1 Action of Groups on Manifolds

1.1 Groups and semi-groups

A group $G$ is a set $\{g_i\}$ with a multiplication law, $G \times G \rightarrow G$ which we write multiplicatively and which satisfies

\begin{align}
\text{closure} & \quad (i) \quad g_1 g_2 \in G \quad \forall \quad g_1, g_2 \in G \\
\text{associativity} & \quad (ii) \quad (g_1 g_2) g_3 = g_1 (g_2 g_3) \quad \forall \quad g_1, g_2, g_3 \in G \\
\text{identity} & \quad (iii) \quad \exists e \in G \quad s.t. \quad e g = g e = g \quad \forall \quad g \in G \\
\text{inverse} & \quad (iv) \quad \forall g \in G \quad \exists g^{-1} \in G \quad s.t. \quad g^{-1} g = g g^{-1}.
\end{align}

If one omits (iv) one gets a semi-group. These frequently arise in physics when irreversible processes, such as diffusion are involved.

1.2 Transformation groups: Left and Right Actions

Usually one is interested in Transformation group acting on some space $X$, typically a manifold, such that there is a map $\phi : G \times X \rightarrow X$ such that $X \ni x \rightarrow \phi_g(x)$ and

\begin{align}
\text{left action} & \quad \phi_{g_1} \circ \phi_{g_2} = \phi_{g_1 g_2}, \quad \phi_e(x) = x.
\end{align}

Such an action is called a left action of the group $G$ on $X$ or sometimes in physics a realization\(^1\). The action of a map $\psi$ would be called a right action if

\(^1\)Often one speaks of a non-linear realization to distinguish the action from a linear action, linear realization or representation of $G$ on $X$. We shall treat the special case of representations shortly. Of course the word ‘action’ has nothing to do with action functionals.
right action \[ \psi_{g_1} \circ \psi_{g_2} = \psi_{g_2g_1}. \] (1.6)

In other words the order of group multiplication is reversed under composition. Our conventions for composition are that the second map is performed first, i.e

\[ f \circ g(x) = f(g(x)) \] (1.7)

which coincides with the standard order for matrix multiplication in the case of linear maps. In terms of arrows, and if we think of in the standard way of reading input to output from left to right, we have

\[ X \rightarrow gX \rightarrow fX. \] (1.8)

It would, perhaps, be more logical to read from right to left

\[ X \leftarrow fX \leftarrow gX. \] (1.9)

It sometimes happens that one needs to consider right actions. This gives rise to various sign changes and so one must be on one’s guard. Of course, given a right action \( \psi \), one may always replace it by a left action provided each group element is replaced by its inverse:

\[ \psi_{g_1}^{-1} \circ \psi_{g_2}^{-1} = \psi_{g_2^{-1}g_1}^{-1} = \psi_{(g_1g_2)^{-1}}. \] (1.10)

1.3 Effectiveness and Transitivity

The action (either left or right) of a transformation group \( G \) on \( X \) is said to be effective if \( \nexists \ g \in G : \phi_g(x) = x \forall x \in X \), i.e which moves no point. We shall mainly be interested in effective actions since if the action of \( G \) is not effective there is a subgroup which is.

The action is transitive if \( \forall x_1, x_2 \in X, \exists g \in G : \phi_g(x_1) = x_2 \). In other words every pair of points in \( X \) may be obtained one form the other by means of a group transformation.

If the group element \( g \) is unique the action is said to be simply transitive. Otherwise the action is said to be multiply transitive. subsectionActions of groups on themselves

One has three important actions

**Left action** \( L_h : g \rightarrow hg, \ h \) regarded as moving \( g \) \hspace{1cm} (1.11)

**Right action** \( R_h : g \rightarrow gh, \ h \) regarded as moving \( g \) \hspace{1cm} (1.12)

**Conjugation** \( C_h : g \rightarrow hgh^{-1}, \ h \) regarded as moving \( g \) \hspace{1cm} (1.13)

Note that the left and right actions are simply transitive and they commute with one another

\[ L_{g_1} \circ R_{g_2} = R_{g_2} \circ L_{g_1}. \] (1.14)

By contrast, \( C_h \), which is in fact a left action, commutes with neither and fixes the identity element or origin \( e \) of the group \( G \)

\[ C_h(e) = e \quad \forall h \in G. \] (1.15)
Conjugation is a natural symmetry of a group. Roughly speaking, it is analogous to change of basis for a vector space. Another way to view it is a sort of rotation about the identity element $e$. Two subgroups $H_1, H_2 \subset G$ are said to be conjugate if there is a $g \in G$ such that for all $h_1 \in H_1$ and $h_2 \in H_2$, $gh_1g^{-1} = h_2$, and thus $gH_1g^{-1} = H_2$. Two subgroups related by conjugation are usually regarded as being essentially identical since one may be obtained from the other by means of a ‘rotation’ about the identity origin $e$.

A subgroup $H$ is said to invariant or normal if $gHg^{-1} = H$, $\forall g \in G$.

### 1.4 Cosets

If $H$ is a subgroup of $G$ it’s set of right cosets $G/H = \{ g \in G : g_1 \equiv g_2 \text{iff } \exists H \in H : g_1 = hg_2 \}$. It’s set of left cosets $H \setminus G = \{ g \in G : g_1 \equiv g_2 \text{iff } \exists H \in H : g_1 = g_2h \}$. Our notation, which may not be quite standard is such that right cosets are obtained by dividing on the right and left cosets by dividing on the left. Note that, because right and left actions commute, $L_g$ the left action of $G$ descends to give a well defined action on a right coset and the right action $R_g$ descends to give a well defined action on a left coset.

### 1.5 Orbits and Stabilizers

The orbit $\text{Orb}_G(x)$ of a transformation group $G$ acting on $X$ is the set of points attainable from $x$ by a transformation, that is $\text{Orb}_G(x) = \{ y \in X : \exists g \in G : \phi_g(x) = y \}$. Clearly $G$ acts transitively on each of its orbits. If $G$ acts transitively on $X$, then $X = \text{Orb}_G(x) \forall x \in X$. One may think of the set of right cosets $G/H$ as the orbits in $G$ of $H$ under the left action of $G$ on itself and the right cosets $H \setminus G$ as the orbits of $H$ under the right action of $G$ on itself.

If $G$ acts multiply transitively on $X$ then $H$ is said to be a homogeneous space and $\forall x \in X$, there is a subgroup $H_x \subset G$ each of whose elements $h \in H_x$ fixes or stabilizes $x$, i.e. $\phi_h(x) = x \forall x \in H_x$. The subgroup $H_x$ is called variously the stabilizer subgroup, isotropy subgroup or little group of $x$. For example, the group $SO(3)$ acts on $S^2 \subset \mathbb{E}^3$ in the standard way and an $SO(2) \equiv U(1)$ subgroup fixes the north pole and south poles. Another $SO(2)$ subgroup in fact a (different) $SO(2)$ subgroup fixes every pair of antipodal points. These subgroups are all related by conjugation. Thus if $y = \phi_y(x)$ so that $x = \phi_y^{-1}(y)$ then

$$\phi_y \circ \phi_h \circ \phi_y^{-1}(y) = y.$$  

That is

$$\phi_{ghg^{-1}}(y) = y, \quad \Rightarrow H_y = gH_xg^{-1}.$$  

Thus, up to conjugation, which corresponds to choosing an origin in $X$, there is a unique subgroup $H$ and one may regard the homogeneous space $X$ as a right coset $G/H$ on which $G$ acts on the left.

#### 1.5.1 Example: De-Sitter and Anti-de-Sitter Spacetimes

De-Sitter spacetime, $dS_4$ and anti-De-Sitter spacetime $AdS_4$ are conveniently defined as quadrics in five dimensional flat spacetimes, with signature $(4,1)$ and $(3,2)$ respectively:

$$dS_4: \quad (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 - (X^5)^2 = 1. \quad (1.18)$$

$$AdS_4: \quad (X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2 - (X^5)^2 = -1. \quad (1.19)$$
They inherit a Lorentzian metric from the ambient flat spacetime which is preserved by $G = SO(4,1)$ or $SO(3,2)$ which acts transitively with stabilizer $H = S0(3,1)$ in both cases. Thus $dS_4 = SO(3,1)/S0(3,1)$ and $dS_4 = SO(3,1)/S0(3,1)$.

1.5.2 Example: Complex Projective space

A physical description of complex projective space, $\mathbb{CP}^n$ is as the space of pure states of an $n+1$ state quantum system with Hilbert space $\mathcal{H} \equiv \mathbb{C}^{n+1}$. Thus a general state is

$$\psi = \sum_{r=1}^{n+1} Z^r|r\rangle, \quad (1.20)$$

where $|r\rangle$ is a basis for $\mathcal{H}$. Now since $|\Psi\rangle$ and $\lambda|\psi\rangle$, with $\lambda \in \mathbb{C} \setminus 0$ define the same physical state the space of states is parametrized by $Z_1, Z_2, \ldots, Z_{n+1}$ but $(\lambda Z^1, \lambda Z^2, \ldots, \lambda Z^{n+1}) \equiv (Z^1, Z^2, \ldots, Z^{n+1})$ with $\lambda \in \mathbb{C} \setminus 0$, which is the standard mathematical definition of $\mathbb{CP}^n$ which is the simplest non-trivial example of a complex manifold with complex charts and locally holomorphic transition functions.

Working in an orthonormal basis, we may attempt to reduce the redundancy of the state vector description by considering normalized states

$$\sum_{r=1}^{n+1} |Z^r|^2 = 1. \quad (1.21)$$

This restricts the vectors $\Psi$ to lie on the unit sphere $S^{2n+1} = SO(2n+2)/SO(2n+1) = U(n+1)/U(n)$. However one must still take into account the redundant, and unphysical, phase of the state vector. The group $G = U(n+1)$ acts transitively on $C\mathbb{CP}^n$ and the stabilizer of each point is $H = U(1) \times U(1)$. We may think of $\mathbb{CP}^n$ as the set of orbits in $S^{2n+1}$ of the action of $U(1)$ given by

$$Z^r \rightarrow e^{i\theta} Z^r. \quad (1.22)$$

Each orbit is a circle $S^1$ and the assignment of each point to the orbit through it defines is called the Hopf map

$$\pi : S^{2n+1} \rightarrow \mathbb{CP}^n. \quad (1.23)$$

The structure we have just described provides perhaps the simplest example of a fibre bundle, in fact a circle bundle. We shall treat them in further detail later.

The most basic case is the spin $\frac{1}{2}$ system which has $n = 1$. One has $\mathbb{CP}^1 \equiv S^2 \equiv SU(2)/U(1) \equiv U(2)/U(1) \times U(1)$. We may introduce two coordinate patches, $\zeta = Z^1/Z^2$ and $\bar{\zeta} = Z^2/Z^1$, one covering the north pole and the other the south pole of $S^2$. On the overlap $\zeta = \frac{1}{\bar{\zeta}}$. In fact these are stereographic coordinates.

1.6 Representations

If $X$ is some vector space $V$ and $g$ acts linearly, i.e by endomorphism or linear maps we have a representation of $G$ by matrices $D(g)$ such that

$$D(g_1)D(g_2) = D(g_1g_2) \quad \text{etc.} \quad (1.24)$$

The representation said to be faithful if the action is effective, i.e. no element of $G$ other than the identity leaves every vector in $V$ invariant. Another way to say this is that a representation is a homomorphism $D$ of $G$ into a subgroup of $GL(V)$ and a faithful representation has no kernel. Given an unfaithful representation we get a faithful representation by taking the quotient of $G$ by the kernel.
1.6.1 Reducible and irreducible representations

The representation is said to be \textit{irreducible} if no vector subspace of \( V \) is left-invariant, otherwise it is said to be \textit{reducible}. The representation is said to be \textit{fully reducible} if the complementary subspace is invariant.

A reducible but not fully reducible representation has

\[
D(g) = \begin{pmatrix} \cdots & 0 \\ \cdots & \cdots & \cdots \\ \end{pmatrix},
\]

(1.25)

and vectors of the form \( \begin{pmatrix} 0 \\ \cdots \end{pmatrix} \) are invariant but those of the form \( \begin{pmatrix} \cdots \end{pmatrix} \) are not. On the other hand for a fully reducible representation one has

\[
D(g) = \begin{pmatrix} \cdots & 0 \\ 0 & \cdots \end{pmatrix},
\]

(1.26)

and both subspaces are invariant. If \( V \) admits a \( G \)-invariant \textit{definite} inner product, \( B \), say, then any reducible representation is fully reducible.

\textbf{Proof} Use \( B \) to project any vector into \( W \), the invariant subspace, and its orthogonal complement \( W_\perp \) with respect to \( B \) Thus for any \( v \in V \) we have \( v = v_\parallel + v_\perp \). Now \( G \) takes \( v_\parallel \) to \( \tilde{v}_\parallel \in W \) and \( v_\perp \) to \( \tilde{v}_\perp \in W_\perp \).

1.6.2 The contragredient representation

If \( D(g) \) acts on \( V \) then \((D^{-1}(g))^t\) acts naturally on \( V^* \) the dual space. In components

\[
D(g) : V^a \to D^a_b(g) V^b = \tilde{V}^a.
\]

(1.27)

\[
D^{-1}(g) : \omega_a \to \omega_c(D^{-1})^c_a = \tilde{\omega}_a
\]

(1.28)

\[
(D^{-1})^t(g) : (\omega^t)^a \to (\omega^t)^a = ((D^{-1})^t)^b_a (\omega^t)^b.
\]

(1.29)

Thus

\[
\omega_a V^a = \tilde{\omega}_a \tilde{V}^a.
\]

(1.30)

The reason we need to transpose and invert is that otherwise the matrix multiplication would be in reverse order. We would have an anti-homomorphism of \( G \) into \( GL(V) \) or in our previous language a right action of \( G \) on \( V \).

1.6.3 Equivalent Representations

Two representations \( D \) and \( \tilde{D} \) acting on vector spaces \( V \) and \( W \) are said to be \textit{equivalent} if there exist an invertible linear map \( B : V \to \tilde{B} \) and \( D = B^{-1} \tilde{D} B \).

\textbf{Example} Suppose that \( W = V^* \), the dual space of \( V \), and \( B \) is a symmetric non-degenerate bilinear form or metric invariant under the action of \( G \), then \( D \) and the contragredient representation \((D^{-1})^t\) are equivalent.

\textbf{Proof} Invariance of \( B \) under the action of \( G \) requires that

\[
D^t B D = B.
\]

(1.31)

Thus

\[
D = B^{-1}(D^{-1})^t B.
\]

(1.32)
In components
\[(D^{-1})^b{}_a = B_{ac}D^c{}_eB^{eb}. \quad (1.33)\]

Thus raising and lowering indices with the metric $B$, which need not be positive definite, gives equivalent representations on $V$ and $V^*$. The maps $B$ and $B^{-1}$ are sometimes referred to as the *musical isomorphisms* and denoted by $\flat$ and $\sharp$ respectively.

In fact there is no need for $B$ to be symmetric, $B = B^t$. It often happens that one have a non-degenerate bi-linear form which is anti-symmetric, $B = -B^t$. In that case $V$ is said to be endowed with a *symplectic structure*, or *symplectic* for short.

### 1.7 *Semi-direct products, group extensions and exact sequences*

Given two groups $G$ and $H$ there is an obvious notion of product
\[G \times H : (g_1, h_1)(g_2, h_2) \rightarrow (g_1g_2, h_1h_2). \quad (1.34)\]

. Frequently however, one encounters a more sophisticated construction: the semi-direct product $G \rtimes H$. The Poincaré group for example, is the semi-direct product of the Lorentz group and the translation group, such that the translations form an invariant subgroup but the the Lorentz-transformsions do not. The construction also arises when one wants to extend a group $G$ by another group $H$. It also arises in the theory of fibre bundles which we will outline later.

Suppose $G$ acts on $H$ preserving the group structure, i.e. there is a map $\rho : G \rightarrow H$ so that $\rho_{g_1}(h) \circ \rho_{g_2}(h) = \rho_{g_1g_2}(h)$ and $\rho_g(h_1\rho_g(h_2) = \rho_g(h_1h_2)$. If, for example, $H$ is abelian, thought of as additively it is a vector space, then then $\rho_g$ would be a representation of $G$ on $H$. In general $\rho$ is homomorphism of $G$ into $\text{Aut}(H)$ the automorphism group of $H$.

We define the product law by
\[G \rtimes H : (g_1, h_1)(g_2, h_2) \rightarrow (g_1g_2, h_1\rho_{g_1}(h_2)), \quad (1.35)\]

which is asociative with inverse
\[(g, h)^{-1} = (g^{-1}, \rho_{g^{-1}}(h^{-1})). \quad (1.36)\]

A standard example is the Affine group $A(n) = GL(n, \mathbb{R} \rtimes \mathbb{R}^n$, which acts on n-dimensional affine space $\mathbb{A}^n \equiv \mathbb{R}^n$ by translations and general linear transformations preserving the usual flat affine connection. A matrix representation is
\[
\begin{pmatrix}
\Lambda^a_b & a^a \\
0 & 1
\end{pmatrix}, \quad \Lambda \in GL(n, \mathbb{R}) \quad a^a \in \mathbb{R}^n. \quad (1.37)
\]

we have
\[
\begin{pmatrix}
\Lambda & a \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\Lambda' & a' \\
0 & 1
\end{pmatrix} =
\begin{pmatrix}
\Lambda\Lambda' & a + \Lambda a' \\
0 & 1
\end{pmatrix}. \quad (1.38)
\]

Restricting to $O(n) \subset GL(n, \mathbb{R})$ we obtain the Euclidean group $E(n) = O(n) \rtimes \mathbb{R}^n$. The Poincaré group is obtained by restriction to the Lorentz group $SO(n - 1, 1) \subset GL(n, \mathbb{R})$.

Glearly $G$ may be regarded as a subgroup of $G \rtimes H$ because of the injection
\[i : G \rightarrow G \rtimes H, \quad g \mapsto (g, e). \quad (1.39)\]

One may also regard $H$ as a subgroup of $G \rtimes H$ because
\[(e, h_1)(e, h_2) = (e, h_1h_2). \quad (1.40)\]
Moreover, \( H \) is an invariant subgroup of \( G \times H \) because
\[
(g, h)(e, h')(g^{-1}, h^{-1}\rho_g(h^{-1})) = (e, h\rho_g(h')\rho_g^{-1}(h^{-1})) \in (e, H) .
\] (1.41)

By contrast, \( G \) is, in general (if \( \rho \neq \text{id} \)), not an invariant subgroup because
\[
(g, h)(g', h')(g^{-1}, h^{-1}\rho_g(h^{-1})) = (g\rho_g^{-1}(h^{-1})\rho_{g'}^{-1}(h^{-1}), h\rho_{g'}(h')h^{-1}\rho_g^{-1}(h^{-1}))
\] (1.42)
\[
= (g\rho_g^{-1}(h^{-1})\rho_{g'}^{-1}(h^{-1}), h\rho_{g'}(h')h^{-1}\rho_g^{-1}(h^{-1}))
\] (1.43)
\[
= (g\rho_g^{-1}(h^{-1})\rho_{g'}^{-1}(h^{-1}), h\rho_{g'}(h')h^{-1}\rho_g^{-1}(h^{-1}))
\] (1.44)
\[
= (g\rho_g^{-1}(h^{-1})\rho_{g'}^{-1}(h^{-1})).
\] (1.45)

Thus even if \( \eta' = e \), the second term is not necessarily the identity. In summary, we have what is called an exact sequence,
\[
1 \longrightarrow G \xrightarrow{i} G \rtimes H \xrightarrow{\pi} H \longrightarrow ,
\] (1.46)

that is the where the maps \( \pi : (g, h) \rightarrow (e, h) \), \( i \rightarrow (g, e) \) satisfy the relation that
\[
\text{Image}(i) = \text{Kernel}(\pi) ,
\] (1.47)

and the arrows at the ends are the obvious maps of identity elements. Note that in this language a map \( \phi : G \rightarrow H \) is an isomorphism if
\[
1 \longrightarrow G \xrightarrow{\phi} H \longrightarrow 1 .
\] (1.48)

It is merely a homomorphism if
\[
G \xrightarrow{\phi} H \longrightarrow 1 .
\] (1.49)

and \( G \) is just a subgroup of \( H \) if
\[
1 \longrightarrow G \xrightarrow{\phi} H .
\] (1.50)

If (1.46) holds, one says that \( G \rtimes H \) is an extension of \( G \) by \( H \). If \( H \subset \text{Z}(G \rtimes H) \), the centre, of one speaks of a central extension. Thus Nil or the Heisenberg group is a central extension of the 2-dimensional abelian group of translations.

1.7.1 The five Lorentz groups

Other examples of semi-direct products arise when one considers the discrete symmetries parity \( P \) and time reversal \( T \). These are two commuting involutions inside \( O(3, 1) \) acting on four-dimensional Minkowski spacetime \( \mathbb{E}^{3,1} \) as
\[
(x^0, x^i) \xrightarrow{P} (x^0, -x^i), \quad (x^0, x^i) \xrightarrow{T} (-x^0, x^i).
\] (1.51)

We abuse notation by calling \( P, T, PT, \) and \( (P, T) \) the \( \mathbb{Z}_2 \) and \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) groups generated by \( P, T, PT \) and \( P, T \) respectively. Only \( PT \) is an invariant subgroup of \( O(3, 1) \), and indeed belongs to the centre of \( O(3, 1) \). We define the following invariant subgroups:

(i) \( SO_0(3, 1) \) is the connected component of the identity in \( O(3, 1) \),
(ii) \( SO(3, 1) \) is the special Lorentz group which is the subgroup with unit determinant.
(iii) \( O \uparrow (3, 1) \) is the otochronous Lorentz-group which preserves time orientation,
(iv) $SO_+(3,1)$ as the subgroup preserving space-orientation.

One has

\[
\begin{align*}
O(3,1)/SO(3,1) &= \mathbb{Z}_2 \times \mathbb{Z}_2 \\
O(3,1)/SO_0(3,1) &= \mathbb{Z}_2 \\
O(3,1)/O \uparrow (3,1) &= \mathbb{Z}_2 \\
O(3,1)/O_+ (3,1) &= \mathbb{Z}_2
\end{align*}
\]

Thus for example

\[
1 \longrightarrow (P,T) \overset{i}{\longrightarrow} O(3,1) \overset{\pi}{\longrightarrow} SO_0(3,1) \longrightarrow 1.
\]

Geometrically, $O(3,1)$, for example, has four connected components and may be regarded as a principal bundle over a a conected base $B = SO_0(3,1)$ with fibres $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = (P,t)$ consisting of 4 disjoint points.

1.8 Geometry of Lie Groups

Lie groups are

(i) Differentiable Manifolds

(ii) groups

(iii) such that group multiplication and inversion are diffeomorphisms

The Lie algebra $\mathfrak{g}$ of $G$ may be identified (as a vector space) with the tangent space of the origin or identity element $e$

\[
\mathfrak{g} = T_e(G).
\]

Now if $C_g$ is conjugation, then it fixes the origin and the derivative map $Ad_g = C_{g*}$ acts linearly on $T_e(g)$ via the what is called the adjoint representation and the contra-gredient representation acts on $\mathfrak{g}^*$ the dual of the Lie algebra via waht is called the co-adjoint representation. In general, these representations are not equivalent although given an invariant metric on $G$ they will be. Now let $e_a$ be a basis for $\mathfrak{g}$ and $e^a$ a basis for $\mathfrak{g}^*$ so that

\[
\langle e^a | e_b \rangle = \delta_b^a.
\]

Since the right action $R_g$ and left action $L_g$ of $G$ on itself are simply transitive we use the pull-forward and push-back maps $R_{g*}, L_{g*}$ and $R_{g}^*, L_{g}^*$ to obtain a set of left-invariant vector fields $L_a$ and right-invariant vector fields $R_a$ and left–invariant one forms $\lambda^a$ and right–invariant one forms $\rho^a$.

Thus

\[
\begin{align*}
R_g e_a &= R_a(g) \\
R_{g*} \rho^a &= e^a,
\end{align*}
\]

and

\[
\begin{align*}
L_g e_a &= L_a(g) \\
R_{g*} \lambda^a &= e^a,
\end{align*}
\]

Thus, for example,

\[
L_{g_1} L_a(g_2) = L_{g_1} L_a(g_2) \star e_a = L_{g_1 g_2} \star e_a = L_{a}(g_1 g_2) \text{ etc...}
\]

Thus every Lie Group is equipped with a global frame field (in two ways). We shall use these frame fields to do calculations. Note that, because left and right actions are
global diffeomorphisms, the frames are everywhere linearly independent. A manifold admitting such a global frame field is said to be parallelizable. Thus every Lie group is parallelizable. Among spheres, it is known that only $S^1$, $S^3$ and $S^7$ are parallelizable. The first two are groups manifolds, $U(1)$ and $SU(2)$ respectively but the third is not. In fact the parallelism of $S^7$ has some applications to supergravity theory and is closely related to the fact that $S^7$ may be regarded as the unit sphere in the octonions $\mathbb{O}$, just as $S^1$ and $S^3$ may be regarded as the unit sphere in the complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$ respectively.

1.8.1 Matrix Groups

The basic example of a Lie Group is $SU(2)$ which we may think of as complex valued matrices of the form
\[
\begin{pmatrix}
  a & b \\
  -\bar{b} & \bar{a}
\end{pmatrix}
\quad |a|^2 + |b|^2 = 1.
\]

If $a = X^1 + iX^2$ and $b = X^3 + iX^4$ this defines a unit 3-sphere $S^3 \subset \mathbb{E}^4$.

Closely related is $SL(2, \mathbb{R})$ which we may think of as real valued matrices of the form
\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\quad ad - bc = 1.
\]

If $a = X^0 + X^3, d = X^0 - X^3, b = X^2 + X^4, c = X^2 - X^4$ this defines a quadric in $\mathbb{E}^2$. The group $SL(2, \mathbb{R})$ acting on $\mathbb{R}^2$ preserves the skew two by two matrix $\epsilon_{AB} = -\epsilon_{BA} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This is because for two by two matrices
\[
\det X = X^1 X^2 \epsilon_{AB} = \epsilon_{12} = 1.
\]

In fact $SL(2, \mathbb{R})$ is the first in an infinite series of Lie groups called the symplectic groups $Sp(2n, \mathbb{R})$ which preserve a skew-symmetric form $\omega_{ab} = -\omega_{ba}$. Just as the orthogonal group $SO(n)$ is the basis of Riemannian geometry involving asymmetric positive definite bi-linear form or metric $g_{ab} = g_{ba}$, the symplectic groups is the basis of symplectic geometry, i.e. the geometry of phase space and involves an anti-symmetric tensor two-form $\omega_{ab} = -\omega_{ba}$.

1.9 Infinitesimal Generators of Right and Left Translations

If a group $G$ acts freely on a manifold $M$ on the left say we focus on the action of a one-parameter subgroup $g(t)$ such that $g(t_1)g(t_2) = g(t_1 + t_2), g(-t) = g^{-1}(t)$ and $g(0) = e$. The orbits in $M$ of of $g(t)$ are curves
\[
x(t) = \phi_g(t)(x).
\]

There is at most one such curve through each point $p \in M$ and it has a tangent vector
\[
K = \frac{dx}{dt}.
\]

Thus we get a vector field $K(x)$ on $M$ for each one parameter subgroup of $G$. A special case occurs when $M$ is a Lie group $G$ and the group action $\phi$ is by left translations. Each one
parameter subgroup of $G$ will have an orbit passing through the origin passing through the unit element or origin $e \in G$ and an initial tangent vector $\frac{de}{dt}|_{t=0}$ which lies in the tangent space of the origin $T_e(G)$ which we identify with the Lie algebra $\mathfrak{g}$. If $e_a$ is a basis for $\mathfrak{g}$, then get a map, called the (left) exponential map from $\mathbb{R} \times T_e(G) \equiv \mathbb{R} \times \mathfrak{g} : (t, V) \rightarrow g_a(t)$ by moving an amount $t$ along the orbit through $e$ with initial tangent $V$ say.

1.9.1 Example Matrix groups

Near the origin we write

$$g = 1 + v^a M_a t + O(t^2).$$

(1.65)

where $M_a$ are a set of matrices spanning $\mathfrak{g}$. Then

$$g(t) = \exp t V^a M_a = 1 + t V^a M_a + \frac{t^2}{2!} (V^a M_a)^2 + \frac{t^3}{3!} (V^a M_a)^3 + \ldots.$$  

(1.66)

Thus the exponential map corresponds to the exponential of matrices.

1.10 Right invariant Vector Fields generate Left Translations and vice versa

By allowing the initial tangent vector $V$ to range over a basis $e_a$ of the initial tangent space $T_e(g) \equiv \mathfrak{g}$ we get family of vector fields $K_a$ on $G$ which are tangent to the curves $g(t)$ and are called the generators of left-translations. The give a global parallelization of $G$. Since left and right actions commute this parallelization is right-invariant. In fact it is clear that this parallelization must coincide with that defined earlier. That is we may set $K_a = R_a$.

It follows that

$$\mathcal{L}_{L_a} \rho^b = 0 = \mathcal{L}_{R_a} \lambda^b.$$  

(1.67)

1.11 Lie Algebra

We can endow $\mathfrak{g}$ with a skew symmetric bracket satisfying the Jacobi identity by setting

$$[e_a, e_b] = -\mathcal{L}_{R_a} R_b|_{t=0} = -[R_a, R_b]|_{t=0}.$$  

(1.68)

We define the structure constants by

$$[e_a, e_b] = C^c_{ab} e_c.$$  

(1.69)

Since the brackets of right-invariant vector fields are determined by their values at the origin we have

$$[R_a, R_b] = -C^c_{ab} R_c.$$  

(1.70)

Indeed this relation may also be taken as a definition of the Lie algebra. Note that the Jacobi Identity

$$[e_a, [e_b, e_c]] + [e_c, [e_a, e_b]] + [e_b, [e_c, e_a]] = 0.$$  

(1.71)

follows from the Jacobi identity for vector fields

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0.$$  

(1.72)
1.12 Origin of the minus sign

The sign conventions we are adopting are standard in mathematics but perhaps unfamiliar to those whose knowledge of Lie algebras comes from some physics textbooks. Moreover it is illuminating to track them down. In order to understand a sign it is usually sufficient to consider a particular example and so we shall consider the special case of matrix groups.

We move from \( e \) to \( x_1 \) along \( g_a(t) \) and from \( x_1 \) to \( x_2 \) along \( g_b(t) \) and compare with moving from \( e \) to \( x_3 \) along \( g_b(t) \) followed by moving from \( x_3 \) to \( x_4 \) along \( g_a(t) \). Now

\[
g_a(t)g_b(t) = \exp tM_a \exp tM_b = 1 + t(M_a + M_b) + \frac{t^2}{2}(M_a^2 + M_b^2) + t^2M_aM_b + \ldots \tag{1.73}
\]

so that

\[
x_4 - x_3 = g_ag_b - g_bg_a = t^2(M_aM_b - MBM_b) + \ldots = t^2[M_a, M_b] + \ldots \tag{1.74}
\]

From the theory of the Lie derivative

\[
x_4 - x_3 = (\mathcal{L}_{R_b}R_a)t^2 + \ldots = [R_b, R_a]T^2 + \ldots \tag{1.75}
\]

1.13 Brackets of left-invariant vector fields

If one repeats the exercise for right actions one must the order of multiplication. This leads to a sign reversal and one has

\[
[L_a, L_b] = C^{c}_{a b}L_c. \tag{1.76}
\]

1.13.1 Example: Right and Left vector fields commute

This is because the former generate left translations and the latter right translations and these commute. Thus infinitesimally they must commute, i.e.

\[
[R_a, L_b] = 0. \tag{1.77}
\]

1.14 Maurer-Cartan Forms

Using the formula

\[
d(\omega(X, Y)) = X\omega(Y) - Y\omega(X) - \omega([X, Y]) \tag{1.78}
\]

and substituting \( \rho^a(R_b) = \delta^a_b \) we get

\[
d\rho^a = \frac{1}{2}C^a_{b c}\rho^b \wedge \rho^c. \tag{1.79}
\]

Similarly

\[
d\lambda^a = -\frac{1}{2}C^a_{b c}\lambda^b \wedge \lambda^c. \tag{1.80}
\]


1.14.1 Example: Matrix groups

Clearly

\[ g^{-1}dg = \lambda^a M_a \]  

(1.81)

is invariant under \( g \to hg \) with \( g \) constant. Now take one for \( d \)

\[ d(g^{-1}) \wedge dg = d\lambda^a M_a. \]  

(1.82)

But for matrices

\[ d(g^{-1}) = -g^{-1}dgg^{-1}. \]  

(1.83)

Thus

\[ d\lambda^a M_a = -\lambda^b M_b \wedge \lambda^c M_c = -\frac{1}{2} \lambda^b \wedge \lambda^c [M_b, M_c] = -\frac{1}{2} C_a^{\phantom{a}bc}\lambda^b \wedge \lambda^c M_a. \]  

(1.84)

One can similarly check the opposite sign occurs for the right-invariant forms

\[ dgg^{-1} = \rho^a M_a \]  

(1.85)

1.15 Metrics on Lie Groups

Given any symmetric invertible tensor \( g_{ab} = g_{ba} \in g^* \otimes_S g^* \) we can construct a left-invariant metric on \( G \) by

\[ ds^2 = g_{ab} \lambda^a \otimes \lambda^b, \]  

(1.86)

because \( G_{ab} \) are constants and \( \mathcal{L}_{\mathcal{R}} \lambda^b = 0 \). In general, however the metric so constructed will not be right-invariant. One calls a metric which is, bi-invariant. Since

\[ L_h^* g = g, \quad \forall g \in G, \]  

(1.87)

it suffices that \( g \) be invariant under conjugation

\[ C_h : g \to hgh^{-1}. \]  

(1.88)

Now acting at the origin \( e \in G \), conjugation rotates the tangent space into itself via the Adjoint representation \( Ad_h = C_h^* : g \to g \). This in turn acts on \( g^* \) by the contra-gradient or co-adjoint representation. A necessary and sufficient condition for the tensor \( g_{ab} \) to give rise to a bi-invariant metric is that it be invariant under the co-adjoint action. One such metric is the Killing metric which we will define shortly. We begin by discussing the infinitesimal version of the Adjoint action, adjoint action \( \text{ad}_X(Y) \) a linear map \( g \to g \) (for fixed \( X \in g \)) defined by

\[ Y \to [X, Y]. \]  

(1.89)

Taking \( X = e_a \) we have

\[ \text{ad}_{e_a}(Y) = [e_a, Y] = C_a, \]  

(1.90)

where \( C_a \) is some matrix or linear map on \( g \) with components

\[ (C_a)^b \phantom{a}_c = C_a^{\phantom{a}b}_c. \]  

(1.91)

The Jacobi identity

\[ [e_a, [e_b, Y]] - [e_b, [e_a, Y]] = [[e_a, e_b], Y], \]  

(1.92)
becomes
\[ C_a C_b - C_b C_a = C_a \,^c \,_b C_c \]  
(1.93)

and therefore the matrices \( C_a \) provide a matrix representation of the Lie algebra called the \textit{adjoint} or \textit{regular representation}.

Multiplying the Jacobi identity by \( C_d \) and taking a trace gives
\[ \text{Tr}(C_d C_a C_b - C_d C_b C_a) = C_{adb} , \]
(1.94)

where
\[ C_{adb} = B_{dc} C_a \,^c \,_b , \]
(1.95)

and
\[ B_{dc} = \text{Tr}(C_d C_c) = \text{Tr}(C_c C_d) = B_{cd} \]
(1.96)
is a symmetric bi-linear form on \( g \) called the \textit{Killing form}.

Using the cyclic property of the trace we have
\[ C_{abc} = -C_{bac} . \]
(1.97)

But because
\[ C_a \,^d \,_b = -C_b \,^d \,_a \]
(1.98)

we have
\[ C_{adb} = -C_{bd_a} , \]
(1.99)

and hence
\[ C_{abc} C_{[abc]} . \]
(1.100)

The tensor \( C_{abc} \) defines a 3-form on \( g \) and hence by left or right translation a 3-form on \( G \) which by use of the Cartan-Maurer formulae may be shown to be closed.

\[ \textbf{1.15.1 Example: } SU(2) \]

On \( SU(2) \), or on \( SL(2, \mathbb{R}) \), we obtain a multiple of the volume form on \( S^3 \) or \( AdS_3 \) respectively.

\[ \textbf{1.16 Non-degenerate Killing forms} \]

From now on in this subsection , unless otherwise stated, we assume that \( B_{ab} \) is non-degenerate, i.e. \( \det B_{ab} \neq - \). Thus we have an inverse \( B^{ab} \) such that \( B^{ab} B_{bc} = \delta^a_c \).

Therefore
\[ C_a \,^c \,_b = B^{ce} C_{aeb} \]
(1.101)

and
\[ C_a \,^c \,_c = C^{ce} C_{ace} = \text{Tr} C_a = 0 . \]
(1.102)

Groups for which \( \text{Tr} C_a = 0 \) are said to be \textit{unimodular}.
1.16.1 Necessary and sufficient condition that the Killing form is non-degenerate

This is that $G$ is semi-simple, i.e. it contains no invariant abelian subgroups. We shall establish the necessity and leave the sufficiency for the reader.

If $G$ did contain an invariant abelian subgroup $H$, let $e_a$ span $\mathfrak{h}$ and $e_i$ span the complement $\mathfrak{k}$ so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$. Since $H$ is abelian

$$C^\alpha_{\mu \beta} = 0 = C^i_{\alpha \beta}. \quad (1.103)$$

Since $H$ is invariant

$$[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}, \quad \Leftrightarrow C^j_{\alpha \beta} = 0. \quad (1.104)$$

One calculates and finds that

$$B^{\alpha \beta} = 0 = B^{\alpha i}. \quad (1.105)$$

Thus

$$B_{ab} \delta_a^a = 0. \quad (1.106)$$

which implies that $\delta_a^a$ lies in the kernel of $B(\,,\,)$.

1.16.2 Invariance of the Killing metric under the co-adjoint action

Let $V^a$ be the components of a vector $V \in \mathfrak{g}$ then under the adjoint action

$$(\delta_{e_a} V)^b = [e_a, V]^b = C^b_{\alpha c} V^c. \quad (1.107)$$

For example if $G = SO(3)$, $\mathfrak{g} = \mathbb{R}^3$ and

$$\delta v = \Omega \times v, \quad \Omega \in \mathfrak{g}. \quad (1.108)$$

let $\omega_a$ be the components of a co-vector $\omega \in \mathfrak{g}^*$. The co-adjoint action must leave invariant $\langle \omega| V \rangle = \omega_a V^a$. Therefore

$$\delta_{e_a} \omega_b = -\omega_c C^c_{a b}. \quad (1.109)$$

The induced action on $B \in \mathfrak{g}^* \otimes \mathfrak{g}^{*}$ is thus

$$\delta_{e_a} B_{eb} = -C^c_{a e} B_{ce} - C^e_{a c} B_{eb} \quad (1.110)$$

$$= -C_{abc} - C^c_{abc} = 0. \quad (1.111)$$

It follows from our earlier work on representations that if $G$ is semi-simple, the Adjoint and the co-Adjoint representation are equivalent since $B$ is an invariant bi-linear form mapping $\mathfrak{g}$ to $\mathfrak{g}^*$.

It also follows that every semi-simple Lie group is an Einstein manifold with respect to the Killing metric.

If we set $\lambda = \frac{1}{2}$ in the previous formulae for the $\nabla_{\mathcal{Z}}$ connection we get for the Ricci tensor

$$R_{eb} = -\frac{1}{4} C^{f}_{d e} C^{d}_{b f} = -\frac{1}{4} B_{de}. \quad (1.112)$$

Moreover, it is clear that since $\nabla_{\mathcal{Z}}$ preserves the Killing metric and is torsion free, then it is the Levi-Civita connection of the Killing metric.

The simplest examples are $G = SU(2) = S^3$ and $G = SL(2, \mathbb{R}) = AdS_3$ which are clearly Einstein since they are of constant curvature.

$$R_{abcd} = -\frac{1}{8} (B_{ac} B_{bd} - B_{ad} B_{bc}). \quad (1.113)$$
1.16.3 Signature of the Killing metric

The Killing form $B$ is, in general, indefinite having one sign for compact directions and the opposite for non-compact directions. By a long standing mathematical convention, the sign is negative definite for compact directions and positive definite for non-compact directions. Thus in the Yang Mills Lagrangian with $g$-valued curvatures $F_{\mu\nu}$,

$$-\frac{1}{2}g_{ab}F^{a}_{\mu}F^{b}_{\mu\nu}$$

(1.114)

gauge invaraince requires that the metric $g_{ab}$ be invariant under the co-adjoint action and if $g_{ab}$ is minus the Killing metric, then the group must be compact in order to have positive energy. In supergravity theories, scalars arise, on which which alllow $g_{ab}$ is allowed to depend. That is $g_{ab}$ becomes spacetime time-dependent. In that case it is possibile to arrage things to that the gauge group is non-compact while retaining positive energy.

We now illustrate the theory with some examples.

1.16.4 Example $SO(3)$ and $SU(2)$

These are clearly compact groups since $\mathbb{RP}^3 \equiv S^3/\mathbb{Z}_2$ is a compact manifold. In our conventions $C_t^j_k = -\delta_{jk}$. Thus $B_{ij} = \epsilon_{krs}\epsilon_{jrs} = -2\delta_{ij}$. It is clear that since $SO(3)$ acts in the usual way on its Lie algebra $\mathbb{R}^3$, any bi-invariant metric must be a multiple of $\delta_{ij}$. For example think of its double cover $SU(2)$ as $2 \times 2$ unimodular unitary matrices $U = (U^{-1})^t$, det $U = 1$. We may set

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad |a|^2 + |b|^2 = 1. \quad (1.115)$$

The left action is $U \rightarrow LU$, $L \in SU(2)$ The right action is $U \rightarrow UR$, $R \in SU(2)$. The metric

$$\det dU = |da|^2 + |db|^2$$

(1.116)

is invariant under $dU \rightarrow LdUR$, and hence bi-invariant.

1.16.5 Matrix Groups

In general, for a matrix group, the Killing metric is proportional to

$$\text{Tr} g^{-1}Dgg^{-1}dg = \lambda^a \otimes \lambda^b \text{Tr}(M_aM_b).$$

(1.117)

For orthogonal groups the infinitesial generators are skew symmetric and $M_a = -M_a^t$ the Killing form is negative definite and these groups are compact.

1.16.6 Example: $SL(2,R)$

As before

$$g = \begin{pmatrix} X^0 + X^1 & X^2 + X^4 \\ X^2 - X^4 & X^0 - X^1 \end{pmatrix}, \quad (X^0)^2 + (X^4)^2 - (X^1)^2 + (X^2) = 1. \quad (1.118)$$

A calculation analogous to that above gives the signature $++1$. The group has topology $S^1 \times \mathbb{R}^2$ with the compact direction corresponding rotations in the 1-2 plane.
1.16.7 Example: \( SO(2, 1) = SL(2, \mathbb{R}/\mathbb{Z}_2) \)

If \( SO(2, 1) \) acts on \( \mathbb{E}^{2,1} \) then

\[
\begin{align*}
\mathbf{e}_3 &= x \partial_y - y \partial_x \quad \text{generates rotations in } x - y \text{ plane,} \quad (1.119) \\
\mathbf{e}_1 &= t \partial_x + x \partial_t \quad \text{generates boosts in } t - x \text{ plane,} \quad (1.120) \\
\mathbf{e}_2 &= t \partial_y + y \partial_t \quad \text{generates boosts in } t - y \text{ plane.} \quad (1.121)
\end{align*}
\]

Thus

\[
\begin{align*}
[\mathbf{e}_1, \mathbf{e}_2] &= \mathbf{e}_3, \quad C_{\mathbf{1} \mathbf{2}}^2 = -1. \quad (1.122) \\
[\mathbf{e}_3, \mathbf{e}_2] &= \mathbf{e}_1, \quad C_{\mathbf{3} \mathbf{2}}^1 = -1. \quad (1.123) \\
[\mathbf{e}_3, \mathbf{e}_1] &= -\mathbf{e}_2, \quad C_{\mathbf{1} \mathbf{2}}^3 = +1. \quad (1.124)
\end{align*}
\]

Therefore the Killing form is diagonal with \( B_{33} = -2, B_{11} = B_{22} = +2 \). This compatible with the fact that the group \( SO(2) \) is compact and the group \( SO(1, 1) \) is non-compact since for example as matrices they are

\[
\begin{pmatrix}
\cos \psi & -\sin \psi \\
\sin \psi & \cos \psi
\end{pmatrix}, \quad 0 \leq \psi < \psi, \quad (1.125)
\]

but

\[
\begin{pmatrix}
\cosh \phi & \sinh \phi \\
\sinh \phi & \cosh \phi
\end{pmatrix}, \quad \infty < \phi < \infty. \quad (1.126)
\]

1.16.8 Example: Nil or the Heisenberg Group

This is upper-triangular matrices

\[
g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = 1 + xX + yY + zZ \quad (1.127)
\]

with

\[
X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.128)
\]

\[
Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.129)
\]

\[
Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.130)
\]

and only non-vanishing commutator being

\[
[X, Y] = Z. \quad (1.132)
\]
Thus $Z$ is in the centre $Z(\mathfrak{g})$ of the algebra, that is it commutes with all elements in $\mathfrak{g}$. The Killing form $B$ is easily seen to vanish. Now

\[
g^{-1} = \begin{pmatrix}
1 & -x & -z + xy \\
0 & 1 & -y \\
0 & 0 & 1
\end{pmatrix},
\]

Thus

\[
g^{-1} dg = X dx + Y dy = Z(dz - xdy).
\]

One finds that

\[
\lambda^x = dx, \quad L^x = \partial_x, \quad (1.135)
\]
\[
\lambda^y = dy, \quad L_y = \partial_y + x\partial_z \quad (1.136)
\]
\[
\lambda^z = dz - xdy, \quad L_z = \partial_z. \quad (1.137)
\]

The non-trivial Cartan-Maurer relations and Lie brackets are

\[
d\lambda^z = -\lambda^x \wedge \lambda^y, \quad [L_x, L_y] = L_z. \quad (1.138)
\]

Using the right-invariant basis one gets

\[
\rho^x = dx, \quad R^x = \partial_x + \partial_z, \quad (1.139)
\]
\[
\rho^y = dy, \quad R_y = \partial_y. \quad (1.140)
\]
\[
\rho^z = dz - ydx, \quad R_z = \partial_z. \quad (1.141)
\]

The non-trivial Cartan-Maurer relations and Lie brackets are now

\[
d\rho^z = +\rho^x \wedge \rho^y, \quad [R_x, R_y] = -R_z. \quad (1.142)
\]

An example of a left-invariant metric on Nil is

\[(dz - xdy)^2 + dx^2 + dy^2. \quad (1.143)\]

The Lie algebra nil is of course the same as Heisenberg algebra $[\hat{x}, \hat{p}] = i\frac{\hbar}{2\pi}\text{id}$.

### 1.16.9 Example: Two-dimensional Euclidean Group E(2)

This provides a manageable example which illustrates more complicated cases such as the four-dimensional Poincaré group. A convenient matrix representation is

\[
g = \begin{pmatrix}
\cos \psi & -\sin \psi & x \\
\sin \psi & \cos \psi & y \\
0 & 0 & 1
\end{pmatrix}.
\]

With $0 \leq \psi < 2\pi$, $-\infty < x < \infty$, $-\infty < y < \infty$. The group is not semi-simple, it is in fact a semi-direct product of $\mathbb{R}^2 \rtimes SO(2)$. The translations form an invariant abelian sub-group The Killing form is degenerate, vanishing on the translations.

The matrix can be thought of as acting on the column vector

\[
\begin{pmatrix}
X^1 \\
X^2 \\
1
\end{pmatrix}. \quad (1.145)
\]
The subspace

\[
\begin{pmatrix}
X_1 \\
X_2 \\
0
\end{pmatrix}
\] (1.146)

is invariant but the complementary subspace

\[
\begin{pmatrix}
0 \\
0 \\
X_3
\end{pmatrix}
\] (1.147)

is not. The representation, which exhibits \(E(2)\) as a subgroup of \(SL(3, \mathbb{R})\), is thus reducible but not fully reducible. Geometrically, this is a projective construction. Euclidean space \(E^2\) is identified with those directions through the origin of \(\mathbb{R}^3\) which intersect the plane \(X_3 = 1\). The full projective group of two dimensions is \(SL(2, \mathbb{R})\) which takes straight lines to straight lines.

### 1.17 Rigid bodies as geodesic motion with respect to a left-invariant metric on \(SO(3)\).

Consider a rigid body, a lamina constrained to slide on a plane \(E^2\). Every configuration may be obtained by acting with \(E(2)\) on some standard configuration in a unique fashion. Thus the configuration space \(Q\) may be regarded as \(E(2)\). A dynamical motion corresponds to a curve \(\gamma(t)\) in \(E(2)\). If there are no frictional forces the motion is free and will be described by geodesic motion with respect to the metric given by the kinetic energy

\[
T = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu,
\] (1.148)

where \(x^\mu\) are local coordinates on \(Q\). The symmetries of the situation dictate that the metric is left-invariant and this gives rise to conservation of momentum and angular momentum.

This example obviously generalizes to a rigid body moving in \(in \text{ vacuo}\) in \(E^3\) with \(E(2)\) replaced by \(E(3)\). The model would still apply if the rigid body moves in an incompressible frictionless fluid, as such has been studied by fluid dynamicists in the nineteenth century. Note that in this case, the kinetic energy gets renormalized compared with its value in vacuum, because one must take into account the kinetic energy of the fluid. There is also an obvious generalization to a rigid body moving in certain curved spaces such as hyperbolic spaces \(H^n = S(n, 1)/SO(n)\) or a spheres \(S^n = SO(n + 1)/SO(n)\) and this idea formed the basis of Helmholtz’s physical characterization of non-Euclidean geometry in terms of the axiom of the free-mobility of a rigid body which he thought of as some sort of measuring rod.

The simplest case to consider is when the body is moving in \(E^3\) but one point in the body is fixed. The configuration space \(Q\) now reduces to \(SO(3)\). Each point in the body has coordinates \(r(t)\) in an inertial or space-fixed coordinate system with origin the fixed point, and where \(r(t) = O(t)x\), where \(x\) co-moving coordinates fixed in the body with respect to some body-fixed frame and \(O(t)\) is a curve in \(SO(3)\) so that

\[
O^{-1}(t) = O^t(t),
\] (1.149)

that is, \(O(t)\) is an orthogonal matrix for each time \(t\). Now
Left actions correspond to rotations of with respect to the body fixed frame

$$\mathbf{r} \rightarrow L\mathbf{r}, \quad \text{i.e. } O \rightarrow LO, \quad L \in SO(3)$$  \hspace{1cm} (1.150)

Left actions correspond to rotations of with respect to the body fixed frame

$$\mathbf{x} \rightarrow R^{-1}\mathbf{x}, \quad \text{i.e. } O \rightarrow OR, \quad R \in SO(3).$$  \hspace{1cm} (1.151)

The kinetic energy should be invariant under rotations about inertial axes and this invariance gives rise to conserved angular momenta. Thus we anticipate that the kinetic energy $T$ is left-invariant but not in general right-invariant. If $\rho(\mathbf{x})$ is the mass density, $T$ is given by

$$T = \int_{\text{Body}} d^3 \mathbf{x} \frac{1}{2} \rho(\mathbf{x}) \dot{\mathbf{r}}^2$$  \hspace{1cm} (1.152)

$$= \int_{\text{Body}} d^3 \mathbf{x} \frac{1}{2} \rho(\mathbf{x}) \dot{\mathbf{r}}^t \dot{\mathbf{r}}$$  \hspace{1cm} (1.153)

$$= \int_{\text{Body}} d^3 \mathbf{x} \frac{1}{2} \rho(\mathbf{x}) (\dot{\mathbf{O}}\mathbf{x})^t \dot{\mathbf{O}}\mathbf{x}$$  \hspace{1cm} (1.154)

$$= \int_{\text{Body}} d^3 \mathbf{x} \frac{1}{2} \rho(\mathbf{x}) \mathbf{x}^t \dot{\mathbf{O}}^t \dot{\mathbf{O}} \mathbf{x}$$  \hspace{1cm} (1.155)

$$= \int_{\text{Body}} d^3 \mathbf{x} \frac{1}{2} \rho(\mathbf{x}) \mathbf{x} (O^{-1}\dot{O})^t (O^{-1}\dot{O}) \mathbf{x}$$  \hspace{1cm} (1.156)

$$= \int_{\text{Body}} d^3 \mathbf{x} \frac{1}{2} \rho(\mathbf{x}) \mathbf{x}^t \omega^i \omega^j \mathbf{x}$$  \hspace{1cm} (1.157)

where

$$\omega_{ij} = (O^{-1}\dot{O})_{ij}.$$  \hspace{1cm} (1.159)

In three dimensions we may dualize

$$\omega_{ik} = (O^{-1}\dot{O})_{ik} \epsilon_{ijk} \omega_j$$  \hspace{1cm} (1.160)

where $\omega_i$ are the instantaneous angular velocities with respect to a body fixed frame and find

$$T = \frac{1}{2} \omega_i I_{ij} \omega_j$$  \hspace{1cm} (1.161)

where

$$I_{ij} \in so(3)^* \otimes S so(3)^* = \int_{\text{Body}} d^3 \rho(\mathbf{x}) (\delta_{ij} \mathbf{x}^2 - x_i x_j) = I_{ji}$$  \hspace{1cm} (1.162)

is the (time-independent) inertia quadric of the body which serves as a left-invariant metric on $SO(3)$. Under right actions, i.e. under rotations of the body with respect to a space-fixed frame

$$I_{ij} \rightarrow R_{ik} I_{kl} R_{lj},$$  \hspace{1cm} (1.163)

where $R_{ik}$ are the components of an orthogonal matrix. We may use this freedom to diagonalize the inertia quadric

$$T = \frac{1}{2} I_x (\omega^x)^2 + \frac{1}{2} I_y (\omega^y)^2 + \frac{1}{2} I_z (\omega^z)^2$$  \hspace{1cm} (1.164)

where $I_x, I_y, I_z$ are called principal moments of inertia.
1.17.1 Euler angles

A geometrical argument shows that any triad may be taken into any other triad by an element of \(SO(3)\) of the form may be written as the product of a rotation through an angle \(\psi\) about the third axis followed by a rotation through an angle \(\theta\) about the new first axis followed by a rotation through an angle \(\phi\) about the new third axis

\[
O(\psi, \theta, \phi) = R_z(\psi)R_x(\theta)R_z(\phi)
\]

\[
= \begin{pmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

with \(0 \leq \phi < 2\pi\), \(0 \leq \theta < \pi\), \(0 \leq \psi < 2\pi\). Thus

\[
O^{-1}(\psi, \theta, \phi) = O(-\phi, -\theta, -\psi).
\]

A covering element of \(SU(2)\) is

\[
S = \exp \frac{\phi \tau_z}{2i} \exp \frac{\theta \tau_x}{2i} \exp \frac{\psi \tau_z}{2i},
\]

so that

\[
S^{-1}dS = \lambda^x \frac{\tau_x}{2i} + \lambda^y \frac{\tau_y}{2i} + \lambda^z \frac{\tau_z}{2i},
\]

with

\[
\lambda^x = \sin \theta \sin \phi d\psi + \cos \psi d\delta \theta,
\]

\[
\lambda^y = \sin \theta \cos \psi d\phi - \sin \phi d\theta,
\]

\[
\lambda^z = d\psi + \cos \theta d\phi,
\]

and

\[
L_x = \cos \psi \partial_\theta - \sin \psi \left( \cot \theta \partial_\psi - \frac{1}{\sin \theta} \partial_\phi \right)
\]

\[
L_y = -\sin \psi \partial_\theta - \cos \psi \left( \cot \theta \partial_\psi - \frac{1}{\sin \theta} \partial_\phi \right)
\]

\[
L_z = \partial_\psi.
\]

The right-invariant basis satisfies

\[
dSS^{-1} = \rho^x \frac{\tau_x}{2i} + \rho^y \frac{\tau_y}{2i} + \rho^z \frac{\tau_z}{2i},
\]

and may be obtained form the observation that inversion \((\phi, \theta, \phi) \rightarrow (-\phi, -\theta, -\psi)\) takes \(S^{-1}dS \rightarrow -dSS^{-1}\) and thus

with

\[
-\rho^x = -\sin \theta \sin \phi d\psi - \cos \phi d\delta \theta
\]

\[
-\rho^y = \sin \theta \cos \phi d\phi + \sin \phi d\theta
\]

\[
-\rho^z = -d\phi - \cos \theta d\psi.
\]
and

\[-R_x = -\cos \phi \partial_y + \sin \phi (\cot \theta \partial_\phi - \frac{1}{\sin \theta} \partial_\psi)\]  
\[-R_y = \sin \phi \partial_y - \cos \phi (\cot \theta \partial_\phi - \frac{1}{\sin \theta} \partial_\psi)\]  
\[-R_z = -\partial_\phi.\]  

We have

\[d\lambda^x = -\lambda^y \wedge \lambda^z, \quad [L_x L_y] = L_z \text{ etc} \]  
\[d\rho^x = \rho^y \wedge \rho^z, \quad [R_x, R_y] = -R_z \text{ etc}.\]  

\[\frac{\partial}{\partial \phi}\] generates left actions of \(SO(2)\) (rotations about the 3rd space-fixed axis)

\[\frac{\partial}{\partial \psi}\] generates right actions of \(SO(2)\) (rotations about the 3rd body-fixed axis)

The kinetic energy metric may be written as

\[ds^2 = I_x (\sin \theta \sin \psi d\phi + \cos \psi d\theta)^2 + I_y (\sin \theta \cos \psi d\phi - \sin \psi d\theta)^2 + I_z (\psi + \cos \theta d\phi)^2.\]  

\[\frac{\partial}{\partial \phi}\] is always a Killing vector corresponding to conservation of angular momentum about the third body fixed axis. In general, the full set of Killing vector fields is \(R_x, R_y, R_z\)

\[\mathcal{L}_{R_i} ds^2 = 0 \quad \text{etc}.\]  

\[\frac{\partial}{\partial \psi}\] is only a Killing vector if \(I_x = I_y = I\) in which case

\[ds^2 = I_z (d\psi + \cos \theta)^2 + I_1 (d\theta^2 + \sin \theta^2 d\phi^2).\]  

In mechanics this is Lagrange’s symmetric top. In geometry \(S^3 = SU(2)\) equipped with such a metric is such is sometimes called a Berger-Sphere. This requires our passing to the double cover by allowing the angle \(\psi\), which is an angle along the \(S^1\) fibres of the Hopf fibration, to run between 0 and \(\pi\). In general, if we take \(0 \leq k = \frac{4\pi}{k}\) we could take a quotient to get the lens spaces \(L(k, 1) \equiv S^3/C_k\), where \(C_k\) is the cyclic group of order \(k\). We shall later learn that these are the possible circle bundles over the 2-sphere \(S^2\), with \(k\) the Chern number. They correspond to all possible Dirac monopoles on \(S^2\) and the metric provides their Kaluza-Klein description of these monopoles.

1.17.2 \(SL(2, \mathbb{R})\) and the Goedel Universe

Using the ‘cheap and dirty’ trick of replacing \(\theta \rightarrow i\), \(\chi_{real}\) results in

\[\lambda^x \rightarrow i\lambda^x = \lambda^1, \quad \text{non-compact}\]  
\[\lambda^y \rightarrow i\lambda^y = \lambda^2, \quad \text{non-compact}\]  
\[\lambda^z \rightarrow i\lambda^z = \lambda^0 \quad \text{compact}.\]  

Up to an overall scale, the family of left-invariant Berger metrics on \(SU(2)\) with an additional \(SO(2)\) right action are given by

\[ds^2 = \lambda(d\psi + \cos \theta d\phi)^2 + (d\theta^2 + \sin^2 \theta d\phi^2).\]
become a family of metrics on $SL(2,\mathbb{R})$

\begin{align*}
    ds^2 &= \lambda (d\psi + \cosh \chi \phi)^2 - (d\chi^2 + \sinh^2 \chi d\phi^2) \quad (1.192) \\
    &= \lambda (dt + 2 \sinh^2(\frac{\chi}{2})d\phi)^2 - (d\chi^2 + \sinh^2 \chi d\phi^2) \quad (1.193)
\end{align*}

with $t = \psi + \phi$. If $\lambda = 1$ we get the standard metric on $Ad_3$. If $\lambda > 1$ we get a family of Lorentzian metrics. For a certain value of $\lambda$ we get a metric, which if multiplied by a line, gives Goedel’s homogeneous solution of the Einstein equations containing rotating dust. If we chose $-\infty < t < \infty$ the manifold is simply connected and in fact homeomorphic to $\mathbb{R}^3$. It may be thought as an $\mathbb{R}$ bundle over the two-dimensional hyperbolic plane $H^2$. To anticipate a later section, the bundle is trivial and $t$ provides a global section. However, if $\lambda > 1$, $t$ is a time function, i.e. a function on a Lorentzian manifold which increases along every timelike curve. We have

\begin{equation*}
g_{\phi\phi} = -\sinh^2 \chi + 4\lambda \sinh^2(\frac{\chi}{2}) \quad (1.194)
\end{equation*}

Now $\phi$ must be periodic period $2\pi$ in order to avoid a conical singularity at $\chi = 0$. Thus the curves $t = \text{constant}, \theta = \text{constant}$ are the orbits of an $SO(2)$ action and hence circles. But if $\lambda > 1$ they are timelike, $g_{\phi\phi} > 0$, for sufficiently large $\chi$. Thus the metric admits closed timelike curves or CTC’s which is incompatible with the existence of a time function.

### 1.17.3 Example: Minkowski Spacetime and Hermitian Matrices

We can set up a one-one correspondence between $E^{3,1} = (t, x, y, z)$ and two by two hermitian matrices $x = x^\dagger$ by

\begin{equation*}
x = \begin{pmatrix} t + z & x + iy \\ x - iy & t - z \end{pmatrix}, \quad \Rightarrow \det x = t^2 - x^2 - y^2 - z^2 \quad (1.195)
\end{equation*}

There is an action of $S \in SL(2, \mathbb{C})$ preserving hermiticity and the determinant

\begin{equation*}
x \rightarrow SxS^\dagger, \quad \Rightarrow \det x \rightarrow \det x \quad (1.196)
\end{equation*}

Thus we get a homomorphism $SL(2, \mathbb{C}) \rightarrow SO_0(3,1)$, the identity component of the Lorentz group. The kernel, i.e. the pre-image of $1 \in SL(2, \mathbb{C})$ is $\pm 11 = \mathbb{Z}_2$ and thus

\begin{equation*}
SO_0(3,1) = SL(2, \mathbb{C})/\mathbb{Z}_2 \quad (1.197)
\end{equation*}

Evidently $SL(2, \mathbb{C})$ acts faithfully on $\mathbb{C}^2$ preserving the simplectic form $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Elements of of $\mathbb{C}^2$ are Weyl spinors. One has $Spin(3,1) = SL(2, \mathbb{C})$. Specialization to $SU(2)$ yields $Spin(3) = SU(2)$, i.e. $SO(3) = SU(2)/\mathbb{Z}_2$.

Working over the reals

\begin{equation*}
x = \begin{pmatrix} t + z & x + y \\ x - y & t - z \end{pmatrix}, \quad \Rightarrow \det x = t^2 - x^2 + y^2 - z^2 \quad (1.198)
\end{equation*}

There is an action of $S \times S^\prime \in SL(2, \mathbb{R} \times SL(2, \mathbb{R})$ preserving the determinant

\begin{equation*}
x \rightarrow SxS^\prime, \quad \Rightarrow \det x \rightarrow \det x \quad (1.199)
\end{equation*}
Thus we get a homomorphism $SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \to SO_0(2, 2)$. The kernel, i.e. the pre-image of $1 \in SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ is $\pm(1, 1) = \mathbb{Z}_2$ and thus

$$SO_0(2, 2) = SL(2, \mathbb{R}) \times SL(2, \mathbb{R})/\mathbb{Z}_2.$$  \hfill (1.200)

We have $Spin(2, 2) = SL(2\mathbb{R}) \times SL(2, \mathbb{R})$ and setting $y = 0$ and $S = S'$ yields $Spin(2, 1) = SL(2, \mathbb{R})$, $SO_0(2, 1) = SL(2, \mathbb{R})/\mathbb{Z}_2$. Elements of $\mathbb{R}^2$ or $\mathbb{R}^2 \oplus \mathbb{R}^2$ are called Majorana spinors for $Spin(2, 1)$ or $Spin(2, 2)$ respectively. Elements of $0 \oplus \mathbb{R}^2$ or $\mathbb{R}^2 \oplus 0$ are called Majorana -Weyl spinors for $Spin(2, 2)$.

### 1.17.4 Example: $SO(4)$ and quaternions

If

$$i = \frac{\tau_1}{\sqrt{-1}}, \quad j = \frac{\tau_2}{\sqrt{-1}}, \quad k = \frac{\tau_3}{\sqrt{-1}},$$  \hfill (1.201)

then we have the algebra

$$ij + ji = jk + k j = ki + ik = 0, \quad (1.202)$$

$$ij - k = jk - i = ki - j = 0, \quad (1.203)$$

$$i^2 + 1 = j^2 + 1 = k^2 + 1 = 0. \quad (1.204)$$

It may be verified that the quaternion algebra $\mathbb{H}$ coincides with the real Clifford algebra, $\text{Cliff}(0, 2)$ of $\mathbb{E}^{0,2}$ generated by gamma matrices $\gamma_1, \gamma_2$ such that $\gamma_1^2 = \gamma_2^2 = -1$. One sets $\gamma_3 = \gamma_1 \gamma_2 = -\gamma_2 \gamma_1$ and identifies $(i, j, k)$ with $(\gamma_1, \gamma_2, \gamma_3)$.

A general quaternion $q \in \mathbb{H}$ and its conjugate $\bar{q}$ are given by

$$q = \tau + xi + yj + zk, \quad \bar{q} = \tau - xi - yj - zk. \quad (1.205)$$

Thus

$$q\bar{q} = \bar{q}q = |q|^2 = \tau^2 + x^2 + y^2 + z^2. \quad (1.206)$$

We may thus identify $\mathbb{H}$ with four-dimensional Euclidean space $\mathbb{E}^4$ and if

$$q \to lqr$$

with $|r| = |l| = 1$, we preserve the metric. We may identify unit quaternions with $SU(2) = Sp(1)$ and similar reasoning to that given earlier shows that $SO(4) = SU(2) \times SU(2))/\mathbb{Z}_2$, $Spin(4) = Sp(1) \times Sp(1)$, where $Sp(n)$ is the group of n times n quaternion valued matrices which are unitary with respect to the quaternion conjugate.

### 1.17.5 Example: Hopf-fibration and Toroidal Coordinates

The Hopf fibration of $S^3$ is the most basic example of a non-trivial fibre bundle. It is useful therefore to have a good intuitive understanding of its properties. One way to acquire this is by stereographic projection into ordinary flat 3-space.

We may think of $S^3$ as $(Z^1, Z^2) \in \mathbb{C}^2$ such that $|Z^1|^2 + |Z^2|^2 = 1$. We set

$$Z^1 = X^1 + iX^2 = e^{i\alpha} \tanh \sigma, \quad X^2 = X^4 + iX^3 = e^{i\beta} \frac{1}{\cosh \sigma}. \quad (1.208)$$

The surfaces $\sigma = \text{constant}$ or $|Z^1| = \sqrt{1 - |Z^2|^2} = \text{constant}$ are called Clifford tori and $\alpha$ and $\beta$ are coordinates on each torus. The Hopf fibration sends $(\alpha, \beta) \to (\alpha + c, \beta + c)$.
and the anti-Hopf fibration sends \((\alpha, \beta) \rightarrow (\alpha + c, \beta - c), c \in S^1\) and the fibres spiral around the torus. There are two degenerate tori, \(\sigma = 0, |Z^1| = 0\), and \(\sigma = \infty, |Z^2| = 0\), which collapse to circles. The induced round metric on \(S^3\) is

\[
\frac{d\sigma^2}{\cosh^2 \sigma} + \tanh^2 \sigma d\alpha^2 + \frac{1}{\cosh^2 \sigma} d\beta^2. \tag{1.209}
\]

This looks more symmetrical if one sets \(\sin \varpi = \tanh \sigma, 0 \leq \varpi \leq \frac{\pi}{2}\)

\[
ds^2 = d\varpi^2 + \sin^2 \omega d\alpha^2 + \cos^2 \varpi d\beta^2. \tag{1.210}
\]

The torus \(\varpi = \frac{\pi}{4}\) is square and a minimal surface. Stereographic projection may be achieved by setting

\[
Z^1 = \sin \chi \sin \theta e^{i\phi}, \quad Z^2 = \cos \chi + i \sin \chi \cos \theta e^{i\phi}, \tag{1.211}
\]

where \(\chi, \theta, \phi\) are polar coordinates on \(S^3\) in which the metric is

\[
ds^2 = d\chi^2 + \sin^2 \chi(d\theta^2 + \sin^2 \phi^2)
\]

\[
= \frac{4}{(1 + r^2)^2}(dr^2 + r^2(d\theta^2 + \sin^2 \phi^2)) \tag{1.212}
\]

with \(r = \tan \frac{\chi}{2}\).

If \((x, y, z) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, \cos \theta)\) are Cartesian coordinates for \(E^3\), on finds that

\[
x + iy = e^{i\alpha} \frac{\sin \sigma}{\cosh \sigma \cos \beta}, \quad z = \frac{\sin \beta}{\cos \sigma + \cos \beta}, \tag{1.214}
\]

and

\[
(\sqrt{x^2 + y^2} - \coth \sigma)^2 + z^2 = \frac{1}{\sinh^2 \sigma}. \tag{1.215}
\]

Thus we obtain the standard orthogonal coordinate system on \(E^3\) consisting of tori of revolution \(\sigma = \text{constant}\) obtained by rotating about the \(z\)-axis circles parameterized by the coordinate \(\beta\) lying in the planes \(\phi = \text{constant}\). Shifting \(\alpha\) corresponds to rotating about the \(z\)-axis. The coordinate \(\beta\). The two degenerate tori \(\sigma = 0\) and \(\sigma = \infty\) correspond to the \(z\)-axis and to a circle lying in the \(xy\) plane respectively. The flat metric turns out to be

\[
ds^2_{\text{flat}} = \frac{1}{(\cosh \sigma + \cos \beta)^2}(d\sigma^2 + d\beta^2 + \sin^2 \sigma d\alpha^2). \tag{1.216}
\]

The reader should verify that the linking number of any disjoint pairs of Hopf fibres is 1.

1.17.6 Example: Kaluza-Klein Theory

Consider the time-independent Schrödinger equation on on the Heisenberg group with left-invariant metric

\[
ds^2 = (dz - zdy)^2 + dx^2 + dy^2. \tag{1.217}
\]

One finds that

\[-\nabla^2 \phi = E\phi \Rightarrow \phi_{xx} + \phi_{zz} + (\partial_x x \partial_z z)^2 \phi = -E\phi. \tag{1.218}\]

If one separates variables \(\phi = \Psi(x, y)e^{i\varepsilon z}\) one gets

\[
\Psi_{xx} - \varepsilon^2 \Psi + (\partial_y + i\varepsilon)^2 \Psi = -E\Psi. \tag{1.219}
\]

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This is of the form
\[
(\partial_x - ieA_x)^2 \Psi + (\partial_y - ieA_y)^2 \Psi = -E\Psi
\]  \hspace{1cm} (1.220)
with
\[
A = -xdy \Rightarrow F = -dx \wedge dy . \hspace{1cm} (1.221)
\]
This is the Schrödinger equation for the Landau problem, i.e. that of a particle of charge \(e\) moving in the plane in a uniform magnetic field. If the coordinate \(z\) is periodic, \(0 \leq z \leq 2\pi L\) then the electric charge \(e\) is quantized \(eL \in \mathbb{Z}\).

A more general case works as follows. Let \(^2\) WE have ametric on \(M\)
\[
ds^2 = (\dot{z} + A_\mu \dot{x}^\mu)^2 + g_{\mu \nu} dx^\mu dx^\nu , \hspace{1cm} (1.222)
\]
with Killing vector field \(\frac{\partial}{\partial z}\) generating an \(S^1 = SO(2)\) isometry groups. The point particle Lagrangian on \(M\)
\[
(\dot{z} + A_\mu \dot{x}^\mu)^2 = g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu . \hspace{1cm} (1.223)
\]
The conserved charge is
\[
\dot{z} + A_\mu \dot{x}^\mu = c . \hspace{1cm} (1.224)
\]
The equation of motion for \(x^m\) turns out to be
\[
\frac{d^2 x^\sigma}{dt^2} + \{ \mu, \nu \} \dot{x}^\mu \dot{x}^\nu = cg^{\sigma \tau} F_{\tau \mu} \dot{x}^\mu \hspace{1cm} (1.225)
\]
with \(F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\). This is a charged geodesic motion in the quotient or base manifold \(B = M/SO(2)\).

As an example consider geodesics of the Berger spheres
\[
L = (\dot{\psi} + \cos \theta \dot{\phi})^2 + \sin \theta^2 \dot{\phi}^2 . \hspace{1cm} (1.226)
\]
We have \(A = \sin \theta d\phi \Rightarrow F = dA = -\sin \theta d\theta \wedge d\phi\). This is a magnetic monopole on the \(B = S^2 = SU(2)/SO(2)\). Electric charge quantization becomes angular momentum conservation in the extra dimension, a view point with a curiously echo of the ideas of Lord Kelvin and other nineteenth century physicists based on the Corioli force, that magnetic forces were essentially gyroscopic in origin.

The reader should verify that the orbits are all little circles on \(S^2\) with a size fixed by the magnitude of the charge, or equivalently the radius. One may repeat the problem on the Hyperbolic plane \(H^2 = PSL(2, \mathbb{R}) = SO_0(2, 1)\). If this is mapped into the unit disc in the complex plane, one finds that all orbits are circles. For weak magnetic field the circles intersect the circle at infinity. For large enough magnetic field, the orbits are closed and lie inside the unit disc. If we pass to a compact quotient \(\Sigma = H^2/\Gamma\), \(\Gamma\) a suitable discrete subgroup of \(PSL(2, \mathbb{R})\), then it is known that the geodesic motion on an energy shell is ergodic. The charged particle motion remains ergodic as long as the orbits on the covering space are closed circles.

\(^2\)Experts will realise that, for simplicity, we have set the ‘gravi-scalar’ to a constant value

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