{n}(t,s) = \frac{\mathbf{x}_2(s) - \mathbf{x}_1(t)}{|\mathbf{x}_2(s) - \mathbf{x}_1(t)|} \quad (105)$$

gives a map $\phi : S^1 \times S^1 \to S^2$ called the Gauss Map. The Gauss-Linking number $\text{Link}(\gamma_1, \gamma_2)$ is defined by

$$\text{Link}(\gamma_1, \gamma_2) = \text{deg} \, \phi = \frac{1}{4\pi} \int_{S^1 \times S^1} \phi^* \eta, \quad (106)$$

where $\eta$ is the volume form on $S^2$. If $\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1$,

$$\eta = \frac{z \, dx \wedge dy + y \, dz \wedge dx + z \, dy \wedge dz}{(x^2 + y^2 + z^2)^{3/2}} \quad (107)$$

and

$$\text{Link}(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_{\gamma_1} \int_{\gamma_2} \frac{(\mathbf{x}_2 - \mathbf{x}_1) \times d\mathbf{x}_1(t) \times d\mathbf{x}_2(s)}{|\mathbf{x}_2 - \mathbf{x}_1|^3} \quad (108)$$

$$= \int_{\gamma_1} B_2(\mathbf{x}_1).d\mathbf{x}_1, \quad (109)$$

where

$$B_2(\mathbf{x}) = \frac{1}{4\pi} \int_{\gamma_2} \frac{(\mathbf{x}_2 - \mathbf{x}) \times d\mathbf{x}_2}{|\mathbf{x}_2 - \mathbf{x}|^3} \quad (110)$$

is, by the Biot-Savart Law, the magnetic field due to a unit current along $\gamma_2$.

The reader should have no difficulty generalizing this example to the linking of a $p$-brane and a $q$-brane in $p + q + 1$ dimensions.
3.1.2 Gauss-Bonnet Theorem

Suppose that $S$ is a closed connected orientable 2-surface immersed into $E^3$ given in any local coordinate neighbourhood on $\Sigma$ by

$$x = x(u, v),$$

(111)

where $u, v$ are local coordinates on $\Sigma$ and

$$n(u, v) = \frac{\partial_u x \times \partial_v x}{|\partial_u x \times \partial_v x|}$$

(112)

is the unit normal. The normal provides the Gauss map $\phi : \Sigma \rightarrow S^2$. A calculation, which the reader is invited to do, shows that

$$\phi^* \eta_{S^2} = K \eta_{\Sigma}$$

(113)

where $K$ is the Gauss-curvature of $\Sigma$, whose induced metric is

$$ds^2 = \partial_i x \partial_j x dx^i dx^j = g_{ij} dx^i dx^j = Edu^2 + 2Fdudv + Gdv^2.$$  

(114)

That is the Riemann tensors is

$$R_{ijmn} = K(g_{im}g_{jn} - g_{in}g_{jm}).$$

(115)

Consideration of the Brouwer degree of $\phi$ then leads to an extrinsic version of the Gauss-Bonnet theorem

$$\frac{1}{2} \chi(\Sigma) = 1 - g(\Sigma) = \frac{1}{4\pi} \int_K \sqrt{EG - F^2} dudv \in \mathbb{Z},$$

(116)

where $\chi$ is the Euler number and $g(\Sigma)$ is the genus or number of handles of the surface $\Sigma$.

The reader should have no difficulty generalizing this argument to an $p$-brane in $p + 1$ dimensions.

3.2 A general framework for classical field and brane-theories

Many physical theories can be cast in a possibly procrustean, framework based on the space of maps $\text{Map}(\Sigma, M) = \{ \phi : \Sigma \rightarrow M \}$ from a manifold $\Sigma$ to another manifold $M$ say.

3.2.1 Particles

The simplest example is when $\Sigma = \mathbb{R}$ and we obtain a point particle moving in a curved spacetime $M$ with $\phi : (\mathbb{R} \rightarrow \mathbb{R}$ given by $x^\mu(\lambda), \lambda \in \mathbb{R}$ being a parameter on the worldline of our particle. $\text{Map}(\mathbb{R}, M)$ is then just Feynman’s space of histories of our particle.
The action (for massless particles) is
\[ S = \frac{1}{2} \int_R N(\lambda) d\lambda \, g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}. \] (117)

Variation with respect to the Lagrange multiplier \( N(\lambda) \) gives the massless condition or constraint.
\[ g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0. \] (118)

Mathematically, we may regard \( N^{-2} d\lambda^2 \) as a metric on \( \Sigma \), and \( N^{-1} d\lambda \) is the volume form or ein-bein on \( \Sigma \) and
\[ g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = (\phi^* g)_{\lambda\lambda} \] (119)
is the pull-back of the metric \( g_{\mu\nu} \) on \( M \) to \( \Sigma \).

### 3.2.2 Strings, Membranes and p-branes

For strings we take \( \Sigma \) to be 2-dimensional. For membranes \( \Sigma \) is three-dimensional. If \( \dim \Sigma = p + 1 \) we get a p-brane\(^3\) Now let \( \lambda^i, i = 1, 2, \ldots, p + 1 \) be local coordinates on \( \Sigma \) and \( \gamma_{ij} \) a metric on \( \Sigma \). The pull-back to \( \Sigma \) of the spacetime metric on \( M \) is given by
\[ (\phi^* g)_{ij} = g_{ij} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda^i} \frac{\partial x^\nu}{\partial \lambda^j} \] (120)
and
\[ \gamma^{ij} g_{ij} = \gamma^{ij} g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda^i} \frac{\partial x^\nu}{\partial \lambda^j} \] (121)
is its trace with respect to \( \gamma \) of the pull-back of \( g \) The volume form on \( \Sigma \) is
\[ \eta_\gamma = \sqrt{\det(\gamma_{ij})} d^{p+1} \lambda, \] (122)
and a suitable action is
\[ S = \frac{1}{2} \int_\Sigma \sqrt{\gamma} d^{p+1} \lambda \gamma^{ij} g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda^i} \frac{\partial x^\nu}{\partial \lambda^j} \] (123)
\[ = \frac{1}{2} \int_\Sigma (\text{Tr}_\gamma \phi^* \eta_\gamma). \] (124) [polyakov]

In string theory this is called the Polyakov form of the action. The independent fields to be varied are the embedding \( \phi \), i.e. one varies the world-volume scalar fields \( x^\mu(\lambda^i) \), and the world volume metric \( \gamma^{ij} \). A different strategy is to set
\[ \gamma_{ij} = g_{ij} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda^i} \frac{\partial x^\nu}{\partial \lambda^j} \] (125)
\(^3\)p is the spatial dimension of the object and \( p+1 \) the dimension of the world-sheet or world-volume. Sometimes, when no notion of time is involved, a p-brane is taken to be the same as a p-dimensional immersed sub-manifold.

17
That is one identifies the world volume metric with the metric induced on the world volume from the spacetime metric, via pull-back. The *Dirac-Nambu-Goto Action* is now taken to be
\[
S = \frac{1}{2} \int _{\Sigma} \eta_{\phi^* g} = \frac{1}{2} \int _{\Sigma} \sqrt{|g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda^i} \frac{\partial x^\nu}{\partial \lambda^j}|} d^{p+1} \lambda.
\]
(126)
This is to be varied with respect to the embedding (or immersion) \( \phi \), that is with respect to the world-volume fields \( x^{m}(\lambda) \). The reader is invited to carry out that exercise. In the mathematical literature solutions of the Euler-Lagrange equations are called *minimal submanifolds* even though they may in fact only be stationary points of the Dirac-Nambu-Goto action functional.

### 3.2.3 Non-linear sigma models

So far we have been thinking of the target \( M \) of the map \( \phi \) as our spacetime and the domain \( \Sigma \) as some particle or extended object moving in it. However we can always change our minds and think of the domain \( \Sigma \) of the map as our spacetime and the target as some sort of field space or internal space. The simplest situation occurs when we think of the metric on \( \Sigma \) as fixed. For example, could take \( \Sigma \) to be Minkowski spacetime \( E^{n-1,1} \), and we take the Polyakov form of action given above. We then get the action of in physics are called *non-linear sigma models* the solutions of whose equations of motion are called in mathematics *harmonic maps* if the metric on \( \Sigma \) is positive definite and *wave maps* if it is Lorentzian. The terminology arises because if \( \{\xi, \gamma\} = E^n \) or \( \{\xi, \gamma\} = E^{n-1,1} \) then the equation of motion reduces to the linear Laplace equation or the wave equation respectively. If the target metric is not flat, the the equations of motion are non-linear. Historically the name non-linear sigma model arose in pion physics. One has four fields \( \pi_i, \sigma \) which are subject (in suitable units) to a constraint
\[
\pi_i \pi_i + \sigma^2 = 1,
\]
(127)
The target space may thus be identified with the round 3-sphere \( S^3 \) with standard round metric. One may also think of \( S^3 \) as the group manifold of \( SU(2) \). The high degree of symmetry, \( SO(4) \equiv SU(2) \times SU(2)/\mathbb{Z}_2 \), contains chiral symmetries and so one also speaks of *chiral models* or *principal chiral models*.

Sigma-models, and indeed all geometric theories based on Map, \( (\Sigma, M) \) admit in general two types of reparametrization invariance, of \( \Sigma \) and of \( M \). In the the metrics admit isometries these give rise, via Noether’s theorems, to conserved currents on \( \Sigma \). For example the reader is invited to check that in the case of a sigma model, if \( M \) admits a Killing vector field \( K \) we can convert it to a *Killing 1-form*
\[
K_{\phi^* g} = K_{\phi^* g} d\lambda^\mu = g_{\mu\nu} K^\nu dx^\mu,
\]
(128)
pulling back \( K_{\phi^* g} \) to \( \Sigma \) gives a conserved current on \( \Sigma \)
\[
J_K = K^* h_y, \quad d \ast J = 0,
\]
(129)
We can then define a conserved Noether charge

\[ Q_K = \int_S \star J_K, \]

(130)

where \( S \) is a suitable spacelike hypersurface in \( \mathcal{M} \). Because \( d \star J_K = 0 \), then by Stokes’s theorem and assuming a suitable behaviour at infinity, the charge \( Q_K \) will not depend on the choice of the spacelike hypersurface \( S \).

3.2.4 Topological Conservation laws

The conservation of Noether charges depends upon the field equations holding, i.e. the fields must be on shell. Topological conservation laws hold independently of field equations and resemble the conservation of magnetic charges whose conservation depends only on Bianchi identities.

Suppose that spacetime which we are regarding as \( \mathcal{M} \) has dimension \( n \) topologically takes the form

\[ \mathcal{M} = S \times \mathbb{R}, \]

(131)

where \( S \) is a hypersurfaces and \( t \in \mathbb{R} \) is a time coordinate. The field \( \phi(x,t) \) gives rise to a one parameter family of maps \( \phi_t : S \to M \). If \( M \) is equipped with a closed \((n-1)\)-form \( \omega \) we can pull it back to \( \mathcal{M} \) to give a closed \((n-1)\)-form

\[ \omega^* = \phi_t^* \omega, \quad d\omega = 0 \Rightarrow d^* \omega = 0. \]

(132)

The topological charge

\[ Q_\omega = \int_S \omega^*, \]

(133)

will, subject to suitable behaviour at infinity, be independent of \( t \) and thus conserved. If \( M \) is \((n-1)\)-dimnsional we can choose for \( \omega \) the volume form \( \eta_M \), in which case the conserved cahrge is proportional to the Brouwer degree \( \deg \phi_t \) which is clearly quanitized and thus constant.

Consider the Skyrme model. This is a non-linear theory of pions \( \pi_i(x,t) \) with higher derivatives. It is based on maps into \( SU(2) \) given by \( x \to U(x) = \exp(i \pi_t \tau_i) \), with \( \tau_i \) being Pauli matrices. Suppose \( \pi_i(x) \) tends to some single value at infinity, independently of direction. Then we may extend the map to \( S^3 \), the one-point compactification of \( E^3 \). Thus we get a map \( S^3 \to SU(2) = S^3 \) and its Brouwer degree is interpreted as Baryon number in this model.

In fact the one-point compactification is a conformal compactification. Consider stereographic projection \( f : S^3 \setminus \text{north pole} \to E^3 \).

Explicitly \( f : (X^1, X^2, X^3, X^4) \to (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta), \quad X^4 = \cos \chi, \sin \chi = \frac{2r}{2r^2+1} \).

We can pull \( U \) back to get a map \( f^* U : S^3 \setminus \text{north pole} \to S^3 \), which we extend continously over the north pole to give our map \( \phi : S^3 \to S^3 \) whose degree we must calculate. A hedgehog configuration is one for which \( \tau_i = h(r)(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) with \( h(0) = 0 \) and \( h(\infty) = \pi \). It has degree one.
Note that, since stereographic projection preserves angles, the compactification is conformal. In fact, if \( g_{\text{round}} \) is the standard metric on \( S^3 \) and \( g_{\text{flat}} \) the standard flat metric on \( E^3 \), then
\[
f^*g_{\text{flat}} = \Omega^2 g_{\text{round}},
\] (134)
with \( \Omega = \frac{1}{2}(4 + r^2) \).

An analogous construction works in Instanton Physics. The Yang-Mill’s equations in four dimensions are conformally invariant and solutions on \( E^4 \) and can be pulled by to \( S^4 \) by stereographic projection.

### 3.2.5 Coupling of p-branes to (p+2) forms

To begin with, consider a charged particle, with charge \( e \) moving along a world line \( \xi \) in a manifold \( M \) with electromagnetic field \( F = dA \in \Omega^1(M) \). One adds to the usual action an interaction term
\[
S_{\text{int}} = \int_{\xi} eA_{\mu} dx^\mu = \int_{\xi} A_{\mu} \frac{dx^\mu}{d\lambda} d\lambda = \int_{\xi} \phi^* A,
\] (135)
where
\[
\phi^* A = A_{\mu} \frac{dx^\mu}{d\lambda} d\lambda.
\] (136)

In this notation we can carry over the same expression of a \( p \)-brane \( \Sigma \), with \( \dim \Sigma = p + 1 \) moving in the background of a closed \( p + 2 \) form \( F + dA \in \Omega^{p+2}(M) \). Moreover \( S_{\text{int}} \) is obviously invariant under abelian gauge transformations \( A \rightarrow A + d\Lambda \), \( \Lambda \in \Omega^p(M) \) up tot a surface term because
\[
S_{\text{int}} \rightarrow S_{\text{int}} + e \int_{\Sigma} \phi^* d\Lambda
\] (137)
\[
= S_{\text{int}} + e \int_{\Sigma} d\phi d\Lambda
\] (138)
\[
= S_{\text{int}} + e \int_{\xi} \phi^* A.
\] (139)

The brane equations of motion will thus be gauge-invariant.

In local coordinates, and using indices, we get the rather formidable expression
\[
S_{\text{int}} = \frac{e}{(p+1)!} \int_{\xi} \sqrt{\gamma} A_{\mu_1 \ldots \mu_p+1} \frac{\partial x^{\mu}}{\partial \lambda^{\mu_1}} \cdots \frac{\partial x^{\mu_{p+1}}}{\partial \lambda^{\mu_{p+1}}} \eta^{\nu_1 \ldots \nu_p+1} \lambda^{\nu_1+1+ \ldots + \lambda^{\nu_{p+1}}}. \tag{140}
\]
is time-consuming to type in Latex and and showing that it is invariant up to a boundary term is rather painful, particularly if the details have to be typed in Latex.

The best known case is \( p = 1 \), which corresponds to string theory. One has a closed 3-form
\[
H = dB \in \Omega^3(M), \quad H_{\mu \nu \lambda} = \partial_\mu B_{\nu \lambda} + \partial_\lambda B_{\mu \nu} + \partial_\nu B_{\lambda \mu} \tag{141}
\]
and

$$S_{\text{string}} = \frac{1}{2} \int (g_{\mu\nu} \gamma^{ij} + B_{\mu\nu} \eta^{ij}) \frac{\partial x^\mu}{\partial \lambda^i} \frac{\partial x^\nu}{\partial \lambda^j} \sqrt{\gamma} d^2 \lambda.$$  \hspace{1cm} (142)

### 3.2.6 Topological Defects

In so-called mean field theory approaches to condensed matter physics, or in cosmology, the space $\phi$ in which the field or order parameter $\phi$ takes its values is typically non-trivial. The simplest case to consider is the global vortex. For example in theories involving a field $C$-valued $\phi(x, y)$ in $2 + 1$ dimensions the energy, is

$$\int_{E^2} d^2 x \left( \frac{1}{2} |\nabla \phi|^2 + V(\phi) \right)$$  \hspace{1cm} (143)

and the potential energy $V(\phi)$ may be $SO(2)$-invariant: $V(\phi) = V(|\phi|)$. Local minima of $V$ define a vacuum manifold $N \subset M = C$ which will be a collection of circles about the origin in $C$. Suppose there is only one, of finite radius on which $V(\phi)$ attains its absolute lower bound. For example take $V(\phi) = \frac{|\phi|^4}{4} - \frac{|\phi|^2}{2}$. The vacuum manifold is the unit circle $|\phi| = 1$

On a circle $S^1_\infty \subset E^2$ near infinity, finite energy forces $|\Phi| = 1$, Thus we get a map $S^1_\infty \to S^1$ whose degree may be interpreted as a conserved vortex number. Note that if the vortex number is non-zero, by an argument identical to the proof of the fundamental theorem of algebra, there must be at least one zero of $\phi$ inside $S^1_\infty$. Such zeros are interpreted as vortex positions.

This example can be trivially generalized to three dimensions with $E^2$, $C \equiv R^2$, $SO(2)$, and $S^1$ replaced by $E^3$, $R^3$, $SO(3)$ and $S^2$, and the words global vortex replaced by the words global monopole.

The defects can also be local is the usual gradient operator $\nabla$ is replaced by a covariant derivative operator $D$ with respect to a $U(1)$ or $SU(2)$ gauge group.

### 3.2.7 Self-interactions of p-forms

The free action for $p$-form $F = dA$ is

$$S = -\frac{1}{2} \frac{1}{p!} \int_M (F, F) \eta = -\frac{1}{2} \frac{1}{p!} \int_M F \wedge \ast F.$$  \hspace{1cm} (144)

The variation is

$$\delta S = \frac{(-1)^{p-1}}{p!} \int_M \delta A \wedge d \ast F.$$  \hspace{1cm} (145)

which yields the linear equation of motion

$$d \ast F = 0.$$  \hspace{1cm} (146)

To get an interacting, i.e. nonlinear theory, we could try an action constructed out of higher exterior powers of $F$. It would have the interesting property of being 'topological' independent of any metric on our manifold $M$. The problem is that such a term would either vanish identically, e.g $F \wedge F \wedge F$ for
a 3-form in nine dimensions, or or not contribute to the equations of motion because it is exact. For example $F \wedge F \wedge F$ in six dimensions. In all dimensions in fact we have

$$F \wedge F \wedge F = d(A \wedge F \wedge F).$$

(147)

However this suggests that in five dimensions

$$S_{\text{Chern–Simons}} = c_{\text{C–S}} \int_M A \wedge F \wedge F, \quad F \in \Omega^2(M),$$

(148)

where $c_{\text{C–S}}$ is a coupling constant, is a possibility. Under a gauge transformation $A \to A + d\Lambda$

$$A \wedge F \wedge F \to A \wedge F \wedge F + d(\Lambda \wedge F \wedge F),$$

(149)

and so the action $S_{\text{Chern–Simons}}$ changes by a surface term which will not affect the equations of motion.

Now up to a surface term

$$\delta S_{\text{Chern–Simons}} = 3c_{\text{C–S}} \int_M \delta A \wedge F \wedge F.$$  

(150)

Thus the non-linear equations of motion are

$$d \star F - 6c_{\text{C–S}} F \wedge F = 0.$$  

(151)

In 3 dimensions the Chern-Simons term gives a mass to the photon. In eleven dimensional supergravity, which has a 4-form, it gives

$$d \star F \propto F \wedge F.$$  

(152)