Part I Special Relativity

G. W. Gibbons
D.A.M.T.P.,
Cambridge University,
Wilberforce Road,
Cambridge CB3 0WA,
U.K.

February 14, 2008

The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union will preserve an independent reality. H Minkowski (1908).

Contents

1 The Schedule

Read as follows:

INTRODUCTION TO SPECIAL RELATIVITY

8 lectures, Easter and Lent terms *[Lecturers should use the signature convention $(+---)$.]*

Space and time The principle of relativity. Relativity and simultaneity. The invariant interval. Lorentz transformations in $(1+1)$-dimensional spacetime. Time dilation and muon decay. Length contraction. The Minkowski metric for $(1+1)$-dimensional spacetime.[4]

4–vectors Lorentz transformations in $(3+1)$ dimensions. 4–vectors and Lorentz invariants. Proper time. 4–velocity and 4–momentum. Conservation of 4–momentum in radioactive decay.[4]

BOOKS

G.F.R. Ellis and R.M. Williams Flat and Curved Space-times Oxford University Press 2000 £24.95 paperback

W. Rindler Introduction to Special Relativity Oxford University Press 1991 £19.99 paperback

W. Rindler Relativity: special, general and cosmological OUP 2001 £24.95 paperback

E.F. Taylor and J.A. Wheeler Spacetime Physics: introduction to special relativity Freeman 1992 £29.99 paperback

1.1 Units

When quoting the values of physical quantities, units in which $c =$ and $\bar{h}$=1, will frequently be used. Thus, at times for example, distances may be expressed in terms of *light year*. Astronomers frequently use *parsecs* which is the distance at which is short for 'paralax second'. It is the distance at which the radius of the earth subtends one second of arc. One parsec works out to be 3.0×10^{13}Km or 3.3 light years. A frequently used unit of energy, momentum or mass is the *electron volt* or eV which is the work or energy required to move an electron through a potential difference of one Volt.

Physical units, masses and properties of elementary particles are tabulated by the Particle Data Group and may be looked up at

http://pdg.lbl.gov

.

Although not necessary in order to follow the course, it is a frequently illuminating and often amusing exercise to go back to the original sources. Many of the original papers quoted here may be consulted on line. For papers in the *Physical Review*, back to its inception in the late nineteenth century go to

http://prola.aps.org

.

For many others, including *Science* and *Philosophical Transactions of the Royal Society* (going back its beginning in the to seventeenth century) go to http://uk.jstor.org

.

2 Einstein's Theory of Special Relativity

is concerned with the motion of bodies or particles whose relative velocities are comparable with that of light

$$\boxed{c = 299,792,458 \quad \mathrm{ms}^{-1}.} \tag{1}$$

In a nutshell, Newton's Second Law remains unchanged in the form

$$\boxed{\frac{d\mathbf{p}}{dt} = \mathbf{F}} \tag{2}$$

where $\mathbf{F}$ the force acting on a particle of momentum $\mathbf{p}$ and mass m[1] , but while according to

$$\boxed{\mathrm{Newton's} \quad \mathrm{Theory} \qquad \mathbf{p} = m\mathbf{v}} \tag{3}$$

in

$$\boxed{\mathrm{Einstein's} \quad \mathrm{Theory} \qquad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}}.} \tag{4}$$

If this were all there is in it, relativity would, perhaps, not be especially interesting. What makes Relativity important is that it entails a radical revision of our elementary ideas of *space* and *time* and in doing so leads to the even more radical theory of *General Relativity* which comes into play when gravity is important. In this course we shall ignore gravity and confine our attention to Special Relativity. For matters gravitational the reader is directed to[34].

To see why relativity has such a profound impact on ideas about space and time, note that we are asserting that there actually is such a thing as *the* velocity of light.

For the benefit of those who have not studied Physics at A-level, or who did, but have now forgotten all they ever knew, the next section contains a review of the elementary physics of light.

3 *Early ideas about light*

Experiments with shadows and mirrors lead to the idea that light is a form of energy that propagates along straight lines called *light rays*. On reflection at a smooth surface $\mathcal{S}$ at rest, it is found that

[1]properly speaking 'rest-mass'

(i) The incident ray, the reflected ray and the normal to the surface at the point of reflection are co-planar

(ii) The incident and reflected rays make equal angles with the normal.

Hero of Alexander showed that these laws may be summarized by the statement that if A is a point on the incident ray, B on reflected ray and $\mathbf{x} \in \mathcal{S}$ the point at which the reflection takes place, then $\mathbf{x}$ is such that the distance

$$d(A, \mathbf{x}) + d(\mathbf{x}, B) \tag{5}$$

is extremized among all paths from AyB, $y \in \mathcal{S}$ to the surface and from the surface to B.

When light is refracted at a smooth surface $\mathcal{S}$ it is found that

(i) The incident ray, the refracted ray and the normal to the surface at the point of refraction are co-planar

(ii) The incident and refracted rays make angles θ_i and θ_r with the normal such that

$$\frac{\sin \theta_i}{\sin \theta_r} = \frac{n_r}{n_i} \tag{6}$$

where the quantities n_i and n_r are characteristic of the medium and may depend upon the colour of the light and are called its refractive index. By convention one sets $n = 1$ for the vacuum.

Pierre Fermat showed that these laws, first clearly enunciated in about 1621 by the Leyden mathematician Willebrod Snellius or Snell in work which was unpublished before his death in 1626, and later by Descartes, although probably known earlier to Thomas Harriot, may be summarized by the statement that if A is a point on the incident ray, B on refracted ray and $\mathbf{x}$ the point at which the refraction takes place, then $\mathbf{x}$ is such that the *optical distance*

$$n_i d(A, \mathbf{x}) + n_r d(\mathbf{x}, B) \tag{7}$$

is extremized among all paths AyB, $y \in \mathcal{S}$ from A to surface and from the surface to B. In other words the differential equations for light rays may be obtained by varying the action functional

$$\int n ds \tag{8}$$

where ds is the element Euclidean distance.

By the time of Galilei its was widely thought that light had a finite speed, c and attempts were made to measure it. Broadly speaking there were two views about the significance of this speed.

The Emission or Ballistic Theory held by Isaac Newton and his followers according to which light consisted of very small particles or corpuscles with mass m speed c and momentum $p = mc$, the speed varying depending upon the medium. On this theory, Snell's law is just conservation of momentum parallel to the surface.

$$p_i \sin\theta_i = p_r \sin\theta_r, \tag{9}$$

whence, assuming that the mass is independent of the medium

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{c_r}{c_i}. \tag{10}$$

The Wave Theory proposed by the dutch physicist Christian Huygens in 1678 , according to which light is a wave phenomenon having a speed c and such that each point on the *wave front* gives rise to a secondary spherical wave of radius ct whose forward envelope gives the wavefront at a time t later.

On this theory, Snell's law arises because the wavelength λ_i of the incident wave and the wavelength of the refracted wave λ_r differ. Applying Huygen's construction gives

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{\lambda_i}{\lambda_r}. \tag{11}$$

Since, for any wave of frequency f , $\lambda f = c$ and since the frequency of the wave does not change on refraction, we have according to the wave theory:

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{c_i}{c_r}. \tag{12}$$

The two theories gave the opposite prediction for the speed of light in a medium. Since refractive indices are never found to be less than unity, according to the emission theory the speed of light in a medium is always greater than in vacuo, while according to the wave theory it is always smaller than in vacuo. One way to distinguish between the two theories was to measure the speed of light in vacuo and in a medium. This was first done by Foucault in 1850, and more accurately by Michelson in 1883 using the rotating mirror method of the former, which will be described shortly. By interposing a tube filled with water in the path of the light, they showed that the speed of light in water was slower than in vacuo [2]. It follows that Hero and Fermat's variational properties may be summarized by the statement that the *time taken* for light to traverse the physical path is extremized.

Another way to distinguish the theories is by their ability to account for the *diffraction of light* by very small obstacles as observed by Grimaldi in 1665 or by experiments on slits, such as were performed by the polymath Thomas Young [3] in 1801. Following a large number of subsequent experiments, notably by Fresnel, by Foucault's time, some form of wave theory was accepted by almost

[2]The argument is in fact slightly indirect since these experiments actually measure the group velocity of light while refraction depends on the phase velocity. The distinction is described later. Given one, and information about the dispersion, i.e. how the refractive index varies with wavelength, one may calculate the other.

[3]Young played an equal role with Champillon in the translation of the Egyptian hieroglyphics on the Rosetta stone.

all physicists. In its simplest form, this postulated that in vacuo, some quantity satisfies the *scalar wave equation*

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \nabla^2\phi\,, \tag{13}$$

which, if c is constant, is easily seen to admit wavelike solutions of the form

$$\phi = A\sin\bigl(2\pi(\frac{x}{\lambda} - ft)\bigr), \tag{14}$$

or more generally

$$\phi = f(\mathbf{k} - \omega t), \tag{15}$$

where $f(\,)$ is an *arbitrary* C^2 function of its argument and $\frac{\omega}{|\mathbf{k}|} = c$. Since equation (13) is a linear equation, the *Principle of Superposition* holds and solutions with arbitrray profiles, moving in arbitrary directions may be superposed. A fact which not only explains many opetical phenomena but also led to the idea of *Fourier Analysis*.Note that solutions (13) are *non-dispersive*, the speed c is independent of the wavelength λ or frequency f.

Until Einstein's work, almost all physicists believed that wave propagation required some form of material medium and that light was no exception. The medium was called the luminiferous aether (or ether) and many remarkable properties were ascribed to it. Many physicist , incorrectly as it turned out, believed ed that it was inextricably linked with the nature of gravitation. Others, like Lord Kelvin, postulated that atoms could be thought of as knotted vortex rings. This seemed to require that the ether was some sort of fluid. A key question became: what is the speed of the earth relative to the aether?

The properties of the ether became even harder to understand when it was established that light could be polarized. This was first noticed by Huygens who was studying the refraction of light through a crystal of calcite also known as Iceland spar. In 1808 Malus discovered that light could be polarized by in the process of reflection.These observations led directly to the idea that light due to some sort of motion transverse to the direction of propagation, and so the quantity ϕ should be some sort of vector rather than a scalar. They also suggested to many that the aether should be some sort of solid.

The realization that light was an electromagnetic phenomenon and the great achievement of the Scottish physicist James Clerk Maxwell (1831-1879) in providing in 1873 a complete, unified and consistent set of equations to describe electromagnetism, which moreover predicted the existence of electromagnetic waves moving at the speed of light and the subsequent experimental verification by the German physicist Heinrich Hertz (1857-1894) around 1887 [4] did nothing to dispel the wide-spread confusion about the aether. Elaborate mechanical models of the aether were constructed and all the while, it and the earth's motion through it, eluded all experimental attempts at detection. The general frustration at this time is perhaps reflected in the words of the president of the

[4] In fact it seems clear that Hertz had been anticipated by the English Electrician D E Hughes in 1879, but the significance of his work was not appreciated until much later [49].

British Association, Lord Salisbury who is reported to have proposed, at one it its meeting held at Oxford, a definition of the aether as the nominative of the verb to undulate.

In a similar vein. discussing the various allegedly physical interpretations, Hertz declared that

> To the question 'What is Maxwell's Theory', I know of no shorter or more definite answer than the following: Maxwell's theory is Maxwell's system of equations. Every theory which leads to the same system of equations, and therefore comprises the same possible phenomena, I would consider a form of Maxwell's theory.

Maxwell's equations have many beautiful and remarkable properties, not the least important of which is invariance not under Galilei transformations as might have been expected if the aether theory was correct, but rather under what we now call Lorentz transformations. This fact was noticed for the scalar wave equation (13) in 1887, long before Einstein's paper of 1905, by Woldemar Voigt(1850-1919) and both Lorentz and Poincaré were aware of the Lorentz invariance of Maxwell's equations but they regarded this as a purely formal property of the equations. As we shall see, Einstein's insight was in effect to see that it is perhaps the single most important mathematical fact about the equations. From it flows all of Special Relativity and much of General Relativity.

3.1 Maxwell's equations

These split into two sets. The first set *always* holds, in vaccuo or in any material medium and independently of whether any electric charges or currents are present. They deny the existence of magnetic monopoles and asserts the validity of Michael Faraday's law of induction.

$$\boxed{\operatorname{div}\mathbf{B} = 0\,, \qquad \operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t\,.}} \tag{16}$$

The second set describe the response of the fields to the pressence of electric charges, charge density ρ and currents, current density $\mathbf{j}$. At the expense of introducing two additional fields they may also be cast in a form which is *always correct*. They assert the validity of Coulomb's law, and Ampére's law, provided it is supplemented by the last, crucial, additional term, called the *displacement current* due to Maxwell himself.

$$\boxed{\operatorname{div}\mathbf{D} = \rho \qquad \operatorname{curl}\mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}\,.} \tag{17}$$

It follows from the identity div curl = 0, that electric charge is conserved

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}\mathbf{j} = 0\,.} \tag{18}$$

In order to close the system one requires *constitutive relations* relating $\mathbf{D}$ and $\mathbf{H}$ to $\mathbf{E}$ and $\mathbf{B}$. In vacuo these are *linear relations*

$$\mathbf{D} = \epsilon_0 \mathbf{E}\,, \qquad \mathbf{H} = \frac{1}{\mu_0}\mathbf{B}\,, \tag{19}$$

where μ_0 and ϵ_0 are two universal physical constants constants called respectively the *permeability* and *permittivity* of free space .Thus, in vacuuo, Maxwell's equations are linear and the principle of superposition holds for their solutions. Thus, in vacuo

$$\boxed{\operatorname{div}\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \operatorname{curl}\frac{1}{\mu_0}\mathbf{B} = \mathbf{j} + \epsilon_0\frac{\partial \mathbf{E}}{\partial t}\,.} \tag{20}$$

If there are no charges or currents present, use of the identity curl curl $=$ grad div $- \nabla^2$ gives

$$\epsilon_0\mu_0\frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla^2\mathbf{E}\,, \tag{21}$$

$$\epsilon_0\mu_0\frac{\partial^2 \mathbf{B}}{\partial t^2} = \nabla^2\mathbf{B}\,. \tag{22}$$

Thus each component of the electric and magnetic field travels non-dispersiveley with velocity

$$\boxed{c = \frac{1}{\sqrt{\epsilon_0\mu_0}}.} \tag{23}$$

The divergence free conditions imply that solutions of the form $\mathbf{E} = \mathbf{E}_0 f(\mathbf{k} - \omega t)$, $\mathbf{B} = \mathbf{B}_0 f(\mathbf{k} - \omega t)$, are transversely (plane) polarized

$$\mathbf{k}.\mathbf{E} = 0\,, \qquad \mathbf{k}.\mathbf{B} = 0 \tag{24}$$

and moreover since

$$\mathbf{B}_0 = \mathbf{n} \times \mathbf{E}_0\,, \qquad \mathbf{E}_0 = \mathbf{B}_0 \times \mathbf{n}, \tag{25}$$

with $\mathbf{n} = \frac{\mathbf{k}}{|\mathbf{k}|}$, the vectors $(\mathbf{E}_0, \mathbf{B}_0, \mathbf{n})$ form a right handed normal but not orthonormal triad. Physically the direction of the polarization is usually taken to be that of the electric field, since this is easier to detect. Thus for any given propagation direction $\mathbf{n}$ there are two *orthogonal polarization states.* in the sense that one may choose the solutions such that $\mathbf{E}_1.\mathbf{E}_2. = 0$ and thus $\mathbf{B}_1.\mathbf{B}_2 = 0$.

4 *The Speed of Light*

4.1 *Roemer's measurement of c*

That light does indeed have a finite speed was first demonstrated, and the speed estimated by the Danish astronomer Olaus Roemer (1614-1710) in 1676 [1]. He observed the phases of Io, the innermost of the four larger satellites or moons of Jupiter (Io, Europa, Ganymede and Callisto in order outward) which had been discovered in 1610 by Galileo Galilei (1564-1642) using the newly invented telescope [5] and of Io's motion around Jupiter is about 1.77 days can be deduced by observing its phases, when it is eclipsed by Jupiter whose orbital period is 11.86 years. In 1688 G D Cassini had published a set of predictions for these but Roemer observed that they were inaccurate by about 15 minutes. The periods are shorter when Jupiter is moving toward the earth than when it is moving away from the earth. Roemer explained this and obtained a value for the speed of light by arguing that when Jupiter is moving toward the earth Io the time between eclipses is shorter than when Jupiter is moving away from us because in the former case light the total distance light has to travel is shorter than in the latter case. He obtained a value of 192,000 miles per second or roughly 310,000 Km per sec.

If we think anachronistically [6], we might say if we think of Io as a clock, its period is Doppler shifted.

4.2 *Fizeau's measurement of c*

The first accurate terrestrial measurement of the speed of light was by the French physicist Fizeau who, in 1849 [6], passed a beam of light through a rotating toothed wheel with 720 teeth, reflected it off a plane mirror 8.633 Km away and sent the light back toward the toothed disc. For a rotation speed of 12.6 turns per second the light was eclipsed giving a speed of about 315,000 Km per second.

4.3 *Foucault's rotating mirror*

In 1850 another French physicist, Foucault [7] reflected light off a mirror which was rotating about an axis parallel to its plane. The reflected light was then sent back in the same direction. If the rate of rotation of the mirror was chosen suitably the light arrived back at its point of departure. From this Foucault deduced a value for the velocity of light of $298,000\,\mathrm{Kms}^{-1}$. As mentioned above, he was also able to establish that the speed of light in water is less than in vacuo.

5 Absolute versus Relative motion

Newton based his theory on the assumption that space was uniform and described by the usual laws of Euclidean geometry. There then arose the issue of whether motion with respect to that background was observable. If it was, then one would have a notion of absolute as opposed to relative motion. Newton argued, using the idea of a suspended bucket of water, that one does have an

[5] The true inventor of the telescope is not known. It seems to have been known to the English cosmologist Thomas Digges and the Oxford mathematician and explorer of Virginia, Thomas Harriot(1560-1621

[6] The Doppler effect was proposed by the Austrian physicist C.J. Doppler in 1842

idea of absolute rotational motion. However according to his laws of motion there is no obvious dynamical way of detecting absolute translational motion. Since his laws imply that the centre of mass of an isolated system of bodies one could define an absolute frame of rest as that in which the centre of mass of for, example the visible stars, is at rest. One later suggestion was that one could take the centre of the Milky way. Lambert had suggested that it was the location of a 'dark regent 'or massive body, a suggestion also made by Mädler. Interestingly we now know that at the centre of the Milky Way there is a massive Black Hole of mass around a million times the mass of the sun, $10^6 M_\odot$. The mass of the Milky way is about $10^{12} M_\odot$. However any such centre of mass frame can only be determined by astronomical observations. It could not be found using purely dynamical experiments beneath cloudy skies here on earth.

Later, physicists, like Ernst Mach[50], began to worry about the logical foundations of Newton's laws. Exactly what was meant by the statement that a particle continues in a state of uniform motion if unaffected by an external force? Wasn't Newton's second law in effect a tautology? etc. L. Lange in 1885 [52] and others [50] had realized that an operational meaning could be given to Newton's laws if one introduces the idea of an *inertial frame of reference*. This Lange thought of as a coordinate system for $\mathbb{R}^3$ which could be determined by the free, mutually non-parallel, motion of three particles. Then the first law could be formulated as the non-trivial and empirically verifiable statement that any fourth free particle would move in this frame with uniform motion. In effect we are to use use the straight line motion of particles to build up what is sometimes called an *inertial coordinate system* or *inertial reference system*. In fact this construction closely resembles various constructions in projective geometry, especially if one adds in time as an extra coordinate. We will discuss this in more detail later.

Mach pointed out that even if one used astronomical observations to determine a fundamental inertial frame of reference which is at rest with respect to the *fixed stars*, that is stars so distant that their *proper motions* are negligible, this raises a puzzle. For example, in principle we can define a non-rotating frame in two different ways,

(i) Using gyroscopes for example which, if they are subject to no external torque will point in a constant direction in an inertial frame of reference, in other words using what has come to be called the *compass of inertia* .

(ii) Using the *fixed stars*. Nowadays we use *quasi stellar radio sources quasars* .

It is then a remarkable coincidence that, as we shall see in detail later, to very high accuracy these two definitions agree. Mach had some, not very specific, suggestions about a possible explanation. Mach's ideas strongly influenced those of Einstein, especially when he was formulating his General Theory of Relativity. They really cannot be pursued in detail without General Relativity and without some understanding of Relativistic Cosmology.

It was against this background that the question of the aether became so important. If it really existed, it would provide an alternative frame of reference, which might,or might not, coincide with the astronomically determined or dynamically determined frames of reference. It could, for example, remove the ambiguity about translational motion. The obvious guess was that it all three frames agreed. But if this was true, then the earth should be moving through the ether and this motion should be detectable.

6 Velocity composition formulae

Given that the speed of light is finite and presumably well defined on would ask, on the basis of Newtonian theory, in what frame? If the there is such a thing as *the* velocity of light, independent of reference frame then the standard velocity addition formula in

$$\boxed{\text{Newtonian} \quad \text{Theory} \qquad v \rightarrow v + u} \tag{26}$$

cannot be right. In fact, as we shall see later, one has a velocity composition (rather than *addition*) formula. In

$$\boxed{\text{Einstein}'\text{s} \quad \text{Theory} \qquad v \rightarrow \frac{v+u}{1+\frac{uv}{c^2}}} \tag{27}$$

so that if $v = c$,

$$v \rightarrow \frac{c+u}{1+\frac{cu}{c^2}} = c. \tag{28}$$

Exercise Show that if $u \le c$ and $v \le c$ then $\frac{u+v}{1+\frac{uv}{c^2}} \le c$.

Exercise Using the formula $\frac{dW}{dt} = \mathbf{F}.\mathbf{v}$, for the rate of doing work W by a force $\mathbf{F}$ acting on a point moving with velocity $\mathbf{v}$, show that the work done in accelerating a particle of mass m from rest to a final velocity $\mathbf{v}$ is $\frac{mc^2}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} - mc^2$.

The theory is called the Theory of Relativity because it deals with *relative velocities* and what is called the *Principle of Relativity.* This idea began, at least in modern times, with Galileo and we shall begin with his version of it.

7 Galilean Principle of Relativity

Suppose a boat is moving with uniform velocity along a canal and we are looking at it. We are asked the following

Question The lookout is in the crow's nest and drops a heavy weight onto the deck. Will it hit the captain below?

Answer Yes.

Reason We pass to a *frame of reference* $\tilde{S}$ moving with the boat. The frame at rest with respect to the canal is an *inertial frame of reference.* Galileo assumed that

The laws of dynamics are the same in all frames of reference which are in uniform motion with respect to an inertial frame of reference

Now if we drop something from rest in frame S it will fall vertically down,
Therefore if we drop something from rest in frame $\tilde{S}$ it will fall vertically down,

The boxed statement is Galilean Relativity follows in Isaac Newton's (1642- account of dynamics because

$$\text{In} \quad \text{frame} \qquad S \qquad m\frac{d^2\mathbf{x}}{dt^2} = \mathbf{F}(\mathbf{x}, t). \tag{29}$$

But to transform to frame $\tilde{S}$ we set

$$\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{u}t, \tag{30}$$

and hence

$$m\frac{d^2\tilde{\mathbf{x}}}{dt^2} = \mathbf{F}(\tilde{\mathbf{x}} + \mathbf{u}t) \qquad \text{in} \quad \text{frame} \quad \tilde{S}. \tag{31}$$

Note that Galileo assumed that the passengers in the boat would use the same coordinate t. In principle one might have thought that one would also have to change the time coordinate to a new coordinate $\tilde{t}$ for this equivalence to work out but both Galileo and Newton agreed that

$$\boxed{\text{Time is an absolute coordinate}} \tag{32}$$

that is, it takes the same value in all inertial frames of reference

$$\tilde{t} = t \tag{33}$$

Formulae (30,33) constitute a

$$\boxed{\text{Gallean} \quad \text{Transformation} \qquad \tilde{t} = t, \qquad \tilde{\mathbf{x}} = \mathbf{x} - \mathbf{u}t.} \tag{34}$$

We have just shown that Newton's equations of motion are invariant under Galilean Transformations. We shall now use Galilean transformations to deduce the **Non-relativistic Velocity Addition Formulae**.

If a particle moves with respect to a frame $\tilde{S}$ such that

$$\tilde{\mathbf{x}} = \tilde{\mathbf{v}}t + \tilde{\mathbf{x}}_0 \tag{35}$$

then

$$\mathbf{x} - \mathbf{u}t = \tilde{\mathbf{v}} + \tilde{\mathbf{x}}_0 \tag{36}$$

Thus

$$\mathbf{x} = (\mathbf{u} + \tilde{\mathbf{v}})t + \tilde{\mathbf{x}}_0. \tag{37}$$

and hence

$$\boxed{\mathbf{v} = \mathbf{u} = \tilde{\mathbf{v}},} \tag{38}$$

gives the velocity with respect to S. Later we will imitate this simple calculation to obtain the velocity addition formula in special relativity.

7.1 Waves and Galilean Transformations

If, in a frame S at rest with respect to the aether, we have a wave of the form

$$\phi = \sin\bigl(\mathbf{k}.\mathbf{x} - \omega t\bigr) \tag{39}$$

Its speed is $c = \frac{\omega}{|\mathbf{k}|}$, its wavelength $\lambda = \frac{2\pi}{|\mathbf{k}|}$ and frequency $f = \frac{\omega}{2\pi}$.

If we submit it to a Galilei transformation it becomes

$$\phi = \sin\bigl(\mathbf{k}.\tilde{\mathbf{x}} - (\omega - \mathbf{u}.\mathbf{k})\mathbf{t}\bigr). \tag{40}$$

In the frame $\tilde{S}$, the wave has the same wavelength but the frequency $\tilde{f} = \tilde{\omega}$ is changed

$$\tilde{f} = f(1 - \frac{u}{c}\cos\theta) \tag{41}$$

and the speed is $\tilde{c}$, where

$$\tilde{c} = c - u\cos\theta. \tag{42}$$

The formula for velocity in the moving frame $\tilde{S}$ is very much what one expects on the basis of a particle viewpoint but note that the angle θ is the angle between the direction of the wave $\mathbf{n} = \frac{\mathbf{k}}{|\mathbf{k}|}$ and the relative velocity $\mathbf{u}$ of the two frames S and $\tilde{S}$. Both frames agree on this as do they on the direction of motion of the wave. *In other words, Galilei's transformation formulae predict that there is no aberration.*

Later, we will obtain the physically correct results using the same method as above, but instead of a Galilei transformation we shall substitute using a Lorentz transformation.

8 Spacetime

Before proceeding, we will pause to develop a way of thinking about kinematics that in fact goes back to Lagrange and D'Alembert. The latter wrote, in his article on dimension in the *Encylopédie ou Dictionaire raisonée des sciences, des arts et des metiers* in 1764

> A clever acquaintance of mine believes that it is possible to think of time as a fourth dimension, so that the product of time and solidity would in some sense be the product of four dimensions; it seems to me that this idea, while debatable, has certain merits-at least the merit of novelty.

The German pioneer of psycho-physics Gustav Theodor Fechner (1801-1887) wrote a popular article entitled 'Der Raum hat vier Dimension'which discusses related ideas. By that time the study of extra spatial dimensions was quiet advanced and the German Astronomer Johann C F Zollner (1834-1882) gained notoriety for claiming that the alleged ability of self-claimed spiritualists to untie knots sealed in glass jars was only explicable if they had been moved into a fourth spatial dimension.

By 1880's the French railway engineer Ibry was using spacetime diagrams in a practical way to construct railway time tables (see illustration on p 55 of [34])

The following examples illustrate the power of the view point in solving this type of mundane problem.

8.1 Example: uniform motion in 1+1 dimensions

A commuter is usually picked up by his/her spouse who drives at constant speed from their house to meet the commuter at 5 o'clock. One day the commuter arrives on an earlier train at 4 o'clock and decides to walk. After a while the commuter is picked up by his/her spouse who has driven to meet him as usual. They arrive back at their house 10 minutes earlier than usual. For how long did the commuter walk?

8.2 Example: uniform motion in 2+1 dimensions

Four ships, A, B, C, D are sailing in a fog with constant and different speeds and constant and different courses. The five pairs A and B, B and C, C and A, B and D, C and D have each had near collisions; call them 'collisions'. Show that A and D necessarily 'collide'.

Hint Consider the triangle in the three-dimensional spacetime diagram formed by the world-lines of A, B and C.

8.3 Example: non-uniform motion in 1+1 dimensions

A mountain hiker sets off at 8.00 am one morning and walks up to a hut where he/she stays the night. Next morning he/she sets of at 8.00 am and walks back down the same track. Show that, independently of how fast or slowly he/she walks there is at least one time on the two days when he/she is at the same point on the track.

For an interesting history of ideas of the fourth dimension before Einstein in art and popular culture ,including H G Wells's ideas about time travel, one may consult the interesting book [23].

9 Minkowski's Spacetime viewpoint

In what follows we shall initially be concerned with the simplified situation in which all motion is restricted to one space dimensions. Thus the position vectors $\mathbf{x}$ have just one component. In this case, it is convenient to adopt a graphical representation, we draw a *spacetime diagram* consisting of points we call *events* with spacetime coordinates $(t, \mathbf{x})$. The two-dimensional space with these coordinates is called *spacetime*.

Passing to another frame of reference is like using *oblique coordinates in spacetime*. However, according to Galilei, all 'observers'use the same time coordinate. Geometrically while the lines of constant x have different slopes in different frames, the lines of constant time are all parallel to each other. This means that two events $(t_1, \mathbf{x}_1)$ and $(t_2, \mathbf{x}_2)$ which are simultaneous in frame S must be simultaneous in frame $\tilde{S}$, that is

$$(\tilde{t}_1, x_1) = (t_{1,1} - \mathbf{ut_1}), \tag{43}$$

$$(\tilde{t}_2, \tilde{\mathbf{x}}_2) = (t_1, \mathbf{x}_2 - \mathbf{ut_2}) \tag{44}$$

thus

$$t_1 - t_2 = 0 \Leftrightarrow \tilde{t}_1 - t_2 = 0. \tag{45}$$

We say that in Newtonian theory *simultaneity is absolute*, that frame independent.

Let's **summarize**

(i) The Laws of Newtonian dynamics are invariant under

$$\boxed{\text{Galilei} \quad \text{transformations} \qquad \tilde{t} = t, \qquad \tilde{\mathbf{x}} = \mathbf{x} - \mathbf{u}t.} \tag{46}$$

(ii)

$$\boxed{\text{velocities} \quad \text{add} \qquad \mathbf{v} = \tilde{\mathbf{v}} + \mathbf{u}.} \tag{47}$$

(iii) Time is absolute.

(iv) Simultaneity is absolute.

10 Einstein's Principle of Relativity

We have discovered that no purely dynamical experiment can determine our absolute velocity. If we are in a closed railway carriage moving *uniformly* we cannot tell, by dropping particles etc, how fast we are traveling.

The natural question to ask is whether we can tell using experiments involving light. If this has speed c relative to some privileged inertial frame S , (identified before Einstein with the mysterious 'Aether'or 'Ether'), it should, according to Galileo, have speed $c-u$ relative to a frame $\tilde{S}$ moving with respect to the aether. By measuring this speed it should be possible to determine u. This was tried in the

10.1 Michelson-Morley Experiment

This is described clearly and in detail in Michelson's own words in [2]. Therefore the present description will be brief. The light travel times $T_\perp$ and $T_\parallel$ of light moving in directions restively perpendicular and parallel to the motion along arms of an interferometer of lengths $L_\perp$ and $L_\|$ are measured. It was argued that in the parallel direction (working in frame $\tilde{S}$)

$$T_\| = L_\| \left[\frac{}{c-u} + \frac{1}{c-u} \right] \Rightarrow T_\| = \frac{2L_\|}{c} \frac{1}{1-\frac{u^2}{c^2}}. \tag{48}$$

On the other hand (working in frame S) it was argued that the total distance the perpendicularly moving light has to travel is, by Pythagoras,

$$\sqrt{L_\perp^2 + (\frac{uT_\perp}{2c})^2} = \frac{2T_\perp}{c} \Rightarrow T_\perp = \frac{2L_\perp}{c\sqrt{1-\frac{u^2}{c^2}}} \tag{49}$$

Thus, for example, if $L_\perp = L_|$ and $T_\perp \neq T_|$ we should be able to measure u. However in 1887 the experiment carried out by the American Physicist MIchelson and Morley [3] revealed that $T_\perp = T_\|$!

Einstein drew the conclusion that *no* experiment, including those using light, should allow one to measure one's absolute velocity, that is he assumed.

The Invariance of the Speed of Light *The velocity of light is the same in all frames of reference which are in uniform motion with respect to an inertial frame.*

In Einstein's own words

> the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. We will raise this conjecture (the purport of which will hereafter be called the "Principle of Relativity") to the status of a postulate and also introduce another postulate, which is only apparently irreconcilable with the former, namely that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body.

If Einstein is correct, then Galilei's transformations cannot be correct. We need a new transformations called *Lorentz Transformations*. They turn out to be (proof shortly)

$$\boxed{\text{Lorentz Transformations} \qquad \tilde{x} = \frac{x - ut}{\sqrt{1 - \frac{u^2}{c^2}}}, \qquad \tilde{t} = \frac{t - \frac{u}{c^2}x}{\sqrt{1 - \frac{u^2}{c^2}}}.} \tag{50}$$

Note that

(i) the time t gets transformed to $\tilde{t}$ as well as x to $\tilde{x}$.

(ii) Simultaneity is no longer absolute

$$\tilde{t}_1 - \tilde{t}_2 = \frac{t_1 - t_2}{\sqrt{1 - \frac{u^2}{c^2}}} - \frac{u}{c^2}(x_1 - x_2) \tag{51}$$

and hence

$$t_1 - t_2 \not\Rightarrow \tilde{t}_1 - \tilde{t}_2, \qquad \text{if} \quad x_1 \neq x_2. \tag{52}$$

(iii) If we take the *non-relativistic limit* $c \to \infty$ in which the speed of light is infinite we Lorentz transformations (50) we recover the Galilei transformations (34).

10.2 Derivation of the Lorentz Transformation formulae

We assume

(i) $(c\tilde{t}, \tilde{x})$ are linear functions of (ct, x)

(ii) $c^2t^2 - x^2 = c^2\tilde{t}^2 - \tilde{x}^2$ and hence the speed of light is invariant because $x = ct \Rightarrow \tilde{x} = \pm c\tilde{t}$.

In this first look at the subject we assume (ii) but in more sophisticated treatments one makes considerably weaker assumptions. A precise statement will be made later. Even at this point it should be clear that we are ignoring trivial *dilations* or *homotheties* $\tilde{x} = \lambda x, \quad \tilde{t} = \lambda t$, for $\lambda \neq 0$ which obviously leave the speed of light invariant. However we do not usually include these in the set of Galilei transformations. We shall also treat space and time translations

$t \to t + t_0$, $x \to x + x_0$ as trivial Thus it is sufficient to consider light rays through the origin of spacetime $(t, x) = (0, 0)$ We shall also regard as trivial *space reversal* $\tilde{x} = -x$, $t = \tilde{t}$ and *time reversal* $\tilde{x} = x$, $\tilde{t} = -t$.

Clearly (50) satisfy (i) and (ii). The converse is obtained by setting

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}, \tag{53}$$

with $A > 0$, $D > 0$ because we are excluding time reversal and space reversal. Substitution gives

$$(Act - B)^2 - (Cct - Dx)^2 - c^2t^2 - x^2 = 0. \tag{54}$$

Thus equating coefficients of t^2 and x^2 to zero, we get

$$(i) \quad A^2 - C^2 = 1 \quad \Rightarrow \quad A = \cosh\theta_1, \quad C = \sinh\theta_1 \tag{55}$$

$$(ii) \quad D^2 - B^2 = 1 \quad \Rightarrow \quad D = \cosh\theta_2, \quad B = \sinh\theta_2 \tag{56}$$

For some θ_1 and θ_2. Now equating the coefficient of xt to zero gives

$$(iii) \quad AB = CD \quad \Rightarrow \quad \cosh\theta_1 \sinh\theta_2 = \cosh\theta_2 \sinh\theta_1 \quad \Rightarrow \quad \theta_1 = \theta_2. \tag{57}$$

Thus

$$\boxed{\begin{pmatrix} c\tilde{t} \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}.} \tag{58}$$

Setting $\tilde{x} = 0$ allows us to see that the origin of the $\tilde{S}$ frame satisfies $x\cosh\theta = ct\sinh\theta$. But if this is to agree with $x = ut$, where u is the relative velocity, we must have

$$\boxed{\frac{u}{c} = \tanh\theta := \beta,} \tag{59}$$

where θ is called the *rapidity*. It follows that

$$\cosh\theta = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} := \gamma \tag{60}$$

and

$$\sinh\theta = \frac{\frac{u}{c}}{\sqrt{1 - \frac{u^2}{c^2}}} = \beta\gamma. \tag{61}$$

The quantities β and γ do not, as far as I am aware, have individual names, and perhaps for that reason γ is often, rather inelegantly, referred to as the *relativistic gamma factor*. The use of the symbols β and γ is both traditional and *almost* universal in the subject. A Lorentz transformation of the form (50) is often called a *boost* which is analogous to a an ordinary rotation. The analogue of the verb rotating is, unsurprisingly, *boosting*. A useful relation, particularly in Tripos questions, is

$$\gamma^2(1 - \beta^2) = 1. \tag{62}$$

10.3 Relativistic velocity composition law

Of course the point is that velocities don't add. Suppose that

$$\text{In frame } \tilde{S} \qquad \tilde{x} = \tilde{v}\tilde{t} - \tilde{x}_0, \tag{63}$$

then using the lorentz transformations (50) we have that

$$\text{In frame } S \qquad \frac{x - uv}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{\tilde{v}(t - \frac{ux}{c^2})}{\sqrt{1 - \frac{u^2}{c^2}}} + \tilde{x}_0. \tag{64}$$

Thus

$$x(1 + \frac{uv}{c^2}) = (\tilde{v} + u)c + t + \tilde{x}_0\sqrt{1 - \frac{u^2}{c^2}}, \tag{65}$$

and hence

$$\boxed{\text{Relativistic velocity composition law} \qquad \tilde{v} = \frac{u + \tilde{v}}{1 + \frac{u\tilde{v}}{c^2}}.} \tag{66}$$

Thus, for example, $\tilde{v} = c \Rightarrow v = c$, which is the invariance of the speed of light.

10.4 *Observational for Einstein's second postulate*

This is that the velocity of light is independent of the velocity of it's source. Many high precision experiments give indirect evidence for it's validity. In addition, *direct* observational support for this includes

(i) The light curves of binary stars. De-Sitter [9] pointed out that if, for example, two stars are in orbit around each other with orbital period T, then if light coming from that portion of the orbit when the star is moving toward us had a larger speed than when it was moving away from us, then light from an earlier part of the motion might even arrive more than half an orbital period *before* light coming from the intermediate portion of the orbit when it is neither moving toward us or away from us. This would lead to significant distortion of the plot of luminosity or of velocity against time.

Consider, for example, the case when we are in the plane a circular orbit of radius R and period P whose centre is a large distance L from us. The relation between time of emission t_e and time of observation t_o expected on the basis of Newtonian theory is, since $R << L$,

$$t_o = t_e + \frac{L - R\sin\frac{2\pi t_e}{P}}{c + v\cos\frac{2\pi t_e}{P}}. \tag{67}$$

In this formula, the quantity v is the extra velocity supposed to be imparted to the photons moving toward us. According to the *Ballistic theory of light* of Newton and Galilei we would apply the usual rules for particles of speed c in the rest frame of the moving stars. Thus we expect $v = v_{\rm orbital} = \frac{2\pi R}{P}$, but in general it could be much smaller. Since $L >> R$ and $c >> v$ we have

$$t_o \approx t_e + \frac{L}{c} - \frac{R}{c}\sin\frac{2\pi t_e}{P} + \frac{Lv}{c^2}\sin\frac{2\pi t_e}{P}\,. \tag{68}$$

The observed Doppler shift is given by

$$\frac{dt_0}{dt_e} = 1 - \frac{v}{c}\cos\frac{2\pi t_e}{P} + \frac{Lv2\pi}{c^2 P}\sin\frac{2\pi t_e}{P}\,. \tag{69}$$

If unless $\frac{Lv2\pi}{Pc^2}$ is small there will be significant distortion of the light curves. Indeed t_o may not be a monotonic function of t_e, in which case, t_e will not be a unique function of t_o. In other words, pulses from different phases of the orbit may arrive on earth at the same time t_o. Such effects have not be seen.

De Sitter himself considered the binary star β-Aurigae.

One example sometimes quoted is, the binary star Castor C. It is 45 light years away and has a period of .8 days. The stars have $v_{\rm orbital} = 130{\rm Kms}^{-1}$. The effect should be very large, but the light curves of the two stars are quite normal [11]. Using pulsating X-ray sources in binary star systems, Brecher [24] was able to conclude that Einstein's second postulate was true to better than 2 parts in a thousand million

$$\frac{v}{v_{\rm orbital}} < 2 \times 10^{-9}. \tag{70}$$

This is certainly an improvement on Zurhellen, who in (1914) obtained a limit of 10^{-6} using ordinary binary stars [29].

(ii) The time of travel over equal distances of gamma rays emitted by a rapidly moving positron annihilating with a stationary electron can be measured as they are found to be equal [12].

(iii) A similar measurement can be done using the decay of a rapidly moving neutral pion which decays into two gamma rays [14].

10.5 Light in a medium: Fresnel Dragging

In a medium, the velocity of light is reduced to $\frac{c}{n}$, where $n \geq 1$ is called the *refractive index*. In general n may depend upon wavelength λ but here we will neglect that effect. Fresnel proposed, in the 1820's, measuring the speed of light in a stream of water moving with speed u relative to the laboratory. Naive Newtonian theory would give a speed

$$\frac{c}{n} + u \tag{71}$$

but experiments by Fizeau in 1851 using the toothed wheel method did not agree with this. If we use the relativistic addition formula in the case that $\frac{u}{c}$ is small we get instead

$$\frac{c}{n} + u(1 - \frac{1}{n^2}) + \dots \tag{72}$$

which *does* agree with Fizeau's experiments. The factor $(1-\frac{1}{n^2})$ is called *Fresnel's dragging coefficient* and had in fact been proposed earlier by the French physicist Fresnel around 1822 using an argument based on wave theory. The experiment was repeated after Einstein had proposed his theory by the 1904? Nobel prize winning Dutchman Zeeman ( ).

10.6 Composition of Lorentz Transformations

We could just multiply the matrices but there is a **useful trick**. We define

$$x_\pm = x \pm ct. \tag{73}$$

Thus

$$x_- = 0 \Rightarrow \text{ we have a right moving light ray } \quad \longrightarrow \tag{74}$$

$$x_+ = 0 \Rightarrow \text{ we have a left moving light ray } \quad \longleftarrow \tag{75}$$

Now Lorentz transformations (50) take the form

$$\tilde{x}_+ = e^{-\theta} x_+ \qquad \Rightarrow \qquad x_+ = e^{+\theta}\tilde{x}_+, \tag{76}$$

$$\tilde{x}_- = e^{+\theta} x_+ \qquad \Rightarrow \qquad x_- = e^{-\theta}\tilde{x}_+. \tag{77}$$

We immediately deduce that the inverse Lorentz transformation is given by setting $u \to -u$, $\Leftrightarrow$ $\theta \to -\theta$,i.e. the inverse of (50) is

$$\boxed{\text{Inverse Lorentz Transformations} \qquad x = \frac{\tilde{x} + u\tilde{t}}{\sqrt{1-\frac{u^2}{c^2}}}, \qquad t = \frac{\tilde{t} + \frac{u}{c^2}\tilde{x}}{\sqrt{1-\frac{u^2}{c^2}}}.} \tag{78}$$

Now consider three frames of reference S, $\tilde{S}$ and $\tilde{\tilde{S}}$ such that we get from S to $\tilde{S}$ by boosting with velocity u_1 and from $\tilde{S}$ to $\tilde{\tilde{S}}$ by boosting with relative velocity u_2. To get from S to $\tilde{\tilde{S}}$ we have to boost with relative velocity u_3. If θ_1, θ_2, θ_3 are the associated rapidities, we have

$$\tilde{x}_\pm = e^{\mp\theta_1} x_\pm, \tag{79}$$

$$\tilde{\tilde{x}}_\pm = e^{\mp\theta_2}\tilde{t}x_\pm. \tag{80}$$

Thus

$$\tilde{\tilde{x}}_\pm = e^{\mp\theta_3} x_\pm, \tag{81}$$

i.e.

$$\boxed{\text{rapidities add} \qquad \theta_3 = \theta_1 + \theta_2.} \tag{82}$$

Using a standard addition formula for hyperbolic functions

$$\tanh\theta_3 = \frac{\tanh\theta_1 + \tanh\theta_2}{1 + \tanh\theta_1 \tanh\theta_2}. \tag{83}$$

That is we re-obtain the velocity composition formula:

$$\frac{u_3}{c} = \frac{\frac{u_1}{c} + \frac{u_2}{c}}{1 + \frac{u_1 u_2}{c^2}}. \tag{84}$$

10.7 Velocity of light as an upper bound

Suppose that $|u_1| < c$ and $|u_2| < c$, then $|u_3| < c$.

Proof Since the hyperbolic tangent function is a one to one map of the real line onto the open interval $(-1, +1)$, we have

$$|u_1| < c \Rightarrow -\infty < \theta_1 < \infty, \tag{85}$$

$$|u_2| < c \Rightarrow -\infty < \theta_2 < \infty. \tag{86}$$

Thus

$$-\infty < \theta_1 + \theta - 2 < \infty \Rightarrow |u_3| < c. \tag{87}$$

Thus no matter how we try, we cannot exceed the velocity of light.

10.8 *'Super-Luminal'Radio sources*

An interesting apparent case of super-luminal velocities but which is perfectly explicable without invoking the existence of anything moving faster than light, has been discovered by radio astronomers. What are called *quasars* or quasi-stellar radio sources exhibit jets of matter symmetrically expelled from a dense central region probably associated with a black hole. For the sake of a simple first look we assume that we can use the geometry of Minkowski spacetime despite the great distances and that the central quasar is located a distance L away from us. We shall also assume that there is a frame in which both the central quasar and ourselves are at rest

We assume, in the simplest case possible, that the matter in the jets are expelled at right angles to our line of sight at time $t = 0$ and therefore at time $t = t_o$ the material in the jets has have travelled a distance vt_e. Light or radio waves coming from the jets will arrive here at time

$$t_o = t_e + \frac{1}{c}\sqrt{L^2 + v^2 t_e^2}\,. \tag{88}$$

The angle α subtended is, for small angles

$$\alpha = \frac{vt_e}{L}. \tag{89}$$

The rate of change with respect to the observation time is

$$\frac{d\alpha}{dt_0} = \frac{v}{L}\frac{dt_e}{dt_o}. \tag{90}$$

We have

$$t_e = \frac{1}{1 - \frac{v^2}{c^2}}\Big(t_o + \sqrt{t_o^2 + (\frac{L^2}{c^2} + t_0^2)(1 - \frac{v^2}{c^2}}\Big). \tag{91}$$

Thus

$$\frac{dt_e}{dt_o} = \frac{1}{1 - \frac{v^2}{c^2}}\Big(1 + \frac{c}{v}\frac{t_o}{\sqrt{t_o^2 + L^2(\frac{1}{v^2} - 1)}}.\Big) \tag{92}$$

For large t_o we get

$$\dot{\alpha} = \frac{v}{L}\frac{1}{1-\frac{v}{c}} \tag{93}$$

Clearly if v is close to c, then $\frac{dt_e}{dt_o}$ can be much bigger than unity. Thus the size of the effect is much larger than one's naive Newtonian expectations. If the jet makes an angle with the line of sight we obtain

$$\dot{\alpha} = \frac{v\sin\theta}{L}\frac{1}{1-\frac{v\cos\theta}{c}} \tag{94}$$

The existence of such apparent superluminal motions was suggested by the present Astronomer Royal in 1966 while a research student in DAMTP[22]. Just over 4 years later, in 1971, the radio astronomers Irwin Shapiro amd Marshall Cohen and Kenneth Kellerman astronomers found, using very long base line interferometry (VLBI) such jets, changing in apparent size over a period of months, in the quasars 3C273 and 3C279. Nowadays the observation of such apparently super-luminal sources is commonplace.

10.9 The two-dimensional Lorentz and Poincaré groups

Clearly Lorentz transformations, i.e. boosts in one space and one time dimension, satisfy the axioms for an abelian group (closure under composition, associativity and existence of an inverse) which is isomorphic to the positive reals under multiplication (one multiplies e^{θ}) or all the reals under addition (one adds θ). This is completely analogous to the group of rotations, $SO(2)$ in two spatial dimensions. The standard notation for the group of boosts is $SO(1,1)$.

If we add in the abelian group of time and space translation translations

$$t \rightarrow t + t_0\,, \qquad x \rightarrow x + a, \tag{95}$$

we get the analogue of the Euclidean group plane, $E(2)$ which is called the *Poincaré group* and which may be denoted $E(1,1)$.

11 The invariant interval

Consider two spacetime events (ct_1, x_1) and (ct_2, x_2) in spacetime. The *invariant interval* τ between them is defined by

$$\boxed{\tau^2 = (t_1 - t_2)^2 - \frac{(x_1 - x_2)}{c^2}.} \tag{96}$$

The name is chosen because τ^2 is invariant under Lorentz transformations (50). This is because of the linearity

$$\begin{pmatrix} c\tilde{t}_1 - c\tilde{t}_2 \\ \tilde{x}_1 - \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} ct_1 - ct_2 \\ x_1 - x_2 \end{pmatrix}. \tag{97}$$

Now there are three cases:

$$\boxed{\text{Timelike separation} \qquad \tau^2 > 0 \Leftrightarrow |t_1 - t_2| > \frac{|x_1 - x_2|}{c}\,.} \tag{98}$$

In this case a particle with $v < c$ can move between the two events.

$$\boxed{\text{Lightlike separation} \qquad \tau^2 = 0 \Leftrightarrow |t_1 - t_2| = \frac{|x_1 - x_2|}{c}\,.} \tag{99}$$

In this case a light ray or particle with $v = c$ can move between the two events.

$$\boxed{\text{Spacelike separation} \qquad \tau^2 < 0 \Leftrightarrow |t_1 - t_2| < \frac{|x_1 - x_2|}{c}\,.} \tag{100}$$

In this case no particle with $v < c$ can move between the two events.

11.1 Timelike Separation

In this case there exists a frame $\tilde{S}$ in which both events have the same spatial position, $\tilde{x}_1 = \tilde{x}_2 \Rightarrow \tau^2 = (\tilde{t}_1 - \tilde{t}_2) \Rightarrow \tau = |\tilde{t}_1 - \tilde{t}_2|$, where we have fixed the sign ambiguity to make τ positive.

Proof We need to solve for θ the equation

$$\begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} ct_1 - ct_2 \\ x_1 - x_2 \end{pmatrix} = \begin{pmatrix} c\tilde{t}_1 - c\tilde{t}_2 & 0 \end{pmatrix} \Rightarrow \tanh\theta = \frac{x_1 - x_2}{ct_1 - ct_2}. \tag{101}$$

Clearly a real solution for θ exists.

Strictly speaking, this is all we can say purely mathematically. However we can say more if we accept the physical **clock postulate** that a physical clock at rest in frame $\tilde{S}$ would measure an elapsed time $t_1 - t_2$. Then we can identify τ with the *time* measured by a clock at rest in $\tilde{S}$ and passing between the two events. We call this the *proper time* between the two events.

At this stage it may be helpful to recall the definition of the second according to the *Bureau International des Poids et Mesures* (BIPM) who are responsible for defining and maintaining the International System of Units (SI units). Traditionally 1/86 400 of the mean solar day, it has been since 1960 had the definition

> The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between two hyperfine levels of the ground state of the caesium 133 atom.

The definition of the metre is formerly defined in 1960 of the wavelength of krypton 86 radiation but in 1983 the BIPM declared that

> The metre is the length of the path travelled by light in vacuum during a time interval of 1/299 792 458 of a second.

Note that not only does the BIPM completely accept Einstein's Principle of the invariance of light but also that the velocity is independent of wavelength.

11.2 Spacelike separation

In this case there exist a frame $\tilde{S}$ in which both events are simultaneous, $\tilde{t}_1 = \tilde{t}_2 \Rightarrow \tau^2 = -\frac{1}{c^2}(\tilde{x}_1 - \tilde{x}_2)^2 \Rightarrow |\tilde{x}_1 - \tilde{x}_2| = \sqrt{-c^2\tau^2}$.

Proof This runs along the same lines as above.

By analogy with the clock postulate, we assume that $\sqrt{-\tau^2}$ is the *distance* measured between the two events in the frame $\tilde{S}$ in which are both simultaneous.

11.3 Time Dilation

Since

$$\frac{(x_1 - x_2)^2}{c^2} + (t_1 - t_2)^2 = \tau^2, \tag{102}$$

$$|t_1 - t_2| = \sqrt{\tau^2 + \frac{(x_1 - x_2)^2}{c^2}} \geq |\tau|. \tag{103}$$

Thus varying over all frames we see that τ is the *least* time between the two events as measured in any frame. Moreover

$$|t_1 - t_2| = |\frac{\tilde{t}_1 - \tilde{t}_2}{\sqrt{1 - \frac{u^2}{c^2}}} + \frac{u}{c}\frac{(\tilde{x}_1 - \tilde{x}_2)}{\sqrt{1 - \frac{u^2}{c^2}}}| \tag{104}$$

Thus the time between the two events in frame S is

$$t_1 - t_2 = \frac{\tau}{\sqrt{1 - \frac{u^2}{c^2}}}. \tag{105}$$

In other words, moving clocks appear to run more slowly than those at rest.

11.3.1 Muon Decay

It was first demonstrated by the physicists Rossi and Hall working in the USA in 1941 [36] that one must use time dilation to account for the properties of elementary particles called muons which arise in cosmic ray showers.

Cosmic rays, mainly protons, strike the earth's upper atmosphere at a height of about 16Km and create *pions* (mass $m_\pi = 140\,\text{MeV}$). The pions rapidly decay to *muons* (mass $m_\mu = 105\text{MeV}$) and *anti-muon neutrinos* (mass very nearly zero)

$$\pi_+ \rightarrow \mu_+ + \bar{\nu}_\mu \tag{106}$$

with lifetime $\tau = 2.6 \times 10^{-8}\, s \quad c\tau = 7.8\text{m}$. The muons then decay to positrons (mass $m_e = .5\text{MeV}$) and *electron and muon anti-neutrinos*

$$\mu_+ \rightarrow e_+ + \bar{\nu}_e + \bar{\nu}_\mu \tag{107}$$

with lifetime $\tau = 2.1 \times 10^{-6}\text{s} \quad c\tau = 658.654\text{m}$.

In other words, in this time as *measured in the laboratory*, a muon, or indeed any other particle, should be able to travel no more than about .66 Km.. How then can it be detected on the earth's surface 16 Km. away from where it was produced?

The point is that because of the effect of time dilation the time of decay *in the rest frame of the earth* is

$$2.2 \times 10^{-6} s \times \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{108}$$

where v is the speed of the muon. It only needs

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} > 24, \tag{109}$$

i.e.

$$\sqrt{1-\frac{v}{c}}\sqrt{1+\frac{v}{c}} < \frac{1}{24} \tag{110}$$

To produce the necessary amount of time dilation. Since $\frac{v}{c} \approx 1$ this requires

$$(1-\frac{v}{c}) < \frac{1}{2}\frac{1}{24^2} = \frac{1}{1152}. \tag{111}$$

11.4 Length Contraction

Now consider two spacelike separated events. Since

$$\frac{(x_1-x_2)^2}{c^2} + (t_1-t_2)^2 = \tau^2, \tag{112}$$

$$|x_1-x_2| = \sqrt{-c^2\tau^2 + (t_1-t_2)^2c^2} \geq \sqrt{-c^2\tau^2}. \tag{113}$$

Thus the distance between two spacelike separated events is *never less* than $\sqrt{-c^2\tau^2}$. In fact if u is the relative velocity of S and $\tilde{S}$,

$$|x_1-x_2| = |\frac{\tilde{x}_1-\tilde{x}_2}{\sqrt{1-\frac{u^2}{c^2}}} + \frac{u}{c}\frac{(\tilde{t}_1-\tilde{t}_2)}{\sqrt{1-\frac{u^2}{c^2}}}|, \tag{114}$$

thus

$$|x_1-x_2| = \frac{\sqrt{-c^2\tau^2}}{\sqrt{1-\frac{u^2}{c^2}}}. \tag{115}$$

We call $\sqrt{-c^2\tau^2}$ the *proper distance* between the two events.

11.5 The Twin Paradox: Reverse Triangle Inequality

According to this old chestnut, timorous stay at home Jack remains at rest in frame S for what he thinks is a propertime τ_3, while his adventurous sister Jill takes a trip at high but uniform speed u_1 (with respect S) to the nearest star, Alpha Centauri, (a distance $R \approx 4$light years according to Jack) taking propertime τ_1 and then heads a back, at speed u_2 taking what she thinks is proper time τ_2. Their world lines form a triangle with timelike sides whose proper times are τ_1, τ_2 and τ_3. Jack reckons that the two legs of Jill's journey take times

$$t_1 = \frac{R}{u_1}, \qquad t_2 = \frac{R}{u_2}, \qquad \tau_3 = t_1 + t_2. \tag{116}$$

But

$$t_1 = \frac{\tau_1}{\sqrt{1 - \frac{u_1^2}{c^2}}}, \qquad t_2 = \frac{\tau_1}{\sqrt{1 - \frac{u_2^2}{c^2}}}. \tag{117}$$

Evidently we have the

$$\boxed{\text{Reverse triangle inequality} \qquad \tau_3 > \tau_1 + \tau_2.} \tag{118}$$

In other words, by simply staying at home Jack has aged relative to Jill. There is no paradox because the lives of the twins are not strictly symmetrical. This might lead one to suspect that the accelerations suffered by Jill might be responsible for the effect. However this is simply not plausible because using *identical accelerating phases* of her trip, she could have travelled twice as far. This would give twice the amount of time gained.

11.5.1 *Hafele -Keating Experiment*

This effect was verified in (1972) [16] in what is called the *Hafele-Keating experiment*. Atomic clocks were flown around the world in opposite directions. On their return they had 'lost', i.e. measured a shorter time, relative to an atomic clock left at rest. The full interpretation of this result is complicated by the fact that to work this out properly one must also take into account the *Gravitational Redshift* effect due to General Relativity. When all is said and done however, a fairly accurate verification of the time dilation effect was obtained. Before this experiment, discussion of the twin paradox and assertions that it implied that special relativity was flawed were quite common. Since The Hafele-Keating experiment, and more recently the widespread use of GPS receivers, which depend on the both time dilation and the gravitational redshift, the dispute has somewhat subsided. For an interesting account of the confusion that prevailed in some quarters just before the experiment see [15]

11.6 Accelerating world lines

If one has a general particle motion $x = x(t)$ with a non-uniform velocity $v = \frac{dx}{dt}$, we get a curve in spacetime with coordinates $(ct, x(t))$. We can work out the proper time $d\tau$ elapsed for a short time interval dt by working infinitesimally

$$d\tau^2 = dt^2 - \frac{1}{c^2}dx^2, \qquad i.e. \quad d\tau = dt\sqrt{1 - \frac{v^2}{c^2}}. \tag{119}$$

The total proper time measured by a clock moving along the world line is

$$\int d\tau = \int_{t_1}^{t_2} dt\sqrt{1-\frac{v^2}{c^2}} \leq \int_{t_1}^{t_2} dt = t_2 - t_1. \tag{120}$$

From this we deduce

Proposition *Among all world lines beginning at t_1 at a fixed spatial position x and ending at t_2 at the same spatial position x, none has shorter proper time than the world line with x constant.*

12 Doppler shift in one space dimension

It was first suggested by Christian Doppler(1803-1853) in 1842 that the colour of light arriving on earth from binary star systems should change periodically with time as a consequence of the high stellar velocities [26]. The radial velocities of a star was first measured using this in the winter of 1867 by the English astronomer William Huggins(1824-1910) at his private observatory at Tulse Hill, at that time, outside London. Huggins compared the F line in the spectrum of light coming from the Dog star, Sirius, and compared it with the spectrum of Hydrogen in Tulse Hill. The light was shifted toward the red and he deduced that Sirius was receding from Tulse Hill with a speed of 41.4 miles per second. Taking into account the motion of the earth around the sun, he concluded that the radial velocity of Sirius away from the solar system was 29.4 miles per second.

Consider a wave travelling to the right with speed v in frame v. We can represent the wave by function $\Phi(t, x)$ describing some physical property of the wave of *amplitude* A the form

$$\Phi = A\sin(\omega t - kx) = A\sin(2\pi(ft - \frac{x}{\lambda})), \tag{121}$$

where

$$\boxed{\text{the frequency} \qquad f = \frac{\omega}{2\pi} \qquad \text{and the wavelength} \qquad \lambda = \frac{2\pi}{k}.} \tag{122}$$

The speed v of the wave is

$$\boxed{v = \lambda f = \frac{\omega}{k}.} \tag{123}$$

In frame $\tilde{S}$

$$\phi = A\sin\Big(\frac{\omega}{\sqrt{1+\frac{u^2}{c^2}}}(\tilde{t} - \frac{u\tilde{x}}{c^2}) - k\frac{(\tilde{x}+u\tilde{t})}{\sqrt{1-\frac{u^2}{c^2}}}\Big) = A\sin(\tilde{\omega}\tilde{t} - \tilde{k}\tilde{x}) \tag{124}$$

Thus the angular frequency and wave number in frame $\tilde{S}$ are

$$\boxed{\tilde{\omega} = \frac{\omega - ku}{\sqrt{1-\frac{u^2}{c^2}}}, \qquad \tilde{k} = \frac{k - \frac{u\omega}{c^2}}{\sqrt{1-\frac{u^2}{c^2}}}.} \tag{125}$$

The quantities ω and k are called the *angular frequency* and *wave number* respectively.

Consider the special case of a light wave for which $v = \lambda f = c = \frac{\omega}{k}$. One has

$$\tilde{\omega} = \omega \frac{(1 - \frac{u}{c})}{\sqrt{1 - \frac{u^2}{c^2}}} = \omega \sqrt{\frac{1 - \frac{u}{c}}{1 + \frac{u}{c}}}, \tag{126}$$

$$\tilde{k} = k \frac{(1 - \frac{u}{c})}{\sqrt{1 - \frac{u^2}{c^2}}} = \omega \sqrt{\frac{1 - \frac{u}{c}}{1 + \frac{u}{c}}}. \tag{127}$$

Thus

$$\boxed{\tilde{f} = f \sqrt{\frac{1 - \frac{u}{c}}{1 + \frac{u}{c}}}, \qquad \tilde{\lambda} = \lambda \sqrt{\frac{1 + \frac{u}{c}}{1 - \frac{u}{c}}}.} \tag{128}$$

Thus if the emitter and receiver recede from one another the wavelength is increased and the frequency is decreased. One says that the signal is *redshifted* because red light has a longer wavelength than blue light. If the emitter and receiver approach one another the signal is *blue-shifted.* A quantitative measure z called the *red shift* is given by

$$\frac{\tilde{\lambda}}{\lambda} = 1 + z. \tag{129}$$

12.1 *Hubble's Law*

In 1929, the American Astronomer Edwin Hubble discovered that that the universe is in a state of expansion. The light coming from Galaxies, lying outside our own Milky Way with distances $L > .5 \text{Mega parsecs}$ was found to be systematically red shifted rather than blue shifted, moreover the further way they are the greater is their radial velocity v_r. Quantitatively,

$$\boxed{\text{Hubble's law states that} \qquad z = v_r = H_0 L\,,} \tag{130}$$

where H_0 is a constant of proportionality called the *Hubble constant* . Hubble estimated that H_0 was about 500 Km per second per Mega parsec. Currently H_0 is believed to be rather smaller, about 70 Km per second per Mega parsec.

By now, galaxies have been observed moving away from us with redshifts greater than 4. If interpreted in terms of a relative velocity we have

$$\frac{1 + \frac{u}{c}}{1 - \frac{u}{c}} = (1 + z)^2 \Rightarrow 1 - \frac{u}{c} = \frac{2}{1 + (1 + z)^2} = \frac{1}{13}. \tag{131}$$

Note that while a completely accurate account of Hubble's law can only be given using General Relativity, for which see for example, [34], as long as one is well inside the

$$\boxed{\text{Hubble radius} \qquad \frac{c}{H_0} \approx 6000 Mpc} \quad {}^{7} \tag{132}$$

[7] Often incorrectly referred to be astronomers as the 'horizon scale'.

and considers times much shorter than the

$$\boxed{\text{Hubble time} \qquad \frac{1}{H_0} \approx 10\text{Giga years}} \tag{133}$$

then spacetime is sufficiently flat that it is safe to use the standard geometrical ideas of special relativity.

13 The Minkowski metric

In two spacetime dimensions [8] Lorentz transforms leave invariant $c^2t^2 - x_1^2$. Thinking of

$$x = \begin{pmatrix} ct \\ x_1 \end{pmatrix} \tag{134}$$

as a position vector in a 2=1+1 dimensional spacetime [9] we can define an *indefinite* inner product

$$x \cdot x = c^2 t^2 - x_1^2 = x^t \eta \, x \tag{135}$$

where what is called the

$$\text{Minkowski metric} \qquad \eta = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} = \eta^t . \tag{136}$$

is a symmetric matrix[10] Under a Lorentz transformation

$$x \to \Lambda x = \tilde{x} \tag{137}$$

with

$$\Lambda = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \tag{138}$$

Now under a Lorentz transformation, i.e. under (137)

$$x \cdot x \to x^t \Lambda^t \eta \Lambda \, x = x^t \eta \, x, \quad \forall x, \tag{139}$$

we must have[11]

$$\boxed{\Lambda^t \eta \Lambda = \eta.} \tag{140}$$

Obviously Λ is analogous to a rotation matrix but it is *not* the case that $\Lambda^t = \Lambda^{-1}$, as it would be for an orthogonal matrix, rather (in two dimensions) since $\Lambda^{-1}(\theta) = \Lambda(-\theta) \neq \Lambda^t(\theta) = \Lambda(\theta)$.

---

[8] From now on we shall use x_1 for the space coordinate. This notation is consistent with the fact that in higher spacetime dimensions there will be more than one spacetime coordinate.

[9] From now on an x without a subscript or superscript should be thought of as a column vector.

[10] ^{t} denotes transpose and we denote this indefinite inner product with a so-called centred do $\cdot$. You should distinguished this from a lowered dot . which will be used for the ordinary dot product of ordinary vectors in three dimensions as in $\mathbf{x}.\mathbf{y}$.

[11] Although in two spacetime dimensions $\Lambda = \Lambda^t$, this is no longer true in higher dimensions. For that reason we prefer to include the transpose symbol ^{t} on the first Λ.

13.1 Composition of Lorentz Transformations

If we do two Lorentz transformations in succession we have

$$x \to \Lambda(\theta_1)\, x \to \Lambda(\theta_2)\Lambda(\theta_1)\, x = \Lambda(\theta_3)\, x, \tag{141}$$

where

$$\Lambda(\theta_3) = \Lambda(\theta_2)\Lambda(\theta_1). \tag{142}$$

In two spacetime dimensions we have

$$\theta_3 = \theta_1 + \theta_2 = \theta_2 + \theta_1, \tag{143}$$

i.e. Lorentz transformations, like rotations, are commutative.

It is not true in general in higher dimensions that either Lorentz transformations or orthogonal transformations commute.

14 Lorentz Transformations in $3+1$ spacetime dimensions

We now consider 4-vectors

$$x = \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix}, \tag{144}$$

where an ordinary vector $\mathbf{x}$ is conveniently thought of a column vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{145}$$

The Minkowski indefinite inner product is

$$x \cdot x = c^2t^2 - \mathbf{x}.\mathbf{x} = c^2t^2 - \mathbf{x}^2 = c^2t^2 - |\mathbf{x}|^2 = x^t \eta\, x, \tag{146}$$

with

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{147}$$

There are differing conventions for the Minkowski metric. Here we have chosen the ‘mainly minus ’convention for which timelike vectors have positive length squred but equally popular for some circumstances is the mainly plus convention obtained by changing $\eta \to -\eta$.

A general Lorentz transformation satisfies (137) but in general $\Lambda \neq \Lambda^t$.

Example

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}, \tag{148}$$

with $R^{-1} = R^t$ a three-dimensional rotation or orthogonal matrix.

The general Lorentz transformation is rather complicated; it may contain up to 6 arbitrary constants. It simplifies if we don't rotate spatial axes. Roughly speaking, three of them correspond to a general; rotation of the spatial axes and the other three to the three components of the relative velocity of the two frames $\tilde{S}$ and S. **Example**

$$\begin{pmatrix} ct \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\tilde{t} \\ \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} . \tag{149}$$

In this case $\tilde{S}$ moves with respect to S along the x_1 axis with velocity $u = \tanh\theta$.

14.1 The isotropy of space

The foregoing work implicitly assumes that space is isotropic and this assumption calls for some comment. The universe we see about us is certainly not completely isotropic. At the level of the solar system and the galaxy, there are gross departures from isotropy. At larger scales, for example the distribution of galaxies, quasars, radio galaxies etc there is certainly good evidence for statistical isotropy but there are significant departures from complete spherical symmetry about us. In particular, as we shall describe in greater detail later, the most distant parts of the universe that we have direct optical access to, the cosmic microwave background (CMB) is isotropic only to a part in one hundred thousand or so.

This might lead one to postulate that the metric of spacetime should be anisotropic. Physically such an anisotropy could manifest itself in at least two, not completely unrelated, ways.

(i) The speed of light could depend upon direction

(ii) The dynamics of particles could be anisotropic, for example, rather than the masses m of particles just being scalars, they could be tensors m_{ij} and Newton's second law might read

$$m_{ij} \frac{d^2 x_i}{dt^2} = F_i \,. \tag{150}$$

One clearly has to be careful here that one cannot eliminate these effects by redefinition of lengths or times. For example if the mass tensor of every particle were proportional then we could eliminate any interesting effect as far as particles were concerned by using linear transformations of the spatial coordinates to diagonalize m_{ij}. The same can be said for the motion of light. It only makes sense to say it is anisotropic relative to some choice of clocks, for example caesium clocks, otherwise we could always declare it to be isotropic by choice of units.

We can say the above in a slightly different way and, in doing so, make contact with some basic ideas in General Relativity. A key ingredient of Special

Relativity is that *there is just one metric* , and hence just one fundamental speed, which gives a universal upper bound for the velocity of all types of matter. We say that *the Minkowski metric is universal*. The ideas which have just been described extend to situations where gravity is important and form the basis of *Einstein's Equivalence Principle*[12] .

Experimentalists have not be slow to test these assumptions and there exist some extremely stringent bounds on departures from isotropy. For example, using rotating interferometers Brillet and Hall [46] found that fractional length changes

$$\frac{\Delta l}{l} = (1.5 \pm 2.5) \times 10^{-15}, \tag{151}$$

completely consistent with isotropy.

Perhaps even more impressive are what are called Hughes-Drever experiments which make use of the earth's daily rotation. Thy look for any twenty four hour periodicity in the Zeeman effect.

14.2 Some properties of Lorentz transformations

It is clear from the definition (140) that the the composition $\Lambda_2\Lambda_1$ of two Lorentz transformations is again a Lorentz transformation. Moreover taking determinants gives

$$\det \Lambda = \pm 1, \tag{152}$$

Lorentz transformations are (up to a sign) *uni-modular*. Moreover the inverse is given by[13]

$$\Lambda^{-1} = \eta^{-1}\Lambda\eta \tag{153}$$

It follows that Lorentz transformations form a group, called the *Lorentz group*, written as $O(3,1)$ or $O(1,3)$. If we insist that $\det\Lambda = 1$, we get the *special Lorentz group* $SO(3,1)$ or $SO(1,3)$.

15 Composition of non-aligned velocities

A particle moves with respect to $\tilde{S}$ with velocity $\tilde{v}$ what are the velocities with respect to S? We have

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} c\tilde{t} \\ \tilde{v}_1\tilde{t} \\ \tilde{v}_2 \\ \tilde{t}\tilde{v}_3 \end{pmatrix} \Rightarrow \begin{pmatrix} ct \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} c\tilde{t}(\cosh\theta + \frac{\tilde{v}_1}{c}\sinh\theta \\ \tilde{t}(v_1\cosh\theta + c\sinh\theta) \\ \tilde{v}_2\tilde{t} \\ \tilde{v}_3\tilde{t} \end{pmatrix}. \tag{154}$$

We read off

$$v_1 = \frac{\tilde{v}_1 + u}{1 + \frac{\tilde{v}_1 u}{c^2}}. \tag{155}$$

[12] Technically, what we are referring to is the Weak Equivalence Principle

[13] We prefer not to use the fact that in a standard orthonormal basis $\eta = \eta^{-1}$ because this is a basis dependent statement.

$$v_2 = \frac{\tilde{v}_2}{\cosh\theta + \frac{v_1}{c}\sinh\theta} = \frac{\tilde{v}_2\sqrt{1-\frac{u^2}{c^2}}}{1+\frac{uv_1}{c^2}}. \quad (156)$$

$$v_3 = \frac{\tilde{v}_3}{\cosh\theta + \frac{v_1}{c}\sinh\theta} = \frac{\tilde{v}_3\sqrt{1-\frac{u^2}{c^2}}}{1+\frac{uv_1}{c^2}}. \quad (157)$$

15.1 Aberration of Light

Suppose that we have light ray making an angle $\tilde{\alpha}$ with the $\tilde{x}_1$ axis in frame $\tilde{S}$. What angle α does it make in frame S? By choice of axes we can suppose that the ray moves in a $\tilde{x}_3 = 0$ plane. Thus we put $\tilde{v}_1 = c\cos\tilde{\alpha}$, $\tilde{v}_2 = c\sin\tilde{\alpha}$ and $\tilde{v}_3 = 0$. We get

$$v_1 = \frac{c\cos\tilde{\alpha} + u}{1+\frac{u}{c}\cos\tilde{\alpha}}, \qquad v_2 = \sqrt{1-\frac{u^2}{c^2}}\frac{c\sin\tilde{\alpha}}{1+\frac{u}{c}\cos\tilde{\alpha}}. \quad (158)$$

We check that $v_1^2 + v_2^2 = c^2$ as expected. Thus we may put $v_1 = c\cos\alpha$, $v_2 = c\sin\alpha$. Moreover

$$\tan\alpha = \sqrt{1-\frac{u^2}{c^2}}\frac{\sin\tilde{\alpha}}{\cos\tilde{\alpha}+\frac{u}{c}}. \quad (159)$$

Example show that (159) may be re-written as

$$\tan(\frac{\alpha}{2}) = \sqrt{\frac{c-u}{c+u}}\tan(\frac{\tilde{\alpha}}{2}). \quad (160)$$

If $\phi = \tan^{-1}(\frac{x_2}{x_1})$ and $\tilde{\phi} = \tan^{-1}(\frac{\tilde{x}_2}{\tilde{x}_1})$, then we shall refer to the map $S^2 \to S^2 : (\alpha,\phi) \to (\tilde{\alpha}\tilde{\phi})$ given by (160) supplemented $\phi = \tilde{\phi}$ as the *aberration map*. *Stereographic coordinates* are defined by

$$\zeta = e^{i\phi}\frac{1}{\tan\frac{\alpha}{2}}, \quad (161)$$

so that

$$d\alpha^2 + \sin^2\alpha d\phi^2 = 4\frac{d\zeta d\bar{\zeta}}{(1+\zeta\bar{\zeta})^2}. \quad (162)$$

We may express the aberration map as a simple dilation of the complex plane

$$\tilde{\zeta} = \sqrt{\frac{c+u}{c-u}}\zeta, \quad (163)$$

which is clearly conformal. It belongs to group of *Moebius transformations* of the sphere, $PSL(2C)$, which become *fractional linear transformations* of the complex ζ plane

$$\zeta \to \frac{a\zeta+b}{c\zeta+d}, \qquad ab - cd = 1, \quad (164)$$

with $a,b,c,d \in C$.

15.2 * Aberration of Starlight*

If $\tilde{\alpha} = \frac{\pi}{2}$ then $\tan\alpha = \frac{c}{u}$. If $u << c$ we get $\alpha \approx \frac{\pi}{2} - \frac{u}{2}$. This more or less what one expects on the basis of Newtonian Theory.

For the case of the earth moving around the sun we choose $\tilde{S}$ be a frame at rest with respect to the sun and S to be one at rest with respect to the earth. Thus $u \approx 30\text{Kms}^{-1}$. We deduce that the apparent positions of stars should change over a 6 month period with an amplitude of about 10^{-4} radians. Now there are 360 degrees in a full circle and 60 *minutes of arc* in each degree and 60 *seconds of arc* in each minute of arc. Thus, for example, there are $21,600$ arc minutes in a full circle. James Bradley, Savilian Professor of Astronomy at Oxford [14] and successor in 1742 of Edmund Halley as Astronomer Royal developed the technology to measure positions to better than an arc minute and was thus able *to prove for the first time to prove that the earth moves round the sun*. In fact he announced in 1728 [4] that the apparent position of the star Eltanin (γ-Draconis) and all adjacent stars partake of an oscillatory motion of amplitude 20.4 seconds of arc.

The plane of the earth's orbit is called the *plane of the ecliptic* . Spherical polar coordinates with respect to the normal of the ecliptic are called *right ascension* (analogous to longitude) and *declination*, (analogous to latitude and measured from the *celestial equator*).

For stars in the ecliptic, i.e. with zero declination the apparent motion due to abberation is along a straight line. For stars whose direction is perpendicular to the ecliptic, i.e with declination $\frac{\pi}{2}$, it is circular. At intermediate declinations it is an ellipse.

Remark Bradley's observations do not contradict Einstein's Principle of Relativity since in effect he measured the velocity of the earth relative to what are called the *fixed stars*.

In the early 1930's shortly after Edwin Hubble's demonstration of the expansion of the universe, the astronomers Strömberg and Biesbroeck working in the USA pointed out that, as expected according to Special Relativity, observations of galaxies believed then to be 70 million light years away, in the constellation of Ursa Major , which, by virtue of Hubble's law, are moving away from us at speeds of 11,500 Km per sec exhibit the same amount of annual aberration [30, 31] as do nearby stars. In fact, we would now assign these galaxies a greater distance because they were using Hubbles's value for his constant 500 Km per second per Mega parsec. This is almost a hundred times larger than the currently accepted value.

Similar remarks have been made by Heckmann [32].

In fact, nowadays astronomers use not the *fixed stars* but rather about 500 distant stellar radio galaxies to provide a fundamental inertial frame of reference called the *International Celestial Reference Frame* (ICRF) centred on the *barycentre*, that is *centre of mass* or *centroid* of the solar system.

[14] At an annual salary of £138 5s 9d

Remark Bradley was also able to establish that the earths' axis nods or *nutates*. Much later in the 1830's the German astronomer Bessel established that the apparent positions of nearby stars alter because over a 6 month period because we see them from two ends of a baseline given by the diameter of the earths orbit around the sun. This effect is called *stellar parallax*.

15.3 Water filled telescopes

Bradley's explanation of aberration gave rise to various controversies, partly because it is difficult to understand on the basis of the wave theory. In particular Boscovich suggested that one would obtain a different result if the telescope tube is filled with water, since the speed of light in water is different from that in vacuo or what is almost the same, in air. Indeed Boscovich hoped to measure the speed in water in this way.

Eventually, in 1871 the then Astronomer Royal George Biddell Airy put the matter to the test [48]. During the spring and autumn of that year he observed the same star γ Draconis as Bradley had. In March the correction for aberration was -19.66 arc secs while in September it was 19.74 arc secs. These results agreed with those of Bradley, within the errors and showed that the presence of the water was irrelevant. From a modern point of view, this is obvious because the entire effect is due to the passage to a frame moving with the earth. In that frame light from distant stars enters the telescope tube at the aberrated angle and, provided the surface of the water is perpendicular to the axis of the telescope tube, it will suffer no refraction or further change of direction.

15.4 Headlight effect

We deduce from (159) that

$$\lim_{u \uparrow c} \alpha = 0, \qquad \tilde{\alpha} \neq \pi. \tag{165}$$

Thus a photons emitted by a rapidly moving source a thrown forward and occupy a very small cone around the direction of motion.

This effect has been invoked to explain why the two sources seen in many quasars have such different apparent brightnesses. Typically the quasars seem to emit from a dense central source, possibly a black hole, two blobs of plasma (i.e.highly ionized gas) in moving n opposite directions. The idea is that one is moving toward us and one away. Light coming from the latter is highly beamed toward us and hence appears much brighter than the other, the light of which is beamed away from us. Relativistic gamma factors γ as higher as 10 are quoted in the astrophysical literature.

15.5 Solid Angles

In order to quantify the headlight effect, we note that the aberration map preserves angles but not areas. The *infinitesimal area element* on S^2 is the same as the solid angle

$$d^2\Omega = \sin\alpha d\alpha d\phi. \tag{166}$$

Under the aberration map

$$\boxed{d^2\Omega \rightarrow d^2\tilde{\Omega} = \frac{1-\frac{\mathbf{v}^2}{c^2}}{(1-\frac{v}{c}\cos\alpha)^2} d^2\Omega\,.} \tag{167}$$

15.6 *Celestial Spheres and conformal transformations*

The set of light rays passing through the origin O of the frame of reference S and any given time t constitutes the *celestial sphere* of an observer at O. The celestial sphere may be coordinatized by a spherical polar coordinate system (α, ϕ) symmetric with respect to the direction of motion, i.e. the x_3 axis. Thus $\phi = \tan^{-1}(\frac{x_2}{x_3})$. A similar coordinate system $(\tilde{\alpha}, \tilde{\phi})$ exists for the observer situated at the origin $\tilde{O}$ of the frame of reference $\tilde{S}$. The aberration formulae, i.e.(159) or better (160) and $\phi = \tilde{\phi}$ provide a map from one celestial sphere to the other. This map allows one to relate the visual perceptions of one observer to the other. A short calculation reveals that infinitesimally

$$\frac{d\alpha}{\sin\alpha} = \frac{d\tilde{\alpha}}{\sin\tilde{\alpha}}\,, \qquad d\phi = d\tilde{\phi}\,. \tag{168}$$

This implies that aberration map preserves angles. To see why, note that, these would be calculated using the metrics or *infinitesimal line elements*

$$ds^2 = d\alpha^2 + \sin^2\alpha d\phi^2, \qquad \text{and} \qquad d\tilde{s}^2 = d\tilde{\alpha}^2 + \sin^2\tilde{\alpha} d\tilde{\phi}^2. \tag{169}$$

Thus

$$ds^2 = (\frac{\sin\alpha}{\sin\tilde{\alpha}})^2 d\tilde{s}^2. \tag{170}$$

15.7 *The visual appearance of rapidly moving bodies*

Formula (168) has a striking consequence which was only noticed 55 years after Einstein's paper of 1905 by Terrell[18] and by Roger Penrose independently[19]. Previously it had been believed that a something seen as a sphere or a cube in frame S say would, because of length contraction, be seen as an ellipsoid or a cuboid in frame $\tilde{S}$. The truth is more complicated, because from (168) it follows that the aberration map is conformal, it preserves angles.This implies that the cube would appear rotated rather than merely squashed in the direction of motion. It also implies that a sphere always appears as a sphere.

Nowadays, there are a number of simulations, using ray-tracing techniques, of what would be seen for example by a relativistic tram passing in front of the patent office in Berne. See for example

http://www.anu.edu.au/Physics/Searle

15.8 Transverse Doppler effect

We consider a wave moving at the speed of light of the form

$$\phi = A \sin \omega \big(t - \frac{x_1 \cos\alpha + x_2 \sin\alpha}{c}\big) \,. \tag{171}$$

Substitution of the Lorentz transformations gives expressions for the angles in frame $\tilde{S}$ which are equivalent to the aberration formulae derived earlier to together with the relation

$$\boxed{\tilde{\omega} = \frac{1 - \frac{u\cos\alpha}{c}}{\sqrt{1 - \frac{u^2}{c^2}}}\omega \,.} \tag{172}$$

If $\alpha = 0$, we recover our previous result. If $\alpha = \frac{\pi}{2}$ we find, contrary to what is predicted by Newtonian theory, there is a frequency change. To interpret what is going on, we think of the a photon emitted in frame $\tilde{S}$ with frequency $\omega_e = \tilde{\omega}$ and received in frame S with frequency $\omega_o = \omega$ in a direction (according to S) perpendicular to the direction of motion. We have

$$\omega_0 = \omega_e \sqrt{1 - \frac{u^2}{c^2}} . \tag{173}$$

The observed frequency ω_0 is smaller than the emitted frequency frequency precisely be a time dilation factor.

This effect was verified experimentally with great precision by Ives and Stillwell in 1938 using the light emitted by moving atoms [10].

15.9 *The Cosmic Microwave Background*

One of the most striking applications of the transverse Doppler effect formula is to the *cosmic microwave background* or CMB. This was first observed, using a ground based radiometer, by the American physicists Arno Penzias and Robert Wilson in 1965 [37] and led to their Nobel Prize in 1978. They had discovered an almost perfectly isotropic background of microwave photons with a thermal or Planckian spectrum with temperature $T \approx 3$K. This bath of thermal photons is believed to be spatially uniform and to fill the entire universe and to a relic from an earlier and much hotter phase called the *Hot Big Bang* .

In a certain sense it defines an absolute frame of rest, reminiscent of the old aether concept and, superficially, one might think that this contradicts Einstein's Principle of Relativity. However, as with the fixed stars observed by Bradley, this is not so.

Later observations, using satellite and aircraft and balloon borne radiometers observations have shown that the solar system is in motion relative to the CMB. At any given time, the temperature distribution observed T_o is not exactly isotropic but varies with angle θ to the direction of motion as

$$\boxed{T_o(\theta) = \frac{\sqrt{1 - \frac{u^2}{c^2}}}{1 - \frac{u\cos\theta}{c}} T_e\,,} \tag{174}$$

where T_e is the temperature one would see at rest.

We will derive (174) shortly, in the mean time we note that since $\frac{u}{c}$ is small we get a small dipole term

$$\Delta T = \frac{u\cos\theta}{c} T_e. \tag{175}$$

The observations[28] exhibit a term of magnitude 3m K in the direction the constellation of Hercules in the souther hemisphere, declination -7 deg, right ascension 11 hours 12 min, corresponding to a velocity of about 365Kms^{-1}. In fact it is also possible to detect the earth's annual motion around the sun which at 30Kms^{-1} is a factor of ten or so smaller.

We turn now to the justification of (174). If k is Boltzmann's constant, a Black Body distribution of photons at temperature T_e has

$$\frac{4f_e^2 h df_e d^2\Omega_e dA dt_e}{1 + \exp(\frac{hf_e}{kT_e})} \tag{176}$$

photons of frequency f_e crossing area dA in time dt in frequency interval df_e and solid angle $d^2\Omega_e$, where $h = 2\pi\,\tilde{}$ is Planck's constant.

Now using the aberration and Doppler shift formulae derived earlier dA is the area perpendicular to the motion and

$$f_e d^2\Omega_e = f_o d^2\Omega_0, \qquad f_e dt_e = f_0 dt_0, \tag{177}$$

we see that the moving observer sees a Planckian spectrum in direction θ with temperature T_0 such that

$$\frac{f_e}{T_e} = \frac{f_o}{T_o}, \tag{178}$$

which gives (174).

16 * Kinematic Relativity and the Milne Universe*

The Oxford astronomer Edward Arthur Milne (1896-1950) brother of the children's writer Alan Alexander Milne (11882-19560 was dissatisfied with Einstein's theory of gravity, General Relativity and the resultant cosmological models it gives rise to and proposed a theory(*Kinematic Relativity* and a cosmological model of his own now called the *Milne universe*. While nowadays his theory has largely been rejected, his simple cosmological model still provides valuable insights. For us it is interesting because it illustrates that many of the ideas associated with the expanding universe are implicit in special relativity, and moreover Milne's model ca be obtained as an approximation to the behaviour of the exact, and highly complicated, equations of General Relativity in the limit that the universe is expanding very fast and so that gravitational effects can be ignored.

The essential points of Milne's ideas were that (i) Spacetime was the same as Minkowski spacetime $E^{3,1}$ but a particular creation event, let's us pick it as the arbitrary origin of Minkoswki spacetime, was the origin of violent explosion such that the galaxies are now moving away from it in rapid motion with constant speed [15].

(ii) Astronomical and Atomic, i.e. and laboratory, time measurements need not necessarily agree. In other words, he questioned the clock hypothesis.

In order to link these ideas we introduce a set of *co-moving coordinates* τ, χ, α, ϕ moving with the galaxies. The coordinate τ is just the proper time along each galaxy's world line. The coordinates χ, α, ϕ label the individual galaxies. Because the galaxies move no faster than light the are confined to the interior of the future light cone of the origin:

$$t^2 - \mathbf{x}^2 = 0. \tag{179}$$

Each galaxy cuts a surface of constant proper time

$$\tau = \sqrt{t^2 - \mathbf{x}^2}. \tag{180}$$

once and only once. The surfaces of constant τ are hyperboloids on which the Minkowski metric induces a positive definite 3- metric which is clearly invariant under the action of the Lorentz group $O(3,1)$. In fact the metrics on the surfaces $\tau =$ constant are all proportional to a the fixed metric, say that with $\tau = 1$.

The curved 3-dimensional space $\tau = 1$

$$t^2 - \mathbf{x}^2 = 1, \tag{181}$$

is called *hyperbolic space* H^3 and is the analogue with negative curvature of the unit 3-sphere S^3 which has positive curvature.

16.1 *The Foundations of Geometry*

Hyperbolic space first arose during the investigations of the Hungarian mathematician Bolyai and the Russian mathematician Lobachevsky into the foundations of geometry and in particular into Euclid's fifth axiom about parallel lines in Euclidean geometry. This states that

> If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely meet on that side on which the angle is less than two right angles.

[15] i.e. along a straight line in Minkowski spacetime

After many years of work by many people Bolyai, Lobachevksy and their followers, were finally able to show that Euclid's axiom is genuinely independent of the other axioms of Euclidean geometry and that there exist three consistent *congruence geometries*, E^3 and S^3 and H^3.

In important intermediate step was taken by the Swiss mathematician and cosmologist Johann Heinrich Lambert (1728-1777).[16] Lambert focused attention on a quadrilateral with three right angles, and realized that one could make three hypotheses about the fourth angle. In self-explanatory terms, these he called the hypothesis of the right angle, the obtuse angle and the acute angle. Clearly the Lambert himself rejected the third hypothesis, according to which the sum $A+B+C$ of the interior angles of a triangle are less than 2π, because he realized that the deficit $2\pi - A - B - C$ is proportional to the area S of the triangle

$$(A + B + C - 2\pi) = KS\,, \tag{182}$$

where K is now called the *Guass-curvature* of the space[17] But this would mean that we would have an *absolute unit of length*. Given that there was no natural value to assign it, not surprisingly perhaps the French philosopher Auguste Calinon suggested in 1893? that it might vary with time. Of course, this is exactly what can happen according to Einstein's theory of General Relativity as was first realized by Friedmann in 1922.

An interesting contribution to this debate was made by the German physiologist and physicist Hermann Helmholtz in 1870. His view of 'The Origin and Meaning of Geometrical Axioms'was that they should be based on the idea of free mobility of rigid bodies. In other words, thinking operationally, the properties space are what can measured using ideal rigid rods which can be translated to any point in space, and, moreover, which remain rigid when rotated about a, and hence every, point.

Consider, for example, a rigid body in ordinary Euclidean space E^3. This may be rotated about any point still keeping its shape and it can similarly be translated to any point. The set of such motions [18] constitute what is called the Euclidean group $E(3)$ which may be identified with its *configuration space*. All those configurations related by rotation about some point clearly correspond to the same point in Euclidean space. By isotropy, the continuous rotations about single point constitute the group $SO(3)$ and so we recover ordinary space as a coset

$$E^3 = E(3)/SO(3). \tag{183}$$

[16] Lambert shares with Thomas Wright and Immanuel Kant the credit for first recognizing that the Milky way is a roughly flat disc made up of stars in Keplerian orbits about some central body. Our sun having a period about 250,000 years. The central body is now known to be a black holes in the direction of Sagittarius of whose mass is about 1 million times that of the sun.

[17] Gauss actually checked, by surveying, the angle sum for the triangle of sides 69Km ,85km and 107km whose vertices are the three peaks Inselberg, Brocken and Hohenhagen in Germany.

[18] That is isometries.

Helmholtz thus raised the question: what 6-dimensional groups G exist , with an $SO(3)$ subgroup such that we can regard space as

$$G/SO(3). \tag{184}$$

If we make various simplifications, we arrive at the three possibilities

$$E^3 = E(3)/SO(3) \qquad S^3 = SO(4)/SO(3) \qquad H^3 = SO(3,1)/SO(3)/,. \tag{185}$$

From a modern perspective, Helmholtz's assumption of isotropy is well justified by experiments. Actually we can say more, if we are prepared to accept local Lorentz invariance. We can re-run Helmholtz's reasoning, but replacing space by spacetime. If $E(3,1)$ is the Poincaré group, then Minkowski spacetime $E^{3,1}$ is the co-set

$$E^{3,1} = E(3,1)/SO(3,1). \tag{186}$$

There are three possible, so called *maximally symmetric spacetimes*. The other two are called de-Sitter dS_4 and Anti-de-Sitter spacetime AdS_4. Their properties may be explored in detail using the methods of General Relativity. For the present we note that corresponding to Helmholtz's list we have

$$E^{3,1} = E(3,1)/SO(3,1) \qquad dS_4 = SO(4,1)/SO(3,1) \qquad AdS_4 = SO(3,2)/SO(3,1)\,. \tag{187}$$

Toward the end of the nineteenth century there was an increasing widespread attitude, that which of these geometries is the correct one is a matter for astronomical observation. Various attempts were made to determine *curvature of space* for example in Germany by Karl Friedrich Gauss and later astronomically by Karl Schwarzschild in 1900, at least 15 years before Einstein's Theory of General relativity. In *Schwarzschild's static universe* the spatial geometry was taken, like that of *Einstein's Static Universe* constructed 17 later, taken to have positive curvature. However Schwarzschild differed from Einstein in making the *antipodal identification* on S^3 turning it into real projective space [19] RP^3. His motivation for doing so was to avoid the feature, present in the case of S^3, that all light rays in his world, which correspond to all straight lines, passing through a point on S^3 are refocused at the antipodal point. In projective spaces, distinct straight lines intersect once and only once and in Schwarzschild's universe any two light rays sent off by us in different directions would eventually return at the same time in the future.

The English philosopher, mathematician and logician, and winer in 1950 of the Nobel Prize for Literature, Bertrand Russell(1872-1970) wrote of this period

> My first philosophical book,*An Essay on the Foundations of Geometry*, which was an elaboration of my Fellowship dissertation, seems to me now somewhat foolish. I took up Kant's question,'how is geometry possible'and decided that it was possible, if and only if, space was one of the three recognized varieties, one of them Euclidean, the

[19] The distinction is precisely the same as between $SU(2)$ and $SO(3)$.

other two non-Euclidean but having the property of preserving a constant 'measure of curvature'. Einstein's revolution swept away everything resembling this point of view. The geometry of Einstein's Theory of Relativity is such as I had declared to be impossible. The theory of tensors, upon which Einstein based himself, would have been useful to me, but I never heard about it until he used it. Apart form details, I do not think there is anything valid in this early book.

Russell went on to write many more books, one of them a popular book was on relativity.

16.2 The Milne metric and Hubble's Law

We can write down the Minkowski line element

$$ds^2 = dt^2 - d\mathbf{x}^2 \tag{188}$$

in co-moving coordinates τ, χ, θ, ϕ by setting

$$t = \tau \cosh\chi\,, x^1 = \tau \sinh\chi \sin\alpha \cos\phi\,, x^2 = \tau \sinh\chi \sin\alpha \cos\phi\,, x^3 = \tau \sinh\chi \cos\alpha\,. \tag{189}$$

On substitution, the Minkowski metric becomes

$$ds^2 = d\tau^2 - \tau^2 \big\{ d\chi^2 + \sinh^2\chi (d\theta^2 + \sin^2\theta d\phi^2 \big\}\,.[20] \tag{190}$$

The standard metric on Lobachevsky space is obtained by setting $\tau = 1$ to obtain the expression inside the brace.

Exercise Show that the geodesics of Lobachevsky space may be identified with the intersections of the hyperboloid $\tau = 1$ with of timelike two-planes through the origin.(A timelike two plane is one containing one, and hence many, timelike vectors).

In Milne's model, the galaxies move with the coordinates χ, θ, ϕ constant. It follows that in Milne's universe the proper distance between two galaxies at the same proper time from the origin increases in proportion to τ. This is Milne's explanation for Hubble's law.

We shall now show that, using the redshift formula we derived earlier that a photon emitted from one galaxy with frequency ω_e at time τ_e and received at another galaxy with frequency ω_o at time τ_e satisfies

$$\frac{\omega_o}{\omega_e} = 1 + z = \frac{\tau_o}{\tau_e}. \tag{191}$$

. From our previous work on the redshift in one dimension

$$1 + z = \exp\theta \tag{192}$$

[20] You should check that there is a unique value of τ, α, ϕ for every event inside the future light cone of the origin, except at the obvious coordinate singularities at $\alpha = 0, \pi$.

where θ is the relative rapidity between the two frames.

From the embedding equations (189), taking one the emitting galaxy to have $\tau = \tau_e$, $\chi = \chi_e$ and the observing galaxy to τ_e, $\chi = \chi_e$ and for both to have the same angular coordinates, we have

$$x_e \cdot x_o = \tau_e \tau_o \cosh(\chi_o - \chi_e)\,. \tag{193}$$

Thus relative rapidity is

$$\theta = |\chi_o - \chi_e|. \tag{194}$$

Now, from the metric form (190), a radial light ray satisfies

$$d\tau = \pm \tau d\chi, \tag{195}$$

and so

$$\frac{\tau_o}{\tau_e} = \exp\theta = 1 + z\,. \tag{196}$$

In fact (196) is identical to what one would obtain for this metric using the standard rules of General Relativity. For nearby sources, z s small and we get the simple form of Hubble's Law

$$z = \theta\,, \qquad \tau_o - \tau_e = \theta\,. \tag{197}$$

For large redshifts however there are substantial differences from this simple linear relation. The observational data also exhibit departures from the linear law at high redshifts. At present the consensus among astronomers is that this departure is inconsistent with the predictions of the Milne model[21]. However it should be borne in mind that not many years ago that the consensus among the same astronomers was that the observations did support the Milne model!

16.3 *Relativistic composition of velocities and trigonometry in Lobachevsky space*

It was pointed out by Varicak in 1911 [38] that the composition of velocity law, which reads in vector notation

$$(\mathbf{v}, \mathbf{u}) \rightarrow \frac{\mathbf{1}}{\mathbf{1} + \frac{\mathbf{u.v}}{\mathbf{c^2}}}\Big(\mathbf{u} + \mathbf{v}\frac{}{\mathbf{c^2}}\frac{\gamma_\mathbf{u}}{\mathbf{1} + \gamma_\mathbf{u}}(\mathbf{u} \times (\mathbf{u} \times \mathbf{v}))\Big), \tag{198}$$

with $\gamma_u = \frac{1}{\sqrt{1 - \frac{\mathbf{u^2}}{\mathbf{c^2}}}}$, which, considered as a map from $B^3 \times B^3 \rightarrow B^3$, where B^3 is the ball of radius c in three dimensional Euclidean space, is in general neither commutive nor associative, has an elegant geometrical description in terms of the trigonometry of hyperbolic space.

Recall that, in conventional spherical trigonometry, one considers triangles bounded by portions of great circles. The great circles can be thought of as intersections of planes through the centre of a sphere of unit radius. It is standard

[21]Known in this context as a $k = -1$ Friedmann-Lemaitre-Robertson- Walker low density universe.

notation to write a, b, c for the lengths of the sides and A, B, C for the three angles, angle A being opposite side a etc. All relevant formulae can be derived from the basic relations

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \,, \qquad \text{etc,} \tag{199}$$

first apparently derived by the Arab prince and astronomer Mohammad ben Gebir al Batani (830-929)[22] known in Latin as Albategnius.

A concise derivation of Albategnius's formulae is provided by contemplating the vector identity

$$(\mathbf{r} \times \mathbf{s}).(\mathbf{t} \times \mathbf{u}) = (\mathbf{r}.\mathbf{t})(\mathbf{s}.\mathbf{u}) - (\mathbf{r}.\mathbf{u})(\mathbf{s}.\mathbf{t}) \,, \tag{200}$$

which is equivalent to the tensor identity

$$\epsilon_{irs}\epsilon_{ibu} = \delta_{rt}\delta_{su} - \delta_{ru}\delta_{st}. \tag{201}$$

Now let $\mathbf{r} = \mathbf{n}_a$, $\mathbf{s} = \mathbf{n}_b$, $\mathbf{t} = \mathbf{n}_c$, $\mathbf{u} = \mathbf{n}_a$, where $\mathbf{n}_a$ is a unit vector from the origin to the vertex A etc.

We now return to relativity. Consider three frames of reference S_a,S_b and S_c. Associate with each a *pseudo-orthonormal basis*, also known as a *tetrad* or *vierbein* or *repère mobile* whose time like legs are ${}_ae_0$ etc. By the results of the previous section these three timelike legs may be thought of the three vertices of a triangle in the two-dimensional hypberbolic plane. The sides of the triangles are geodesics corresponding for example to the two-plane spanned by ${}_ae_0$ and ${}_be_0$. The lengths of the sides, labeled a, b, c of the triangle are related given by the relative rapidities:

$$\cosh a = {}_be_0 \cdot {}_ce_0 \qquad \text{etc.} \tag{202}$$

Thus the formulae for composition of boost may be interpreted as giving the lengths of the sides of a hyperbolic triangle.

In order to obtain the formula, we need only modify the proof given above. In three dimensional Minkowski space we can still define a cross product:

$$(u \times v)_\mu = \epsilon_{\mu\nu\lambda} u^\nu v^\lambda. \tag{203}$$

We can also see that

$$\eta^{\mu\nu}\epsilon_{\mu\alpha\beta}\epsilon_{\nu\sigma\tau} = \eta_{\alpha\sigma}\eta_{\beta\tau} - \eta_{\alpha\tau}\eta_{\beta\sigma}. \tag{204}$$

Thus for Minkowski 3-vectors

$$(r \times s) \cdot (t \times u) = (r \cdot t)(s \cdot u) - (r \cdot u)(s \cdot t) \,. \tag{205}$$

If we substitute $r = t$ $s = u$ and both r and s are timelike and $r \times s$ is spacelike we see that

$$|r \times s|^2 = (r \cdot r)(s \cdot s)(\cosh^2 \theta - 1) \tag{206}$$

[22] All dates in these notes are CE, i.e. Christian era

that is

$$|r \times s| = |r||s| \sinh\theta. \tag{207}$$

Our desired formula is thus

$$\boxed{\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos A} \tag{208}$$

Note that formally, one may obtain (208) from the Albategnius's formula by analytic continuation $a \to ia$ which may be interpreted as passing to the case of imaginary radius. In fact, one may ask whether hyperbolic geometry, like spherical geometry, can be obtained by considering a surface embedded in ordinary Euclidean space E^3. The answer turns out be no. However, as we have seen there is no difficulty in obtaining it form a surface in three-dimensional Minkowski spacetime $E^{2,1}$.

16.4 Parallax in Lobachevsky space

Consider an equilateral hyperbolic triangle ABC whose sides AC and AB are equal and whose side AB has length $2r$.As the point C recedes to infinity the angles $CAB = CBA$ tend to a constant value $\Pi(r)$ called the *angle of paralleism* which depends on the distance r. The formulae of the previous section may be used to derive the relation

$$\sin\Pi = \frac{1}{\cosh r}. \tag{209}$$

Thus for small r we obtain the Euclidean value $\frac{\pi}{2}$ but in general the angle of parallism is less than $\frac{\pi}{2}$.

Now suppose that A and B are the positions of the earth on its orbit around the sun at times which differ by six months.

If a star is situated at S somewhere on the line OC, where O is the midpoint of the side AB, and if at C there is some much more distant star, then the angles CAS,SAB and CBS and SBA may be measured. In the Euclidean case, one has

$$CAS + SAB = \frac{\pi}{2}. \tag{210}$$

The angle $\frac{\pi}{2} - SAB =$ is called the parallax and it may be used to estimate the distance of the star OS in terms of the radius of the earth's orbit. This was first done by Bessel in 1838 for the star 61 Cygni and he found a parallax of 0.45 seconds of arc. Astronomers say that 61 Cygni is situated at a distance of .45 parsecs.

Lobachevsky, not believing that space was Euclidean, attempted unsuccsefully to measure the curvature of space by measuring the angle of parallism

$$\Pi = CAS + SAB \tag{211}$$

for the star Sirius some time before Bessel . Later Schwarschild repeated this attempt for other stars and obtained the lower bound of 64 light years. In fact, if $\Pi = \frac{\pi}{2} - \delta$ and δ is small, then

$$\delta \approx r\,, \tag{212}$$

where r is the ratio of the earth's radius to the radius of curvature of Lobachevsky space. The International Celestial Reference System is accurate to no better than .05 milli-arc-seconds. Thus one could expect to get a lower bound for the curvature of space in this way using present day technology, no better than about 10^5 parsecs.

17 *Rotating reference frames*

We know both from elementary experience and from Newtonian dynamics that we can tell by means of local experiments if the reference frame we are using is rotating with respect to an inertial frame of reference, for example that determined by the fixed stars. As Newton himself observed, the water in a rotating bucket at rest on a rigidly rotating platform rises up the sides due to 'apparent 'centrifugal forces'. No trip to Paris is complete without a visit to the Panthéon to view Foucault's pendulum demonstrating the rotation of the earth, even on those days when the skies are covered by cloud and astronomical methods are not available.

Thus we do not expect the Minkowski metric of flat spacetime, when written in *co-rotating coordinates* $t, z, \rho, \tilde{\phi}$ to take the same form

$$ds^2 = dt^2 - dz^2 - d\rho^2 - \rho d\phi^2, \tag{213}$$

as it does in non-rotating cylindrical polars t, z, ρ, ϕ. For particles in uniform rigid rotation about the z-axis, $\phi = \tilde{\phi} + \omega t$, where ω is the rate of rotation in radians per sec, and $\tilde{\phi}, \rho, z$ labels the particles. Their velocity, relative to an inertial frame is $v(\rho) = \omega\rho$.

Substitution and completion of a square gives the so-called *Langevin form* of the flat Minkowski metric

$$\boxed{ds^2 = (1-\omega^2\rho^2)\big(dt - \frac{\rho^2 d\tilde{\phi}}{1-\omega^2\rho^2}\big)^2 - dz^2 - d\rho^2 - \frac{\rho^2}{1-\omega^2\rho^2} d\tilde{\phi}^2 \,.} \tag{214}$$

Clearly the Langevin form of the metric breaks down at the *velocity of light cylinder* $\rho = \omega^{-1}$. For $\rho > \omega^{-1}$ the particles on the platform would have to travel faster than light. Any physical platform must have smaller proper radius than ω^{-1}. Inside the velocity of light cylinder, $\rho = \omega^{-1}$, the metric is independent of time. Thus all distances are independent of time. In other words, the system really is in a state of rigid rotation.

17.1 Transverse Doppler effect and time dilation

We can read off immediately that for a particle at rest on the platform,

$$d\tau = \sqrt{1-\omega^2\rho^2}dt. \tag{215}$$

Thus means that a signal of duration dt consisting of n pulses sent from $\rho = \rho_e$ with frequency f_e, so that $f_e = \frac{n}{d\tau_e}$ will be received at $\rho = \rho_e$ with frequency

$f_o = \frac{n}{d\tau_o}$. Thus

$$\boxed{\frac{f_0}{f_e} = \sqrt{\frac{1-\omega^2\rho_e}{1-\omega^2\rho_o}} = \sqrt{\frac{1-v_e^2}{1-v_o^2}}\,.} \tag{216}$$

Thus, for example, photon emitted from somewhere on the platform and absorbed at the centre will be redshifted. In fact this is just the transverse Doppler effect in a different guise.

The effect was first demonstrated experimentally in 1960 by Hay, Schiffer, Cranshaw and Egelstaff making use of the Mössbauer effect for a 14-4 KeV γ rays emitted by a Co^{57} source with an Fe^{57} absorber [33].

17.2 *The Sagnac Effect*

The Langevin form of the flat Minkowski metric is invariant neither under time reversal nor reversal of the co-moving angle $\tilde{\phi}$ but it is invariant under simultaneous reversal of both. This gives rise to a difference between the behaviour of light moving in the direction of rotation compared with that moving in the opposite direction. The effect is really rather elementary but it has given rise to considerable discussion.

A light ray, passing along a light pipe for example, satisfies

$$dt = \frac{\rho^2\omega d\tilde{\phi}}{1-\omega^2\rho^2} + dl \tag{217}$$

where

$$dl^2 = \frac{dr^2+dz^2}{1-\omega^2\rho^2} + \frac{\rho^2 d\tilde{\phi}^2}{(1-\omega^2\rho^2)^2}\,. \tag{218}$$

If the light ray executes a closed curve C , as judged on the platform in the *pro-grade* (in the direction of rotation $d\tilde{\phi} > 0$) it will take a longer $t_=$ than the time taken t_- if it traverses the curve in the *retro-grade* sense ($d\tilde{\phi} < 0$). The time difference between these times is

$$t_+ - t_- = 2\omega \oint_C \frac{\rho^2 d\tilde{\phi}}{1-\omega^2\rho^2} = 2\omega \int\int_D \frac{\rho d\rho d\tilde{\phi}}{(1-\omega^2\rho)^2} \tag{219}$$

with $\partial D = C$. If the curve C is well inside the velocity of light cylinder and so $\omega^2\rho^2 << 1$, we find

$$t_+ - t_- = 4\omega A, \tag{220}$$

where A is the area of the domain D enclosed by the curve C.

The difference between the two travel times really has nothing much to do with Einstein's theory of Special Relativity; it may be ascribed to the simple fact that light has to travel further in one direction than in the other. The effect is now usually named after Sagnac. If C is taken to be around the equator, at sea level $t_+ - t_- = 414.8$ns. This is substantial and must be taken into account in the calibration of GPS receivers.

The Sagnac effect was first demonstrated using interferometry by Harres in 1911 and Sagnac in 1913 [35]. The method was later used to measure the absolute rotation rate of the earth by Michelson and Gale in 1925. Using an optical loop 2/5 miles wide and 2/5 mile long, they verified the shift of 236/1000 of a fringe predicted. Current laser technology allows the measurements of $0.00001\,\mathrm{deg}\,h^{-1}$. Of the experiment, Michelson said

> Well gentlemen, we will undertake this, although my conviction is strong that we shall prove only that the earth rotates on its axis, a conclusion which I think we may be said to be sure of already

17.3 Length Contraction

It is a striking mathematical fact is that if we use dl^2 in (218) as a spatial line element in the disc, i.e. set $dz = 0$ we find that

$$dl^2 = \frac{1}{4\omega^2}\Big(d\chi^2 + \sinh^2\chi d\tilde{\phi}^2\Big), \tag{221}$$

where $\omega\rho = \tanh(\frac{\chi}{2})$. The curved metric in brackets in (221) is that of the unit *pseudo-sphere*, i.e. two-dimensional hyperbolic space with constant Gauss curvature $= -1$.

However the *physical* line element or proper distance on the rotating disc , is *not* (218) but rather (from (214)

$$ds^2 = d\rho^2 + \frac{\rho^2 d\tilde{\phi}^2}{1-\omega\rho^2}. \tag{222}$$

The metric (222) is also curved. Note that radial directions, orthogonal to the motion agree with those in an inertial frame, but, as expected, circumferential distances are increased, relative to those in an inertial frame by a factor

$$\frac{1}{\sqrt{1-v^2(\rho)}}. \tag{223}$$

Because of this increase, it is not possible to embed isometrically the surface with coordinates $\rho, \tilde{\phi}$ and metric (222) as a surface of revolution in ordinary Euclidean 3-space E^3.

17.4 Mach's Principle and the Rotation of the Universe

The Austrian physicist and philosopher Ernst Mach was much exercised by ideas of absolute motion and absolute rest. In effect he pointed out that inertial frames of reference' in which Newton's laws hold, and those at rest with the 'fixed stars'need not necessarily coincide. In fact they do, to high accuracy, and this requires some sort of explanation. Nowadays this can be provided using the theory of *Inflation*. This requires developing some General Relativity, but it is possible to use Special Relativity to quantify the extent of the agreement.

The English physicist Stephen William Hawking, 17th Lucasian professor pointed out that if the universe were rotating about us, then distant light received here should suffer a direction dependent transverse Doppler shift . The photons which have travelled furthest in their journey toward us are part of the Cosmic Microwave Background (CMB). If the universe were rotating with angular velocity ω we would expect a temperature variation with angle of the form

$$T = \frac{T_0}{\sqrt{1 - \frac{\omega^2 r^2 \cos^2\theta}{c^2}}}\,, \tag{224}$$

where θ is the angle made by the line of sight with the axis of rotation. Roughly speaking we may take $\frac{r}{c}$ as the age of the universe and so the angle $\Delta\phi$ turned through in that time , in radians, is related to the maximum variation of temperature across the sky by

$$\Delta\phi = \sqrt{2\frac{\Delta T}{T}}. \tag{225}$$

Given that the measured temperature fluctuations are about one part in 10^5, the universe cannot have rotated by more than one hundredth of a turn since its beginning.

18 General 4-vectors and Lorentz-invariants

In general we set

$$v = \begin{pmatrix} v_0 \\ \mathbf{v} \end{pmatrix}, \text{ etc, and} \qquad v \cdot u = v^t \eta u = u^t \eta v = v_0 u_0 - \mathbf{v}.\mathbf{u} \quad . \tag{226}$$

One may check that $v \cdot w$ is a Lorentz invariant using (137) but it may also be seen from the following elementary but illuminating

Proposition *If v and w are 4-vectors then $v \cdot u = u \cdot v$ is a Lorentz invariant.*

Proof Evidently

$$(v+u)\cdot(v+u) = v\cdot v + v\cdot u + u\cdot v + u\cdot u = v\cdot v + u\cdot u = 2u\cdot v. \tag{227}$$

The left hand side is Lorentz-invariant and the first two terms on the left hand side are Lorentz. Thus $u \cdot v$ is Lorentz-invariant. In other words we can a calculate with the Minkowski inner product in the same way we would for any quadratic form.

18.1 4-velocity and 4-momentum

The world line of a particle in spacetime is a curve $f : \lambda \to x(\lambda)$ and is specified by giving its spacetime coordinates $t = t(\lambda)$ and $\mathbf{x} = \mathbf{x}(\lambda)$ as a function of some parameter [23] λ along the curve. One might thing it more natural to

[23] In this section λ has nothing to do with wavelength.

describe the motion by giving $\mathbf{x}$ as a function of t, and indeed this is possible for the particular choice $\lambda = t$ but this is distinguishes the time coordinate from the spatial coordinates but as we have discovered this is against the spirit of Relativity. Moreover, as we shall see, there are advantages in not making that choice.

18.2 4-velocity

We can then define the *tangent 4-vector* of the curve f by

$$T(\lambda) = \frac{dx}{d\lambda} = \begin{pmatrix} \frac{cdt}{d\lambda} \\ \frac{d\mathbf{x}}{d\lambda} \end{pmatrix}. \tag{228}$$

If we insist that under a Lorentz transformation the parameter λ is unchanged $\lambda \rightarrow \lambda$, the T will transform under Lorentz transformations (137) as a 4-vector.

Note that this would not be true if we made the choice $\lambda = t$. Suppose that $T \cdot T > 0$, then the curve f is said to be *timelike* and we can make the choice $\lambda = \tau$ where τ is proper time along the world line. We then define the *velocity 4-vector* often called the *4-velocity* by

$$U = \frac{dx}{d\tau} = \begin{pmatrix} \frac{cdt}{d\tau} \\ \frac{d\mathbf{x}}{d\tau} \end{pmatrix}. \tag{229}$$

It follows from the definition of proper time that

$$U \cdot U = c^2 (\frac{dt}{d\tau})^2 - (\frac{d\mathbf{x}}{d\tau})^2 = c^2 (\frac{d\tau}{d\tau})^2, \tag{230}$$

i.e.

$$\boxed{U \cdot U = c^2.} \tag{231}$$

In units in which $c = 1$, U is a unit timelike vector. We shall always assume that t is a strictly monotonically increasing function of λ, i.e. $T_0 > 0$, which in the timelike case means that $U_0 > 0$. Such a timelike tangent vector is called *future directed*

18.3 4-momentum and Energy

This is defined by

$$\boxed{p = m\,U,} \tag{232}$$

where m is a positive constant called the *rest mass* of the particle. If U is future directed then 4-momentum will also future directed, $p_0 > 0$. We have

$$\boxed{p \cdot p = m^2 c^2.} \tag{233}$$

Now

$$p_0 = \frac{mcdt}{d\tau} = \frac{mc}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} := \frac{E}{c}, \qquad \mathbf{p} = m\frac{d\mathbf{x}}{d\tau} = \frac{m\mathbf{v}}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}}. \tag{234}$$

In other words

$$\boxed{p = \begin{pmatrix} \frac{E}{c} \\ \mathbf{p} \end{pmatrix}}. \tag{235}$$

The quantity E is the energy of the particle, as will be justified shortly. Now

$$p \cdot p = \frac{E^2}{c^2} - \mathbf{p}^2 = m^2 c^2, \tag{236}$$

thus

$$\boxed{E = \sqrt{m^2 c^4 + \mathbf{p}^2 c^2}.} \tag{237}$$

and moreover

$$\boxed{\mathbf{p} = \frac{E}{c}\mathbf{v}.} \tag{238}$$

18.4 Non-relativistic limit

For small $\frac{\mathbf{v}^2}{c^2}$ we have up to terms of $\mathcal{O}(\mathbf{v}^3)$,

$$\mathbf{p} = m\mathbf{v} + \dots, \qquad E = mc^2 + \frac{1}{2}m\mathbf{v}^2 + \dots . \tag{239}$$

We call mc^2 the *rest mass energy*. Note that E is non-zero even if the particle is at rest. Therefore it is reasonable to define the *kinetic energy* by

$$T = E - mc^2. \tag{240}$$

18.5 Justification for the name energy

If we suppose a general equation of motion of the form

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \tag{241}$$

then the rate of doing work W on the particle is

$$\frac{dW}{dt} = \mathbf{F}.\mathbf{v} = \mathbf{v}.\frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\mathbf{v}.\mathbf{p}) - \mathbf{p}\frac{d\mathbf{v}}{dt}. \tag{242}$$

That is

$$dW = d(\mathbf{v}.\mathbf{p}) - \mathbf{p}.d\mathbf{v}. \tag{243}$$

In the special case that we set

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}}, \tag{244}$$

we find

$$\frac{dW}{dt} = \frac{d}{dt}\left(\frac{m\mathbf{v}^2}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}}\right) - \frac{m\mathbf{v}}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}}.\dot{\mathbf{v}} \tag{245}$$

$$= \frac{d}{dt}\left(\frac{m\mathbf{v}^2}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} + mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}}\right) = \frac{dE}{dt}. \tag{246}$$

That is

$$\boxed{\mathbf{F}.\mathbf{v} = \frac{dE}{dt}.} \tag{247}$$

It is reasonable therefore to regard E or $T = E - mc^2$ as the energy of the particle. In fact we usually include the rest mass energy mc^2 in the energy because in energetic nuclear processes in which particles decay into other particles of different rest masses, for instance, the rest mass term must be included in the total energy budget.

That E really is the type of energy you might pay your electricity bill to acquire gas been demonstrated by timing rapidly electrons to find their velocity and then absorbing them into a calorimeter to measure their energy in calories. Sadly for those who dream of perpetual motion, Einstein's formula (237) was verified [13].

18.6 *Hamiltonian and Lagrangian*

Quite generally in dynamics, for example when considering excitations in condensed matter systems, we define what is called the *Hamiltonian function* $H(\mathbf{p})$ of a free particle whose spatial momentum is $\mathbf{p}$ by

$$dH = \mathbf{v}.d\mathbf{p}, . \tag{248}$$

If the system is to be conservative, H must be an exact differential and we have

$$H(\mathbf{p}) = W. \tag{249}$$

We also define the *Lagrangian function* $L(\mathbf{v})$ of a particle with spatial velocity $\mathbf{v}$ as the *Legendre transform* of the Hamiltonian, i.e.,

$$L + H = \mathbf{p}.\mathbf{v}, \tag{250}$$

so that from (248, 243)

$$\mathbf{v} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}. \tag{251}$$

In the special case of a relativistic particle one has

$$H = \sqrt{m^2c^4 + c^2\mathbf{p}^2}, \qquad L = -mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}}. \tag{252}$$

19 Particles with vanishing rest mass

Einstein's theory allows for the possibility of particles which cannot be treated in Newton's mechanics, those whose speed is strictly constant.The constant value of the speed can only be, by Einstein's Principle of Relativity, exactly that of light. It turns out that by using only momentum $\mathbf{p}$ and energy E as the basic variables the basic equations still make sense if we set $m = 0$.

We have in general $\mathbf{v} = \frac{E}{c}\mathbf{v}$ and so if $|\mathbf{v}| = c$, we have

$$\boxed{E = c|\mathbf{p}|} \tag{253}$$

Now since

$$p = \begin{pmatrix} \frac{E}{c} \\ \mathbf{p} \end{pmatrix}, \tag{254}$$

we have

$$\boxed{p \cdot p = 0}. \tag{255}$$

Particles of this type include the *photon* which is responsible, according to quantum electro-dynamics, for electromagnetic phenomena and more speculatively the *graviton* which, according to quantum gravity, is responsible for gravitational phenomena. In addition there are three types of *neutrinos*, ν_e, ν_μ, ν_τ associated with the electron, muon and tau particle respectively.

19.1 Equality of photon and neutrino speeds

At 7:35:40 UT[24] on 23 February 1987 electron neutrinos $\bar{\nu}_e$ from the Large Magelllanic Cloud arrived in Japan and were detected using the KAMIOKANDE neutrino telescope. By 10:38 UT the same day, the first optical brightening of what is now known as the supernova SN1987A were seen. Thus the travel time for neutrinos and photons (160,000 years) differed by less than 3 hours. It follows that their speeds differ by less than two parts in an American billion (10^9) [17].

For most purposes therefore one may regard neutrinos as being massless,like the photon. Other experiments however based on neutrinos arriving here on earth from the sun indicate that they do have a very small mass, of the order 10^{-4} eV.

Example If the neutrino actually has a mass m and energy E, and the SN1987 is at a distance L from us, then if $T_\gamma = \frac{L}{c}$ is the transit time of the photon and $T_\nu = \frac{L}{v}$ that of a neutrino, we have

$$\frac{mc^2}{E} = \sqrt{\frac{(T_\nu - T_\gamma)(T_\nu + T_\gamma)}{T_\nu^2}}. \tag{256}$$

20 Particle decays collisions and production

20.1 Radioactive Decays

Perhaps the simplest process one may consider is the decay of a particle of rest mass m_1 into particles of rests mass m_2 and m_3. To get a Lorentz-invariant law

[24]Universal time , i.e. almost exactly Greenwich Mean Time, GMT

of decay we express it in terms of 4-vectors. The simplest possibility would be

$$\boxed{p_1 = p_2 + p_3.} \tag{257}$$

The components of (257) give four equations: the conservation of

$$\boxed{\text{Energy} \quad \frac{E_1}{c} = \frac{E_2}{c} + \frac{E_3}{c}, \quad \text{and} \quad \text{Momentum} \quad \mathbf{p}_1 = \mathbf{p}_2 + \mathbf{p}_3.} \tag{258}$$

Suppose the decaying particle is at rest in frame S. Then

$$p_1 = (m_1 c, 0) \Rightarrow \mathbf{p}_2 + \mathbf{p}_3 = 0, \tag{259}$$

and hence $|\mathbf{p}_2| = |\mathbf{p}_3| = p$. The two particles produced move off with equal and opposite momentum. We also have

$$E_1 = m_1 c^2 = \sqrt{m_2^2 c^4 + p^2 c^2} + \sqrt{m_3^2 c^4 + p^2 c^2} \Rightarrow m_1 \geq m_2 + m_3. \tag{260}$$

Particle 1 can only decay into particle 2 plus particle 3 if its rest mass exceeds the sum of the rest masses of the products. If this is true, then a solution for p always exists. Put another way, the kinetic energy liberated $T = T_2 + T_2 = (m_1 - m_2 - m_3)c^2$ and this must be positive and this must come from the original rest mass energy.

In general one expects that unless there is some reason, for example a conservation law like that of electric charge, that heavy particles will always be able to decay into lighter particles. Only the particle with least rest mass can be stable. This is the electron. It could, in principle decay into two photons, but photons carry no electric charge and hence this is impossible.

20.2 Impossibility of Decay of massless particles

Suppose that

$$p_1 = p_2 + p_3, \tag{261}$$

with p_1 massless, i.e.

$$p_1 \cdot p_1 = 0. \tag{262}$$

It follows that

$$p_2^2 + p_3^2 + 2p_2 \cdot p_3 = 0 = m_2^2 c^2 + m_3^2 + 2p_2 \cdot p_3. \tag{263}$$

But this is impossible because if for p_2 and p_3 future directed timelike or lightlike

$$p_2 \cdot p_3 \geq 0. \tag{264}$$

To prove (264) note that

$$p_2 \cdot p_3 = \frac{E_2 E_3}{c^2} - \mathbf{p}_2 . \mathbf{p}_3 = \sqrt{m_1^2 c^2 + \mathbf{p}_2^2} \sqrt{m_3^2 c^2 + \mathbf{p}_3^2} - \mathbf{p}_2 . \mathbf{p}_3 \tag{265}$$

but the left hand side is bounded below by

$$|\mathbf{p}_2||\mathbf{p}_3| - \mathbf{p}_2.\mathbf{p}_3 \geq 0, \tag{266}$$

since, by the usual Schwarz inequality, $\mathbf{v}.\mathbf{u} \leq |\mathbf{v}||\mathbf{u}|$, for any pair of 3-vectors $\mathbf{v}$ and $\mathbf{u}$.

Note that equality can only be attained if p_2 and p_3 are two parallel lightlike vectors. However the decay of a photon for example into two collinear photons should perhaps be better thought of as superposition. Moreover in quantum mechanical terms it is a process with vanishing small phase space volume.

20.3 Some useful Inequalities

For a timelike or light 4-vector v we define $|v| = \sqrt{v \cdot v}$. The working in the previous subsection can be re-arranged to show

Proposition(Reverse Schwarz Inequality) *If p_2 and p_3 are two future directed timelike or lightlike vectors, then*

$$p_2 \cdot p_3 \geq |p_2||p_3|. \tag{267}$$

Similarly

Proposition(Reverse triangle Inequality) *if p_1, p_2 and p_3 are all timelike or light like sides of a triangle then*

$$|p_1| \leq |p_2| + |p_3|. \tag{268}$$

The reverse triangle inequality can be extended to give

Proposition *If $p - 1, p_2, \dots p_{n+1}$ are future directed timelike or null and*

$$p_1 = p_2 + \dots + p_{n+1} \tag{269}$$

then

$$|p_1| \geq |p_1| + \dots + |p_{n+1}|,. \tag{270}$$

A slight extension of the same working yields

Proposition*If $p \cdot q \geq 0$ for all future directed timelike and lightlike 4-vectors p, then q is future directed timelike or lightlike.*

Finally

Proposition(Convexity of the future light cone) *If p and q are future directed and timelike then so is any positive linear combination $ap + bq$, $a \geq 0$, $b \geq 0$.*

20.4 Impossibility of emission without recoil

A particle of rest mass m cannot emit one or more particles keeping its rest mass constant. Thus free electron cannot emit one or more photons. To see why not, suppose

$$p_1 = p_2 + p_3, \tag{271}$$

with $|p_|| = |p_2|$ and p_3 future directed and timelike. We get

$$p_1^2 = p_2^2 + p_3^2 + 2p.p_3, \Rightarrow p_3^2 + 2p_2 \cdot p_3 = 0, \tag{272}$$

which is impossible. This has an application to so called *bremsstrahlung radiation* emitted by an accelerated electron. This can only occur if there is some other body or particle to take up the recoil.

20.5 Decay of a massive particle into one massive and one massless particle

For example one could consider pion($m_{\pi^-} = 140$ Mev) decaying into a muon ($m_\mu = 105$ Mev) and an anti-muon-neutrino.

$$\pi^- \rightarrow \mu^- + \bar{\nu}_\mu \tag{273}$$

We have $p_2^2 = 0$, thus

$$p_1 = p_2 + p_3, \Rightarrow (p_1 - p_3)^2 = 0 = p_1^2 + p_3^2 - 2p_1 \cdot p_3. \tag{274}$$

This gives

$$m_1^2 c^2 + m_3^2 c^2 = 2(\frac{E_1 E_2}{c^2} - \mathbf{p}_1 . \mathbf{p}_3). \tag{275}$$

Suppose we are in the rest frame of particle 1 so $p_1 = (m_1 c, 0)^t$. We get

$$m_1^2 c^2 = m_3^2 = 2m_1 E_3, \Rightarrow E_3 = \frac{m_1^2 + m_3^2}{2m_1} c^2. \tag{276}$$

The relativistic gamma factor of the third particle is given by

$$\gamma = \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} = \frac{E_3}{m_3 c^2} = \frac{1}{2}(\frac{m_1}{m_3} + \frac{m_3}{m_1}). \tag{277}$$

20.6 Decay of a massive particle into two massless particles

For example one might consider the decay of a neutral pion ($m_{pi^0} = 135$MeV) into two photons (symbol γ) with lifetime $\tau = 8.4 \times 10^{-17} s$.

$$\pi^0 \rightarrow 2\gamma \tag{278}$$

We have (because $p_2^2 = p_3^2 = 0$)

$$p_1 = p_2 + p_3, \Rightarrow p_1^2 = 2p_2 \cdot p_3 \,. \tag{279}$$

Thus

$$m_1^2 c^2 = 2(\frac{E_2 E_3}{c^2} - \mathbf{p}_2.\mathbf{p}_3). \tag{280}$$

Now for a massless particle

$$\mathbf{p}_2 = \mathbf{n}_2 \frac{E_2}{c} \tag{281}$$

where $\mathbf{n}_3$ is a unit vector in the direction of motion. Thus if $\mathbf{n_2}.\mathbf{n}_3 = \cos\theta$, where θ is the angle (not rapidity!) between the directions of the two massless particles, we have

$$\frac{m_1^2 c^4}{2E_2 E_3} = (1 - \mathbf{n}_2.\mathbf{n}_3) = (1 - \cos\theta) = 2\sin^2(\frac{\theta}{2}). \tag{282}$$

Thus

$$\sin^2(\frac{\theta}{2}) = \frac{m_1^2}{4E_2 E_3}. \tag{283}$$

There are two simple cases

(i) Particle 1 decays from rest. We have $E_2 = E_3 = \frac{1}{2} m_1^2 c^2$. This implies $\theta = \pi$.

(ii) Particle 1 is moving along say the 1-axis with relativistic gamma factor $\gamma = \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}}$ and the two photons are emitted symmetrically making an angle $\frac{\theta}{2}$ to the 1-axis. We have $E_2 = E_3 = \frac{1}{2} m_1 c^2 \gamma$, whence

$$\sin^2(\frac{\theta}{2}) = \frac{1}{\gamma^2}. \tag{284}$$

If particle 1 is moving very fast, $\gamma >> 1$ then θ will be very small. This is the headlight effect.

21 Collisions, centre of mass

We have in general

$$p_1 + p_2 = p_3 + p_4. \tag{285}$$

The 4-vector $p_+ p_= m_{\rm com} U$, with $U \cdot U = -c^2$ defines the *centre of mass energy*, that is $m_{\rm com} c^2$ is the total energy in the *centre of mass frame*, i.e. in the frame in which $\mathbf{p}_1 + \mathbf{p}_2 = 0$.

Th relation

$$(p_1 + p_2)^2 = (p_3 + p_4)^2 = m_{\rm com}^2 c^2 \tag{286}$$

often brings simplifications to the algebra.

As an example, if a particle of mass m and 3-momentum $\mathbf{p}$ collides with a particle of mass m which is at rest, then

$$m_{\rm com} = \sqrt{2\frac{E}{c^2}M + M^2 + m^2}, \tag{287}$$

where $E = \sqrt{m^2c^4 + \mathbf{p}^2c^2}$ is the energy of the incident particle.

The quantity $m_{\rm com}c^2$ is usually taken as a measure of the *available energy* in such collisions. For large E it is much smaller than E, rising as $E^{\frac{1}{2}}$ rather than linearly with energy. This is because so much energy must go into the recoil. An extreme case is provided by Ultra-High Energy Cosmic rays. These hit protons (mass 938MeV) in the upper atmosphere with energies up to $10^{21}eV = 10^9$TeV. They are by far the fastest particles known to us. However energy available for nuclear reactions is no more than about 10^3TeV.

21.1 Compton scattering

In this process, for the discovery of which Compton was awarded the Nobel prize in 1927, an X-ray photon is scattered of an electron which is initially at rest. We have $p_1^2 = p_3^2 = 0$, $p_2^2 = p_4^2 = m_e^2c^2$, $p_2 = (m_ec, 0)^t$. Now

$$(p_1 - p_3) = p_4 - p_1, \Rightarrow (p_1 - p_3)^2 = (p_4 - p_2)^2. \tag{288}$$

This gives

$$-2p_1 \cdot p_3 = 2m_e^2c^2 - 2p_4 \cdot p_2\,. \tag{289}$$

This gives, using the method for photons used earlier

$$-2\frac{E_1E_3}{c^2}(1 - \cos\theta) = 2m_e^2c^2 - \frac{2E_4m_ec}{c^2} \tag{290}$$

But energy conservation gives

$$E_4 = E_1 + m_ec^2 - E_3. \tag{291}$$

Substitution and simplification gives

$$(1 - \cos\theta) = m_ec^2\Big(\frac{1}{E_3} - \frac{1}{E_1}\Big). \tag{292}$$

According to quantum theory $E_1 = hf_1 = \frac{hc}{\lambda_3}$, where h is Planck's constant, f_1 is the frequency and λ_1 the wavelength of the incident photon and f_3 and λ_3 that of the scattered photon. We get

$$(1 - \cos\theta) = \frac{m_ec}{h}(\lambda_3 - \lambda_1). \tag{293}$$

Clearly the wavelength of the scattered photon is longer than that of the incident photon because kinetic energy has been imparted to the electron.

21.2 Production of pions

Protons in cosmic rays striking the upper atmosphere may produce either neutral (π^0) or positively charged (π^+) pions according to the reactions

$$p + p \rightarrow p + p + \pi^0 \qquad \text{or} \qquad p + p \rightarrow p + n + \pi^+ \tag{294}$$

respectively, where n is the neutron. Since the mass of the proton is 1836.1 times the electron mass and that of the neutron 1838.6, which is why the latter can decay to the former according to the reaction

$$n \rightarrow p + e^- + \bar{\nu}_e \tag{295}$$

in about 13 minutes, we ignore the difference and call the common mass M. Numerically it is about 938MeV. Despite the fact that the mass of the neutral pion π^o is 264 times the electron mass and that the charged pions $\pi^{\pm}$ are both 273.2 times the electron mass, the latter cannot decay into the former by conservation of electric charge. In either case, we call the mass m. Its value is about 140Mev. If T is the kinetic energy of the incident proton and p is its momentum, then equating the invariant $(p_1 + p_2)^2 = (p_3 + p_4 + p_5)^2$ and using the inequality

$$|p_3 + p_4 + p_5| \geq (m_3 + m_4 + m_5)c, \tag{296}$$

we get

$$(T + 2Mc^2)^2 - c^2p^2 \geq (2M + m)^2c^4. \tag{297}$$

Using what is sometimes called the *on-shell condition*

$$(\frac{E}{c} + Mc)^2 - p^2 = M^2c^2, \tag{298}$$

one finds that the T^2 terms cancel and one obtains the *threshold*

$$T \geq 2mc^2(1 + \frac{m}{4M}). \tag{299}$$

This is about 290Mev in the present case.

21.3 Creation of anti-protons

If a proton p collides at sufficiently high speed against a stationary proton a proton-anti-proton pair can be created as in the following reaction

$$p + p \rightarrow p + p + (p + \bar{p}) \tag{300}$$

($\bar{p}$ denotes an anti-proton). One might have thought that the least kinetic energy T required for this process is $2m_pc^2$, but this is not correct. Most of the incident kinetic energy goes into the kinetic energy of the recoiling proton. In fact the *threshold* , i.e., the minimum energy required, is $6m_pc^2 = 5.6$MeV. To see why, note that 4-momentum conservation gives

$$p_1 + p_2 = p_1 + p_2 + p_3 + p_4 \tag{301}$$

thus

$$|p_1 + p_2| = |p_1 + p_2 + p_3 + p_4| \geq |p_1| + |p_2| + |p_3| + |p_4| = 4m_p c. \qquad (302)$$

(we use the fact that $m_p = m_{\bar{p}}$.) Now $p_1 = (m_p c, 0)$ and $p_2 = (\frac{E}{c}, \mathbf{p})$, where E is the total energy, including rest mass energy of the incident proton and $\mathbf{p}$ is 3-momentum. Thus

$$|p_1 + p_2| = \sqrt{(\frac{E}{c} + m_p c)^2 - \mathbf{p}^2} \geq 4m_p c^2. \qquad (303)$$

But

$$\mathbf{p}^2 = \frac{E^2}{c^2} - m_p^2 c^2. \qquad (304)$$

Simplifying gives

$$E \geq 7m_p c^2, \Rightarrow T \geq 6m_p c^2 \,. \qquad (305)$$

The first production of anti-protons on earth was achieved by Chamberlain and Segré at the Berkley Bevatron in California. This linear accelerator was built to be capable of accelerating protons up to energies of 6.6 BeV [25] Chamberlain and Segré received the Nobel Prize in 1959 for this work.

21.4 Head on collisions

In this case we have

$$p_1 = (Mc, 0)^t \qquad p_2 = (\frac{E}{c}, p, 0.0)^t \qquad (306)$$

with

$$E^2 - c^2 p^2 = m^2 c^4. \qquad (307)$$

By 'head on'we mean

$$p_3 = (\frac{E_2}{c}, p_M, 0, 0)^t \qquad p_4 = (\frac{E_4}{c}, p_M, 0, 0)^t \qquad (308)$$

with

$${E_3}^2 - c^2 {p_m}^2 = m^2 c^2, \qquad \text{and} \qquad {E_4}^2 - c^2 {p_M}^2 = M^2 c^2. \qquad (309)$$

The two conservation equations are

$$E + M = \sqrt{c^2 p_m^2 + c^4 p_m^2} + \sqrt{c^2 p_M^2 + c^4 M^2}, \qquad p = p_m + p_M. \qquad (310)$$

Our strategy is to eliminate p_m and solve for p_M in terms of E. This leads to some moderately heavy algebra so we will go through all the steps, if only to illustrate the superiority of 4-vector methods. We set $c = 1$ during the intermediate stages of the calculation.

[25] A BeV is nowadays called a Gev.

Using momentum conservation, the energy conservation equation gives

$$E + M - \sqrt{p_M^2 + M^2} = \sqrt{(p - p_M)^2 + m^2}. \tag{311}$$

Squaring gives

$$(E+M)^2 - 2(E+M)\sqrt{p_M^2 + M^2} + p_M^2 + M^2 = (p - p_M)^2 + m^2. \tag{312}$$

Thus

$$E^2 + 2ME + 2M^2 + p_M^2 - 2(E+M)\sqrt{p_M^2 + M^2} = p^2 - 2p_m p + m^2. \tag{313}$$

But $E^2 = p^2 + m^2$, and hence,

$$2EM + 2pp_M + 2M^2 = 2(E+M)\sqrt{p_M^2 + M^2}. \tag{314}$$

Dividing by two and squaring once more gives

$$M^2(E+M)^2 + 2M(M+E)pp_M + p^2 p_M^2 = (E+M)^2 M^2 + (E+M)^2 p_M^2. \tag{315}$$

Thus, taking out a factor of p_M,

$$2M(E+M)p = ((E+M)^2 - p^2)p_M \tag{316}$$

Finally we get, restoring units,

$$\boxed{p_M = \frac{2Mp(\frac{E}{c^2} + M)}{2M\frac{E}{c^2} + M^2 + m^2}, \qquad p_m = \frac{p(m^2 - M^2)}{2M\frac{E}{c^2} + M^2 + m^2},} \tag{317}$$

with

$$E = \sqrt{m^2c^4 + c^2p^2}. \tag{318}$$

You should check that in the non-relativistic limit, $c \to \infty$, one recovers the usual Newtonian formulae. Just as in that case, if the incident particle is more massive than the particle it hits (i.e.$m > M$) it moves forward after the collision, while if it is less massive it reverse its direction. If the two particles are perfectly matched, (i.e. $m^2 = M^2$) all the incident energy will be transferred to the target particle. By contrast if, for example, the target is very massive, (i.e. $M \to \infty$), the incident particle is reflected back with the same speed it arrives with.

If the incident energy is very large compared with the rest masses of both itself and the target then

$$p_M \to p, \qquad p_m \to \frac{(m^2 - M^2)c}{2M}. \tag{319}$$

All the incident momentum is transferred to the target.

21.5 Example: Relativistic Billiards

It is well known to players of billiards that if ball is struck and the process is *elastic*, i.e. no energy is lost, and the collision is not exactly head-on, then the two balls move off at an angle $\theta = 90\,\text{deg}$ between each others direction, as seen in the frame of reference of the table.

In relativistic billiards this is not so. The angle θ depends on the ratio of the energies imparted to the two balls and the incident kinetic energy T. If the balls have mass m and each emerges form the collision carrying the same energy, then one finds

$$2\sin^2\frac{\theta}{2} = \frac{4mc^2}{T + 4mc^2}. \tag{320}$$

If $T << mc^2$ we recover the non-relativistic result. By contrast, if $T >> mc^2$ we find $\theta \to 0$. This is another manifestation of the headlight effect.

21.6 Mandelstam Variables.

Consider a four-body scattering

$$a + b \to c + d, \tag{321}$$

with particles of masses m_a, m_b, m_c, m_d. Conservation of 4-momentum gives

$$p_a + p_b = p_c + p_d, \tag{322}$$

where p_a, p_b, p_c, p_d are all taken to be future directed. One has

$$p_a \cdot p_a = m_a^2\,, \qquad \text{etc.} \tag{323}$$

The *energy momentum in the centre of mass frame* is given by

$$p_a + p_b = p_c + p_d \tag{324}$$

and thus the energy available for any reaction, i.e. the *centre of mass energy* is $\sqrt{s}$, where

$$s = (p_a + p_b)^2 = (p_c + p_d)^2. \tag{325}$$

Because $p_a + p_b$ is non-spacelike s is non-negative, $s \geq 0$. The *energy momentum transferred* from particle a to particle c

$$t = (p_a - p_c)^2 = (p_b - p_d)^2\,. \tag{326}$$

By the reverse Schwartz inequality $t \leq (m_a - m_c)^2$. In particular if $m_a = m_c$, the *momentum transfer* $p_a - p_c$ is spacelike. Now the four vectors $p_a, p_b, -p_c, -p_d$ are not linearly dependent, they lie in a timelike three-plane. Their endpoints thus define a tetrahedron. In a tetrahedron, the lengths of opposite sides are equal and s and t give the lengths squared of two of the three possible pairs of opposite sides. The remaining length squared is given by

$$u = (p_a - p_d)^2 = (p_b - p_c)^2\,. \tag{327}$$

From three vectors linearly independent vectors with 12 components one expects to be able to form only $6 = 12 - 6$ independent Lorentz scalars and hence the seven quantities $m_a, m_b m_c, m_d, s, t$, can not be independent. A simple calculation shows that

$$s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2 \, . \tag{328}$$

If the masses m_a, m_b, m_c, m_d are fixed s, t, u may be thought of as a set over three over complete coordinates on a two-dimensional space of scattering states since they are constrained by the relation (328).

Example all four masses equal.

In the centre of mass frame the ingoing 3-momenta are equal and opposite as are the outgoing momenta. Their common magnitude is $\sqrt{\mathbf{p}^2}$ and the scattering angle is θ, one has

$$s = 4(m^2 + \mathbf{p}^2) \qquad t = -2\mathbf{p}^2(1 - \cos\theta) \, , \qquad u = -2\mathbf{p}^2(1 + \cos\theta) \, . \tag{329}$$

The allowed range of s is thus $s \geq m^2$ and of t $0 \geq t \geq -4\mathbf{p}^2$.

It is convenient to regard s, t, u as *triangular coordinates* in the plane. In particular we regard them as giving the oriented perpendicular distances from the sides of an equilateral triangle of height $m_a^2 + m_b^2 + m_c^2 + m_d^2$. The sides of the triangle are thus given by $s = 0$,$t = 0$ and $u = 0$. Not all points in the plane correspond to physically allowed values of s, t, u. For example, in the case of four equal masses, the physical region lies outside the equilateral triangle and occupies an infinitely large 60 deg sector starting from the vertex $s = 4m^2$ and bounded by two half lines given by $t = 0$ and $u = 0$ obtained by producing the two sides adjacent to that vertex.

It is possible to give physical meaning but this requires the idea of *anti-particles.*

22 Mirrors and Reflections

22.1 *The Fermi mechanism*

Fermi proposed a mechanism for accelerating cosmic rays. The details have something in common with a well known thought experiment in which photons are confined within a cylinder and work done on them by means of a slowly (i.e. *adiabatically*) moving piston. What Fermi had in mind was a large cloud of mass M moving slowly with velocity $u << c$. Incident on it is a relativistic particle with momentum p and energy $E = pc$. The particle is scattered back with momentum $-p'$ and energy $E' = E + \delta E = -p'$. The velocity energy of the cloud becomes $u' = u + du$ but it;s rest mass is unchanged. In this approximation, energy and momentum conservation become

$$\frac{1}{2}Mu^2 + E = \frac{1}{2}M(u + \delta u)^2 + E', \qquad Mu - p = M(u + \delta u) + p' \, . \tag{330}$$

We have not include the rest mass energy Mc^2 because it cancels on both sides of the energy conservation equation. One gets

$$Mu\delta u = -E' - E, \qquad \text{and} \qquad M\delta u = -(p' - p) = \frac{E + E'}{c}. \tag{331}$$

Thus

$$\boxed{M\delta u \approx -\frac{2E}{c}, \qquad \delta E = 2E\frac{u}{c}\,.} \tag{332}$$

The first equation of (332) tells us by how much the cloud slows down, and the second that the energy of the particles is multiplied by a factor $(1 + 2\frac{u}{c})$ which is greater than unity as long as u is positive. Note that this factor depends only on u. It does not depend on the mass of the cloud.

Fermi imagined particles bouncing backwards and forward between two clouds which were slowly approaching each another. The energy of the trapped particle would go up like a geometrical progression and it would seem that very high energies could be achieved. In practice, while it is easy to see why particles might scatter off such clouds, it is not so easy to see how they would get trapped between two clouds and so Fermi's theory has fallen out of favour. However it is interesting as an illustration of scattering. In fact the cloud behaves like a mirror and the effect may be understood heuristically as a manifestation of the Doppler effect. The incoming particle has energy $E(1 + \frac{u}{c})$ with respect to a rest frame sitting on the cloud. on the cloud with frequency $E(1 + \frac{u}{c})$ In the rest frame of the cloud it is re-emitted with this frequency in the opposite direction and this is seen in the original rest frame as having energy $(1+\frac{u}{c})(1+\frac{u}{c})E \approx (1+2\frac{u}{c})E$.

In fact reflection problems of this type can also be solved by composing Lorentz transformations.

Now if L is the distance between the two clouds, then the time between bounces is $\frac{2L}{c}$ and in this time the distance has diminished by an amount $\delta L = -\frac{uL}{c}$. Thus during an adiabatic change

$$\boxed{\frac{\delta L}{L} = -\frac{\delta E}{E}, \qquad EL = \text{constant.}} \tag{333}$$

If, for example, we think of a photon with frequency f or wavelength λ find from (333) $f \propto \frac{1}{L}$ or $\lambda \propto L$.

22.2 *Relativistic Mirrors*

Suppose the mirror occupies the region of spacetime

$$e_1 \cdot x > -d\,, \tag{334}$$

where e_1 is a unit spacelike vector, corresponding geometrically to the *normal* to the timelike hyperplane,

$$e_1 \cdot x = -d, \tag{335}$$

and d a constant giving the distance of the plane from the origin.

If a particle, for example a light ray, with 4-momentum p is incident on the mirror and *elastically* or *specularly* reflected off the mirror the reflected particle, has 4-momentum

$$R_1(p) = p + 2e_i(e.p)\,. \tag{336}$$

The possibly unfamiliar sign in (336) is because the normal satisfies

$$e_1 \cdot e_1 = -1 \tag{337}$$

Note that the *reflection operator* $R_1(p)$ leaves the rest mass unchanged since

$$R_1(p) \cdot R_1(p) = p \cdot p\,. \tag{338}$$

In other words $R_1(p)$ is an *isometry*, it leaves Minkowski lengths unchanged.

In components, if

$$e_1 = (\beta\gamma, \gamma, 0, 0)^t\,, \qquad p = (E, -p_1, p_2, p_3)^t, \tag{339}$$

where β and γ have there usual meanings in terms of the velocity v of the mirror, and we take p_1 to be positive so that the incident particle is moving from right to left. Then reflected particle has momentum

$$R_1(p) = (E\frac{1+v^2}{1-v^2} + \frac{2v}{1-v^2}p_1, +p_1\frac{1+v^2}{1-v^2} + \frac{2v}{1-v^2}E, p_2, p_3)^t. \tag{340}$$

As an example, consider the mirror at rest, $v = 0$.

$$R_1(p) = (E, -p_1, p_2, p_3)\,. \tag{341}$$

The energy E is unchanged and only the component of momentum perpendicular to the mirror is reversed. Another interesting case is of a light ray or photon moving perpendicular to the mirror. Thus $p = (E, -E, 0, 0)^t$, and

$$R_1(p) = \Big(\sqrt{\frac{1+v}{1-v}}\Big)^2 (E, E, 0, 0)^t \tag{342}$$

We see that the light ray is reflected backward with two factors of the Doppler shift as described in the previous section.

Note that if the mirror is moving and the incident photon is not moving exactly perpendicularly to the mirror then *the angle of reflection will not equal the angle of incidence*.

22.3 *Corner Reflectors on the Moon*

During the first Apollo landing on the moon in 1969 a corner reflector was left on the lunar surface. Within weeks laser photon pulses sent from the Lick Observatory in California reflected off the reflector and received back in California. Over the past 30 years or so the number of reflectors and the precision has been increased so that at any given time, the distance to the moon can be determined to better than 1cm.

A corner reflector effect three successive reflections in there mutually perpendicular mirrors, the walls of an *orthant* in the rest frame of the reflector. If the walls have spacelike normals e_1, e_2, e_3, then the effect of three reflections is given by

$$R_1R_2R_3(p) = P(p) = e_0(p\cdot)e_0 + e_1(e_1 \cdot p) + e_2(e_2 \cdot p) + e_3(e_3 \cdot p). \qquad (343)$$

The operator $P(p)$ is called *spatial parity* and reverses the spatial components of any 4-vector it acts on. Thus, according to an observer in its rest frame, a the corner reflector send back a photo in precisely the direction it comes from.

22.4 Time reversal

22.5 Anti-particles and the CPT Theorem

23 4-acceleration and 4-force

Given a timelike curve $x = x(\tau)$, where τ is propertime along the curve,, we define its *acceleration 4-vector* by

$$\boxed{a = \frac{dU}{d\tau} = \frac{d^2x}{d\tau^2}\,.} \qquad (344)$$

But

$$U \cdot U = c^2, \Rightarrow \frac{dU}{d\tau} \cdot U + U \cdot \frac{dU}{d\tau} = 0, \Rightarrow 2U \cdot \frac{dU}{d\tau} = 0\,. \qquad (345)$$

Thus 4-acceleration and 4-velocity are orthogonal

$$\boxed{a \cdot U = 0\,.} \qquad (346)$$

Thus, since U is timelike, a must be spacelike. Geometrically, U is the *unit tangent vector* of the world line and a is its *curvature vector*.

23.1 Relativistic form of Newton's second law

This may be written as

$$\boxed{m\frac{d^2U}{d\tau^2} = G, \qquad \text{or} \qquad ma = G\,,} \qquad (347)$$

where the *4-force* G is not an arbitrary 4-vector but must be orthogonal to U,

$$\boxed{a \cdot G = 0\,.} \qquad (348)$$

23.2 Energy and work done

We have

$$p = mU, \Rightarrow \frac{dp}{d\tau} = G, \Rightarrow \frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\frac{d}{dt}\begin{pmatrix}\frac{E}{c}\\ \mathbf{p}\end{pmatrix} = \begin{pmatrix}G_0\\ \mathbf{G}\end{pmatrix}. \tag{349}$$

This becomes

$$\frac{dE}{dt} = c\sqrt{1-\frac{\mathbf{v}^2}{c^2}}G_0, \qquad \frac{d\mathbf{p}}{dt} = \sqrt{1-\frac{\mathbf{v}^2}{c^2}}\mathbf{G}. \tag{350}$$

Thus

$$\boxed{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}\mathbf{G} = \mathbf{F},} \tag{351}$$

where **F** is the old fashioned Newtonian force.

Now

$$G\cdot U = G_0\frac{c}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} - \frac{\mathbf{G}.\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} = 0. \tag{352}$$

Thus

$$\boxed{G_0 = \frac{1}{c}\frac{\mathbf{F}.\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}} \tag{353}$$

which gives

$$\boxed{\frac{dE}{dt} = \mathbf{F}.\mathbf{v}\,,} \tag{354}$$

which is precisely the result we used earlier.

23.3 Example: relativistic rockets

These have variable rest-mass, $m = m(\tau)$. The equation of motion is

$$\frac{d(mU)}{d\tau} = J, \tag{355}$$

where J is the rate of emission of 4-momentum of the ejecta. Physically J must be timelike, $J \cdot J > 0$, which leads to the inequality

$$\frac{\dot{m}}{m} > |a|. \tag{356}$$

Thus to obtain a certain acceleration, as in the Twin-Paradox set-up over a certain proper time requires a lower bound on the total mass of the fuel used

$$\ln\left(\frac{m_{\text{final}}}{m_{\text{initial}}}\right) < \int |a| d\tau. \tag{357}$$

In two dimensional Minkowski spacetime $E^{1,1}$

$$U^a = (\cosh\theta, \sinh\theta) \Rightarrow |a^a| = \frac{d\theta}{d\tau}, \tag{358}$$

where θ is the *rapidity*.We find

$$\frac{m_{\text{final}}}{m_{\text{initial}}} < \sqrt{\frac{1+v_{\text{initial}}}{1-v_{\text{initial}}}}\sqrt{\frac{1-v_{\text{final}}}{1+v_{\text{final}}}} = \frac{1}{1+z} \tag{359}$$

Consider for example two observers, one of whom is at rest and and engaged in checking Goldbach's conjecture that every even number is the sum of two primes using a computer. The second observer, initialy at rest with respect to the first observer $v_{\text{initial}} = 0$, decides to use time dilation to find out faster by accelerating *toward* the stationary observer thus acquiring a velocity v_{final} and *blue shift* factor $1+z$. The increase in the rate of gain of information is bounded by the energy or mass of the fuel expended.

24 The Lorentz Force

We will illustrate the general theory in the previous section by means of the simplest way of solving the constraint $G \cdot U = 0$. We set

$$G = e\eta FU, \tag{360}$$

where e is a constant and F is a 4×4 matrix. (the inclusion of η is for later convenience. Since η is invertible, and in fact $\eta^2 = 1$, we could absorb it into the definition of F). Now, using the fact that $\eta^2 = 1$,

$$U \cdot G = u^t \eta\eta FU = U^t FU. \tag{361}$$

Thus we may satisfy our constraint by demanding that

$$\boxed{F = -F^t}. \tag{362}$$

We call F the *Faraday tensor*. The word 'tensor 'will not be explained here since we won't need at this point. We give the components of F suggestive names.

$$F = \begin{pmatrix} 0 & \frac{E_1}{c} & \frac{E_2}{c} & \frac{E_3}{c} \\ -\frac{E_1}{c} & 0 & -B_3 & B_2 \\ -\frac{E_2}{c} & B_3 & 0 & B_1 \\ -\frac{E_3}{c} & -B_2 & -B_1 & 0 \end{pmatrix}. \tag{363}$$

Thus

$$\eta F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \frac{E_1}{c} & \frac{E_2}{c} & \frac{E_3}{c} \\ -\frac{E_1}{c} & 0 & -B_3 & B_2 \\ -\frac{E_2}{c} & B_3 & 0 & B_1 \\ -\frac{E_3}{c} & -B_2 & -B_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{E_1}{c} & \frac{E_2}{c} & \frac{E_3}{c} \\ \frac{E_1}{c} & 0 & B_3 & -B_2 \\ \frac{E_2}{c} & -B_3 & 0 & -B_1 \\ \frac{E_3}{c} & B_2 & B_1 & 0 \end{pmatrix} \tag{364}$$

$$G = e\eta FU = e\begin{pmatrix} 0 & \frac{E_1}{c} & \frac{E_2}{c} & \frac{E_3}{c} \\ \frac{E_1}{c} & 0 & B_3 & -B_2 \\ \frac{E_2}{c} & -B_3 & 0 & -B_1 \\ \frac{E_3}{c} & B_2 & B_1 & 0 \end{pmatrix}\begin{pmatrix} c\gamma \\ v_1\gamma \\ v_2\gamma \\ v_3\gamma \end{pmatrix} = \begin{pmatrix} e\gamma\mathbf{v}.\frac{\mathbf{E}}{c} \\ e\gamma(E_1 + v_2B_3 - v_3B_2) \\ e\gamma(E_2 + v_3B_1 - v_1B_3) \\ e\gamma(E_3 + v_1B_2 - v_2B_1) \end{pmatrix} \tag{365}$$

or in 3-vector notation

$$\boxed{\frac{d\mathbf{p}}{dt} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B}),\ \frac{dE}{dt} = e\mathbf{E}.\mathbf{v}\,.} \tag{366}$$

These are just the Lorentz force equations for a particle carrying an electric charge e in an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$.

24.1 Example: particle in a uniform magnetic field

In a vanishing electric field, $\mathbf{E} = 0$, the energy E and hence the speed v is constant. Thus $\mathbf{p} = m\gamma\mathbf{v}$, where the relativistic gamma factor γ is constant. If the magnetic field is uniform, independent of time, and aligned, for example, along the x_3 direction, we have $p_3 =$ constant $\Rightarrow x_3(t) = v_3t + x_3(0)$, where v_3 is the constant component of the velocity in the 3-direction. Thus $v_1^2 + v_2^2 := v_\perp^2 =$ constant. Now we have

$$\dot v_1 = \frac{e|\mathbf{B}|}{m\gamma}v_2, \qquad \dot v_2 = -\frac{e|\mathbf{B}|}{m\gamma}v_1 \tag{367}$$

Thus, with a choice of origin for time

$$\dot x_1 = v_1 = -v_\perp cos\omega t, \qquad \dot x_2 = v_2 = v_\perp\sin\omega t, \tag{368}$$

with

$$\boxed{\omega = \frac{\omega_L}{\gamma}\,,} \tag{369}$$

and

$$\boxed{\omega_L = \frac{e|\mathbf{B}|}{m}\ \text{is the Larmor Frequency.}} \tag{370}$$

The projection of the motion in the x_1, x_2 plane is circular. Up to a translation

$$x_1 = R\sin\omega t, \qquad x_2 = -R\cos\omega t, \tag{371}$$

with

$$v_\perp = R\omega. \tag{372}$$

Thus if $p_\perp = m\gamma v_\perp$, $p_\perp = \sqrt{p_1^2 + p_3^2}$,

$$\boxed{p_\perp = e|\mathbf{B}|R\,.} \tag{373}$$

This result is used by cosmic ray physicist, who measure the radius R of particles the tracks of particles, to obtain their momentum. Numerically, (373)

$$p = 300|\mathbf{B}|R, \tag{374}$$

with p in eV, $|\mathbf{B}|$ in Gauss and R in cm. The radius of the earth is 6,400 Km and its magnetic moment 8×10^{25}Gausscm^2. Thus only particles of 59.5 GeV or more can be expected to reach the surface of the earth.

Example The relation (373) was used by Bucherer in 1909 [20] to check the relativistic formula relating energy and momentum. Bucherer produced electrons of known kinetic energy by sending through a known potential difference V and then sent them through known magnetic fields and measured the radii of their orbits. He found agreement with the relation

$$eV = \sqrt{m_e^2 c^4 + e^2 c^2 \mathbf{B}^2 R^2} - m_e c^2. \tag{375}$$

24.2 Uniform electro-magnetic field and uniform acceleration

If F is a constant matrix we can integrate the equation of motion rather easily. In this denote we shall denote $\frac{d}{d\tau}$ by a dot. The equation of motion is [26]

$$a = \ddot{x} = \frac{e}{m}\eta^{-1} F \dot{x}. \tag{376}$$

Now

$$\frac{d}{\tau}(a \cdot a) = 2a \cdot (\dot{a}) = 2a^t \eta \dot{a}\,. \tag{377}$$

But

$$\dot{a} = \frac{e}{m}\eta F \dot{U} = \frac{e}{m}\eta F a. \tag{378}$$

Thus $a^t \eta \dot{a} = \frac{e}{m} a^t \eta \eta^{-1} F a = \frac{e}{m} a^t F a = 0$, Because F is antisymmetric, $F = -F^t$. Thus the magnitude of the acceleration is constant. Of course its direction changes.

Now the first integral of (377) is

$$\dot{x} = \frac{e}{m}\eta^{-1} F x + U_0. \tag{379}$$

Let's set $U_0 = 0$ and consider the case of a purely electric field along the x^1 axis.

$$\eta^{-1} F = \begin{pmatrix} 0 & \frac{|\mathbf{E}|}{c} \\ \frac{|\mathbf{E}|}{c} & 0 \end{pmatrix}. \tag{380}$$

The equations becomes, with

$$\frac{cdt}{d\tau} = \frac{e|\mathbf{E}|}{mc} x_1, \qquad \frac{dx_1}{d\tau} = \frac{e|\mathbf{E}|}{m} t. \tag{381}$$

Thus, with a choice of origin of proper time τ,

$$ct = A \sinh\left(\frac{e|\mathbf{E}|\tau}{mc}\right), \qquad x_1 = A \cosh\left(\frac{e|\mathbf{E}|\tau}{mc}\right), \tag{382}$$

[26] The inverse is explicitly included to make contact with the index notation we will introduce later

where A is a constant of integration. The world line is a hyperbola

$$c^2t^2 - x_1^2 = A^2. \tag{383}$$

The magnitude of the acceleration is

$$|a| = \frac{|A\mathbf{E}|}{mc}. \tag{384}$$

25 4-vectors, tensors and index notation

In advanced work, particularly when passing to Einstein's theory of General Relativity, it is helpful to adopt a notation which is a natural extension of elementary Cartesian tensor analysis (see e.g. [21]). The notation is universally used in physics and engineers throughout the world and despite the initial impression that it is rather complicated, rather more so than the matrix notation we have been using so far, experience shows that when the basic conventions have been absorbed, it provides both a very compact notation and one which allows for very efficient calculations. All legal expressions are automatically *covariant*, , i.e have well defined transformation rules under Lorentz transformations, and, from mathematical point of view, it allows one to write down mathematically well defined formulae and make well defined mathematical constructions without needing expertise in abstract algebra or needing to be familiar with the complicated basis independent definitions introduced in books on multi-linear algebra. What, for example pure mathematicians call functoriality is almost guaranteed. In fact the notation was introduced and widely adopted by pure mathematicians during the first half of the twentieth century and then abandoned by them, in favour of coordinate free notations. Such notations have many merits but they often require detailed explanations to unpack them. The wise words of Arthur Cayley(1821-1895), inventor of matrices and explorer of higher dimensions, invariants and co-variants speaking in a related context seem appropriate:

> My own view is that quaternions are merely a particular method, or say a theory, in coordinates. I have the highest admiration for the notion of a quaternion;but, ... as I consider the full moon far more beautiful than any moonlit view, so I regard the notion of a quaternion as far more beautiful than any of its applications. As another illustration ... I compare a quaternion formula as a pocket-map - a capital thing to put in one's pocket, but for use must be unfolded:the formula, to be understood, must be translated into coordinates.

25.1 Contravariant vectors

One labels the components of a 4-vector in some basis with *indices which take values* $0, 1, 2, 4$ *and which are placed upstairs*. In common with most modern books these indices will be denoted by letters from the Greek alphabet. Instead of 0 one sometimes uses 4. Thus the following notation should be thought of as conveying the same information.

$$x \qquad x^\mu \qquad \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \qquad \begin{pmatrix} x^0 \\ x^i \end{pmatrix} . \tag{385}$$

The usual time coordinate is given by $x^0 = ct$ but in advanced work we usually set $c = 1$ so we will always think of x^0 as the time coordinate. Of course, mathematically, the first is the abstract 4-vector, the second the set of its components in a basis (i.e. in a particular frame of reference) and the third its representation as a column vector. The real 4-dimensional space of 4- vectors will be called V.

The Lorentzian inner product is written as

$$x \cdot y = x^t \eta x = x^\mu \eta_{\mu\nu} y^\nu . \tag{386}$$

Evidently $\eta_{\mu\nu}$ are the components of a the quadratic form η in the basis.

Now, as with any vector space, Lorentz transformation may be be viewed *passively*: as a change of basis or *actively* as a linear map $\Lambda : V \to V$. In either case we have

$$\boxed{x^\mu \to \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu .} \tag{387}$$

Thus $\Lambda^\mu{}_\nu$ are the components of the linear map or endomorphism Λ. The first, upper, so-called *contravariant* index labels rows and the second, so called *covariant* index labels the columns of the associated matrix. Note that the Einstein summation convention applies in the modified form that *contractions are allowed only between a covariant index and a contravariant index*, i.e. between an upstairs and a downstairs index.

25.2 Covariant vectors

Now what about row vectors, e.g $z = y^t$, where y is a column vector?. We write the components of z with indices downstairs and so all of the following should convey the same information

$$z \qquad z_\mu \qquad (\, z_0 \quad z_1 \quad z_2 \quad x_3 \,) \qquad (\, z_0 \quad z_i \,) . \tag{388}$$

Thus

$$zx = z_\mu x^\mu = z_0 x^0 + z_1 x^1 + z_2 x^2 + z_3 x^3 = z_0 x^0 + z_i x^i . \tag{389}$$

We would like zx to be invariant Lorentz transformations and hence it must transform like

$$z \to \tilde{z} = z\Lambda^{-1}, \qquad \tilde{z}_\nu = z_\mu (\Lambda^{-1})^\mu{}_\nu \Rightarrow z_\mu \Lambda^\mu{}_\nu = \tilde{z}_\nu . \tag{390}$$

Clearly column vectors and row vectors transform in the opposite way, one with Λ and the other with $(\Lambda^{-1})^t$. We say they transform *contragrediently.*

Alternatively we refer to column vectors as *contravariant* vectors and row vectors as *covariant* vectors. Another way to say this is that x^μ are the components of an element of the four-dimensional vector space V of 4-vectors, and z_μ, the components of the four-dimensional dual vector space $V^\star$, i.e. the space of linear maps from V to R.

25.3 Example: Wave vectors and Doppler shift

If one looks back at our derivation of the Doppler effect, we wrote

$$\Phi = A\sin(kx - \omega t) = A\sin(\tilde{k}\tilde{x} - \tilde{\omega}\tilde{t}) \tag{391}$$

and deduced the transformation rules for the angular frequency ω and wave vector k (125 using the invariance of the phase

$$\tilde{k}\tilde{x} - \tilde{\omega}\tilde{t} = k - \omega t. \tag{392}$$

In our present language we see that we can think of ct, x as a contravariant vector x^μ and $\omega, -k$ as a covariant vector k_μ. Thus

$$\Phi = A\sin(k_\mu x^\mu). \tag{393}$$

Now we see that our Lorentz transformation rule (50) is that for a contravariant vector and our Doppler shift rule (125) is that for a covariant vector. The invariance of the phase is the statement that

$$k_\mu x^\mu = \tilde{k}_\mu \tilde{x}^\mu. \tag{394}$$

Abbreviating the term covariant vector to *covector* and contravariant vectors to *contravectors* , , we can say that a *wave covector* k_μ belongs to the vector space $V^\star$ dual to the vector space V of contravectors.

Geometrically, the surfaces of constant phase are *hyperplanes* in Minkowski spacetime $E^{3,1}$

$$\phi = k_\mu x^\mu = \text{constant}. \tag{395}$$

The wave covector k_μ corresponds to the co-normal

$$k_\mu = \frac{\partial \phi}{\partial x^\mu} \tag{396}$$

to the 3-dimensional hypersurfaces of of constant phase.

25.4 Contravariant and covariant second rank tensors

Now consider how a quadratic form given by

$$x^t Q x = x^\mu Q_{\mu\nu} y^\nu \tag{397}$$

changes under a Lorentz transformation

$$x \to \tilde{x} = \Lambda x, \qquad \text{i.e.} \qquad \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu, \qquad \tilde{y}^\mu = \Lambda^\mu{}_\nu y^\nu\,. \tag{398}$$

If we define the transformed quadratic form by

$$x^t Q y = \tilde{x}^t \tilde{Q} \tilde{y}, \tag{399}$$

then

$$\tilde{Q} = (\Lambda^{-1})^t Q \Lambda^{-1} \Rightarrow \Lambda^t \tilde{Q} \Lambda = Q \qquad \text{or} \qquad \Lambda^{\alpha}{}_{\mu} \tilde{Q}_{\alpha\beta} \Lambda^{\beta}{}_{\nu} = Q_{\mu\nu} \,. \tag{400}$$

The Lorentz invariance condition (137) reads

$$\boxed{\Lambda^{\alpha}{}_{\mu}\, \eta_{\alpha\beta} \Lambda^{\beta}{}_{\nu} = \eta_{\mu\nu} \,.} \tag{401}$$

$$(\Lambda^t)_{\mu}{}^{\alpha} = \Lambda^{\alpha}{}_{\mu} \qquad \text{and} \qquad \tilde{\eta} = \eta. \tag{402}$$

We say that $\eta_{\mu\nu}$ are the components of a (*symmetric*) second rank covariant tensor η since they transform in the same fashion as the *tensor product* or *outer product* $x_\mu y_\nu$ of two covariant vectors x_μ and y_μ.

The components of the inverse of the metric are

$$(\eta^{-1})^{\mu\nu} = \eta^{\mu\nu} = \eta^{\nu\mu}, \tag{403}$$

and satisfy

$$\boxed{\eta^{\mu\alpha} \eta_{\alpha\nu} = \delta^{\mu}_{\nu} \,,} \tag{404}$$

where δ^{μ}_{ν} is the *Kronecker delta*, i.e. the unit matrix, whose trace or contraction is $\delta^{\mu}_{\mu} = 4$. We say that $\eta_{\mu\nu}$ are the components of a (*symmetric*) second rank contravariant tensor η since they transform in the same fashion as the tensor product or outer product of two contravariant vectors x^μ and y^ν.

The Minkowski metric $\eta_{\mu\nu} = \eta_{\nu\mu}$ [27] can be thought of as a *symmetric second tensor* , i.e. mathematically speaking a symmetric bilinear map $V \times V \to R$. The Faraday tensor $F_{\mu\nu} = -F_{\nu\mu}$ is an example of an *antisymmetric second tensor* , i.e. mathematically speaking an antisymmetric bilinear map $V \times V \to R$. Under a Lorentz transformation its components change

$$F_{\mu\nu} \to \tilde{F}_{\mu\nu} \; s.t. \; F_{\mu\nu} = \Lambda^{\alpha}{}_{\mu} \tilde{F}_{\alpha\beta} \Lambda^{\beta}{}_{\nu} = \tilde{F}_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} \,. \tag{405}$$

Note that the transformation rule is exactly the same as for the metric $\eta_{\mu\nu}$. The same rule holds for an arbitrary second rank tensor $Q_{\alpha\beta}$, symmetric, antisymmetric or with no special symmetry. The components of an n-th rank covariant tensor transform , i.e. a tensor with n indices downstairs or mathematically speaking, a multi-linear real valued map from the n-fold Cartesian product $V \times \ldots \times \to R$ transform analogously. The symmetry or anti-symmetry of a tensor is a Lorentz-invariant. In the case of rank the symmetric and antisymmetric parts

$$Q_{\alpha\beta} = Q_{(\alpha\beta)} + Q_{[\alpha\beta]}, \tag{406}$$

[27] Note that from now on we are will be indulging in the standard abuse of language which refers to an object by its components.

with

$$Q_{(\alpha\beta)} = \frac{1}{2}(Q_{\alpha\beta} + Q_{\beta\alpha}), \qquad Q_{[\alpha\beta]} = \frac{1}{2}(Q_{\alpha\beta} - Q_{\beta\alpha}) \tag{407}$$

transform separately into themselves. The proof proceeds by ' index shuffling'. For example in the anti-symmetric case

$$\tilde{Q}_{[\mu\nu]} = Q_{\alpha\beta}\Lambda^{\alpha}{}_{[\mu}\Lambda^{\beta}{}_{\nu]} = Q_{\alpha\beta}\Lambda^{[\alpha}{}_{[\mu}\Lambda^{\beta]}{}_{\nu} = Q_{[\alpha\beta]}\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}. \tag{408}$$

An identical argument with square brackets replacing round brackets applies in the symmetric case.

25.5 The musical isomorphism

Note that if v^μ transforms like a contravariant vector and $Q_{\mu\nu}$ is a second rank covariant tensor the

$$Q_{\mu\nu}x^\nu, \qquad \text{and} \qquad Q_{\mu\nu}x^\mu \tag{409}$$

are covariant vectors which coincide or coincide up to a sign of q is symmetric or antisymmetric respectively.

Thus, in the case of Minkowski space, the distinction between contravariant and contravariant vectors is more apparent than real, because one may pass from one to the other by *index lowering* and *index raising* using the metric $\eta_{\mu\nu}$ or inverse metric $\eta^{\mu\nu}$ respectively. We use a notation in which the same 'kernel letter ' is used for vectors and tensors which are identified using index raising or lowering.

Thus we write, for example

$$\boxed{x^\mu = \eta^{\mu\nu}x_\nu \Rightarrow x_\mu = \eta_{\mu\nu}x^\mu\,.} \tag{410}$$

In other words the metric η effects an isomorphism between the vector space V of contravariant 4-vectors and its dual vectors space $V^\star$ of covariant 4-vectors.

Thus

$$\boxed{p\cdot q = q\cdot p = \eta_{\mu\nu}p_\mu q^\nu = p_\mu q^\nu = p^\mu q_\nu\,.} \tag{411}$$

Note that the *order* of indices is still important. $F^\mu{}_\nu = \eta^{\mu\alpha}F_{\alpha\nu}$ and $F_n u^{\,\mu} = F_{\mu\beta}\eta^{\beta\mu}$ should be distinguished.

One sometimes uses the musical symbols $\sharp$ and $\flat$ to denote index raising and lowering respectively and so the isomorphism is referred to as the *musical isomorphism*.

25.6 De Broglie's Wave Particle Duality

Tn

In 1924 the French aristocrat Louis-Victor 7^e duc de Broglie(1892-1987) proposed, in his doctoral dissertation that just as light, believed since the interference experiments of Thomas Young, to be a wave phenomenon, has, according to Albert Einstein's photon hypothesis (for which he was awarded the Nobel prize in 1922) some of the properties of particles, so should ordinary particles, like electrons, and indeed all forms of matter of waves, according to the universal scheme,

$$\boxed{\text{Energy and frequency} \quad E = hf \qquad \text{Wavelength and momentum} \quad p = \frac{h}{\lambda}\,,} \tag{412}$$

where h is Planck's constant. The American physicist Clinton Josephson Davidson(1881-1958) and the English physicist George Paget Thomson(1892-19750) were awarded the Nobel prize in 1937 for the experimental demonstration of the diffraction of electrons. George Thomson was the son of the 1906 Nobelist Joseph John Thomson (1856-1940) who established the existence of the electron. It was said of the pair that the father received the prize for proving that electrons are particles and the son for proving that they waves. No parallel case appears to be known, and indeed may not be possible, in the case of the mathematicians equivalent of the Nobel prize, the Fields medal.

Note that de Broglie's proposal allows us to reconcile the two opposing theories of refraction, the emission and the wave theory described earlier. One may indeed think of Snell's law as expressing conservation of momentum parallel to the refracting surface as long as one uses de Broglie's relation $p = \frac{h}{\lambda}$ for the momentum rather than Newtons formula $p = mc$.

An important part of de Broglie's preposterous proposal, for which he was awarded the Nobel prize in 1929, was that he could show that it is covariant with respect to Lorentz transformations. With the formalism we have just developed this is simple. Defining $\tilde{} = \frac{h}{2\pi}$, his proposal becomes

$$\boxed{p^\mu = \tilde{}k^\mu = \tilde{}\eta^{\mu\nu} k_\mu\,,} \tag{413}$$

In other words his wave-particle duality is equivalent to the musical isomorphism.

25.7 * Wave and Group Velocity: Legendre Duality*

I order to reconcile de Broglie's proposal with our usual ideas it is necessary to recall some facts about wave motion. In fact, these facts are also relevant for some of the optical experiments mentioned earlier. In general, monochromatic wave motion, that is waves of a single fixed wavelength have a single well defined frequency and conversely [28] Thus the phase travels with the *phase velocity*

$$v_p = f\lambda = \frac{\omega}{k}. \tag{414}$$

In general the motion is *dispersive* , which means that the phase velocity v_p depends on wavelength λ. For example, for light, we define the refractive index by $v_p = nc$ and a little familiarity with prisms and the rainbow soon convinces

[28]In some situations, such as in condensed matter physics, it may be that the frequency is a multivalued function of wavelength. In what follows, we exclude this possibility.

one that refractive index depends upon wavelength, $n = n(\lambda)$. In other words the *dispersion relation* $\omega = \omega(\mathbf{k})$ is not, in general $\omega = c|\mathbf{k}|$, but more general.

Now pure monochromatic waves never exist in practice. The best one can arrange is a superposition of a group or *wave packet* of waves with almost the same frequency

$$\Phi(x,t) = \int A(\mathbf{k}') e^{i\big(\mathbf{k}'.\mathbf{x} - \omega(\mathbf{k}')t\big)} d^3k' \tag{415}$$

where $A(\mathbf{k}')$ is peaked near $\mathbf{k}' = \mathbf{k}$.

We set

$$\mathbf{k}' = \mathbf{k} + \mathbf{s} \qquad \omega(\mathbf{k}') = \omega(\mathbf{k}) + \mathbf{v}_g.\mathbf{s} + \mathcal{O}(|\mathbf{s}|^2) \tag{416}$$

where the

$$\boxed{\text{group velocity} \qquad \mathbf{v}_g = \frac{\partial \omega}{\partial \mathbf{k}}\,.}\,. \tag{417}$$

One now performs a stationary phase or saddle point evaluation of the integral. This amounts to assuming that

$$A(\mathbf{s}) = e^{-\frac{a^2}{2}|\mathbf{s}|^2}. \tag{418}$$

One finds that

$$\Phi \propto e^{i\big(\mathbf{k}.\mathbf{x} - \omega(\mathbf{k})t\big)} e^{-\frac{1}{2a}|\mathbf{x} - \mathbf{v}_g t|^2}. \tag{419}$$

One sees that the peak of the wave packet moves with the group velocity, *not* the phase velocity.

Note that de Broglie's proposal is compatible with Hamiltonian mechanics. If we set

$$H = \tilde{}\,\omega, \qquad \mathbf{p} = \tilde{}\,\mathbf{k}, \tag{420}$$

then (417) and (251) become identical.

Now let's turn to the special case of a relativistic particle. Using units in which ˜ $= c = 1$, the dispersion relation is

$$\omega = \sqrt{m^2 + \mathbf{k}^2}. \tag{421}$$

Thus

$$\boxed{v_p = \sqrt{1 + (\frac{m}{|\mathbf{k}|})^2}, \qquad \mathbf{v}_g = \frac{\mathbf{k}}{\sqrt{m^2 + \mathbf{k}^2}}.} \tag{422}$$

The group velocity $\mathbf{v}_g$ coincides with what we have been thinking of the velocity $\mathbf{v}$ of the relativistic particle and is never greater in magnitude than the speed of light c. By contrast the phase velocity is always greater than that of light. If $v_g = |\mathbf{v}_g$, then and we have the strikingly simple relation

$$\boxed{v_p v_g = c^2\,.} \tag{423}$$

In Hamiltonian mechanics, the passage between momentum and velocity is via the Legendre transform. The Legendre transform is a *duality* or *involution* because the Legendre transform of a Legendre transform gets you back to yourself. The musical isomorphism is also an involution. These facts are of course related. We can consider an arbitrary a covariant Lagrangian $L(v^\mu)$ and covariant or *super Hamiltonian* $\mathcal{H}(p_\mu)$ such that

$$\mathcal{H} + L = p_\mu v^\mu, \tag{424}$$

and

$$p_\mu = \frac{\partial L}{\partial v^\mu}, \qquad v^\mu = \frac{\partial \mathcal{H}}{\partial p_\mu}. \tag{425}$$

To obtain the standard, experimentally well verified Lorentz-invariant relation between energy and momentum, we choose

$$L = \frac{m}{2}\eta_{\mu\nu} v^\mu v^\nu \Leftrightarrow \frac{1}{2m}\eta^{\mu\nu} p_\mu p_\nu\,, \tag{426}$$

we have

$$p_\mu = m\eta_{\mu\nu} v^\nu \Leftrightarrow v^\mu = \frac{1}{m}\eta^{\mu\nu} p_\nu\,. \tag{427}$$

It is illuminating to look at this from the Galilean perspective. [29]

Unlike the case with the Lorentz group, Galilean boosts form a three-dimensional invariant subgroup subgroup of the full Galilei group. Under its action, the four quantities $\mathbf{x}, t$ transform linearly as

$$\begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \mathbf{u} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix}. \tag{428}$$

which gives a reducible but not fully reducible representation since the subspaces t = constant are left invariant. The phase $\mathbf{k}.\mathbf{x} - \omega t$ is left invariant and so the wave vector $\mathbf{k}$ and frequency ω transform under the contragredient representation (i.e. under the transpose of the inverse)

$$\begin{pmatrix} \mathbf{k} \\ \omega \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ -\mathbf{u}^{\mathbf{t}} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{k} \\ \omega \end{pmatrix}. \tag{429}$$

These two representations are not equivalent, essentially because no non-degenerate metric is available to raise and lower indices. This is one way of understanding the difference between the predictions about aberration made according to the particle and wave viewpoint in Galilean physics.

25.8 The Lorentz equation

Having set up the notation, we are now in a position to write down the equation of a relativistic particle of mass m and charge e moving in an electro-magnetic field $F_{\mu\nu} = -F_{\nu\mu}$

$$m\frac{d^2 x^\mu}{d\tau} = eF^\mu{}_\nu \frac{dx^\nu}{d\tau},. \tag{430}$$

[29] In what follows we shall use some standard group-theoretic terminology which will not be defined here. An understanding of the rest of this section is not necessary for the rest of the lectures.

26 Uniformly Accelerating reference frames: Event Horizons

Uniform translational motion is, according to Special Relativity, unobservable. Uniformly accelerated motion however is observable. To reveal some of its effects, we may pass to an accelerated system of coordinates, often called *Rindler coordinates*

$$x^0 = \rho \sinh t \,, \qquad x^3 = \rho \cosh t \,, \tag{431}$$

which, if $0 < \rho < \infty$, $-\infty < t < \infty$, cover only one quarter of two-dimensional Minkowski spacetime $E^{1,1}$, the so-called *Rindler wedge*

$$x^1 > |x^0|. \tag{432}$$

In this wedge the flat Minkowski metric takes the static form

$$ds^2 = \rho^2 dt^2 - d\rho^2 \,. \tag{433}$$

From our previous work, we see that The curves $\rho =$ constant have constant acceleration $a = \frac{1}{\rho}$. We shall refer to these curves as *Rindler observers*. They are in fact the orbits of a one parameter family of Lorentz boosts, $t \rightarrow t + t_0 \Leftrightarrow x^{\pm} = x^1 \pm x^0 \rightarrow e^{\pm t_0} x^{\pm}$. Note that the propertime τ_{Rindler} along a Rindler observer is given by $\tau_{\text{Rindler}} = \rho t$.

The acceleration of the set of Rindler observers goes to infinity on the boundary of the Rindler wedge, i.e. on the pair of null hypersurfaces surfaces $x^0 = \pm x^1$. These surfaces are called the *future and past horizons* of the Rindler observers. That is because the past, respectively future, light cones of all the points on the worldline of a Rindler observer, and thus necessarily their interiors, lie to the past, respectively future of these null hypersurfaces. In other words the future horizon is the boundary of the set of events that can ever causally influence a Rindler observer and the past horizon the boundary of the set of events which a Rindler horizon may causally influence. Thus the nature of all events for which $X^0 > x^1$ can never be known to Rindler observers. On the other hand, there is no boundary to the past of an *inertial observer*, i.e. a timelike geodesic. For example a timelike observer with say $x^1 =$ constant > 0 will simply pass through the future horizon and out of the Rindler wedge in finite propertime $\tau_{\text{Inertial}} = x^1$. A simple calculation shows that a light ray emitted from the event (x^0, x^1) will be received by a Rindler observer at a propertime

$$\tau_{\text{Rindler}} = -\frac{1}{\rho} \ln \left(\frac{x^1 - \tau_{\text{Inertial}}}{\rho} \right) . \tag{434}$$

According to the Rindler observer, the light coming from the Inertial observer is increasing redshifted. The motion appears to be slower and slower. So much so, that the redshift becomes infinite as the Inertial observer is on the point of passing through the future event horizon and according to the Rindler observer the Inertial observer never actually passes through in finite time.

The rather counter-intuitive phenomena described above have a very precise parallel in the behaviour of the event horizons of black holes. The fact that they may occur in such as simple situation as that two-dimensional Minkowski spacetime shows that although apparently paradoxical, there is nothing logically inconsistent about them.

27 Causality and The Lorentz Group

.

27.1 Causal Structure

We may endow Minkowski spacetime with a *causal structure* , that is a partially ordering, called a *causal relation* which is reflexive and transitive. In a general, time orientable, spacetime, one says that x causally precedes y and writes

$$x \preceq y \tag{435}$$

if the event x can be joined to the event y by a future directed timelike or null curve. Thus

$$(i) \qquad x \preceq y \quad \text{and} \quad y \preceq z \Rightarrow x \preceq z \quad \text{and} \quad (ii) \qquad x \preceq x. \tag{436}$$

There is an obvious dual relation, written as $x \succeq y$ in which past and future are interchanged. A stronger relation, called *chronology* can also be introduced. We say that x chronologically precedes y if there is a future directed timelike curve joining x to y and write

$$x \prec y\,. \tag{437}$$

In Minkowski spacetime the curves may be taken to be straight lines, i.e. geodesics. We write

$$\boxed{x \succeq y \qquad \Leftrightarrow \qquad x^0 - y^0 \geq \sqrt{(x^1-y^1)^2 + (x^2-y^2)^2 + (x^3-y^3)^2}\,.} \tag{438}$$

27.2 The Alexandrov-Zeeman theorem

Our derivation of the Lorentz group earlier depended upon the assumption of linearity. In fact this may be removed.

Alexandrov[54] and independently Zeeman[53] have shown that any continuous map of Minkowski spacetime into Minkowski spacetime, as long as it is higher than 1+1 dimensional which preserves the light cone of the origin must in fact be linear. It follows that such a transformation is the product of a dilation and a Lorentz transformation. In other words, in four spacetime dimensions, one may characterize the eleven dimensional group consisting of the Poincaré group semi-direct product dilatations as the automorphism group of the causal structure of Minkowski spacetime. Since the proof entails special techniques we will not give it here.

In $1+1$ dimensions, things are very different. In light cone coordinates the metric is

$$ds^2 = dx^+ dx^- \,. \tag{439}$$

The light cone and causal structure is clearly left unchanged under

$$x^\pm \rightarrow \tilde{x}^\pm = f^\pm(x^\pm)\,, \tag{440}$$

where the two functions $f^\pm$ are arbitrary monotonic C^1 functions of their argument. Thus the group of causal automorphisms of two-dimensional Minkowski spacetime is infinite dimensional. It is the product of two copies of the infinite dimensional group $\mathrm{Diff}(R)$ of invertible and differentiable maps of the real line into itself. This fact plays an important role in what is called String Theory.

27.3 Minkowski Spacetime and Hermitian matrices

We may identify four-dimensional Minkowski spacetime with the space of 2×2 Hermitian matrices $X = X^\dagger$ according to the scheme

$$\boxed{X = \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix}\,.} \tag{441}$$

Now if X and Y are lightlike separated, then

$$\det\left(X - Y\right) = 0 \tag{442}$$

One can say more,

$$\boxed{X \succeq Y \qquad \Leftrightarrow \qquad X - Y \quad \mathrm{is\, non-negative\, definite}} \tag{443}$$

The Minkowski metric may be written as

$$ds^2 = \det\, dX\,. \tag{444}$$

Now consider the group $GL(2, C)$ acts on Hermitian matrices by conjugation

$$X \rightarrow SXS^\dagger = \tilde{X}\,, \qquad \Rightarrow \qquad \tilde{X} = \tilde{X}^\dagger\,. \tag{445}$$

Moreover if we insist that

$$\det S = 1, \tag{446}$$

we obtain the group, $SL(2, C)$ of 2×2 uni-modular complex valued matrices, which is 6 dimensional. We have exhibited a homomorphism from $SL(2, \mathbf{C})$ to $SO(3, 1)$ [30]The kernel of this homomorphism is easily seen to be the group Z_2 given by $S = \pm 1$. Thus the homomorphism is a double covering, S and $-S$ give the same element of the identity component $SO_0(3, 1)$.

We could pursue this homomorphism further, but at this point we prefer to return to the causal structure (443) on Hermitian matrices. Orderings of this type were studied by Hua. They have other applications, including to providing a natural ordering, corresponding to *purity* on density matrices in quantum mechanics. A special case of Hua's formalism is the case of 2×2 matrices. In tis case he is able to re-obtain the Alexandrov-Zeeman result [55].

[30]Strictly speaking since $SL(2, C)$ is connected it is onto the connected component $S_0(3, 1)$.

28 Spinning Particles and Gyroscopes

28.1 Fermi-Walker Transport

We begin by considering a timelike curve C with unit tangent vector $u = \frac{dx}{d\tau}$ and a vector e defined along the curve satisfying

$$\frac{de}{d\tau} + u(e \cdot \dot{u}) = 0, \tag{447}$$

where $\dot{u} = \frac{du}{d\tau}$ is the acceleration of the curve C. One has

$$\frac{d}{d\tau}(e \cdot u) = 0, \qquad \frac{d}{d\tau}(e \cdot s) = -\frac{1}{2}(e \cdot u)(e \cdot \dot{u}). \tag{448}$$

Thus if s satisfies (447) along C, we say that it is *Fermi-Walker transported* along C.

From (448) it follows that if s is orthogonal to u, $(u \cdot s) = 0$, at one point on the curve C then it is orthogonal to u at all points of C. Moreover its length $|e| = \sqrt{-e \cdot e}$ will be constant along e. If e_1 and e_2 are Fermi-Walker transported along C, then

$$\frac{d}{d\tau}(e_1 \cdot e_2) = -(e_1 \cdot u)(e_2 \cdot \dot{u}) - (e_2 \cdot u)(e_1 \cdot \dot{u}). \tag{449}$$

Thus, if e_1 and e_2 are initially orthogonal to u and each other they will remain orthogonal to u and to each other. Introducing a third vector e_3 we can arrange that $e_0 = u, e_1, e_2, e_3$ we may constrict in this way a pseudo-orthonormal frame along the curve C.

Physically we can think of e_a, $a = 0, 1, 2, 3$ as a locally non-rotating frame defined along the accelerating worldline C.

28.2 Spinning particles and Thomas precession

Let s be the spin vector of a particle whose 4-velocity is u. In a local rest frame, s should be purely spatial, so

$$s \cdot u = 0\,. \tag{450}$$

In the absence of an external torque, we postulate that its components are constant in a Fermi-Walker transported frame, along the world line i.e.

$$\frac{ds}{d\tau} + u(s \cdot \dot{u}) = 0. \tag{451}$$

Note that if the world line of the particle is accelerating, even in the absence of an external torque, the spin, while staying constant in magnitude, will change in direction. This is called *Thomas precession.* Its existence was pointed out in 1927[39]. If $s \cdot u = 0$, then we can write (451) as

$$\frac{ds^\mu}{d\tau} + U^\mu{}_\nu s^\nu = 0, \tag{452}$$

with

$$U_{\mu\nu} = u_\mu(\dot{u})_\nu - (\dot{u})_\mu u_\nu. \tag{453}$$

One may think of $U_{\mu\nu}$ as an infinitesimal rotation. Thomas regards this as the result of the commutator of two successive, non parallel boosts.

If an external torque H is applied the equation becomes

$$\boxed{\frac{ds}{d\tau} + u(s \cdot \dot{u}) = H\,, \qquad \text{where} \qquad u \cdot H = 0\,.} \tag{454}$$

28.3 Bargmann-Michel-Telegdi Equations

In (1926) Goudsmit and Uhlenbeck, studying the fine structure of atomic spectral lines and their behaviour in external magnetic fields, the Zeeman effect, realized that the electron at rest has both an *intrinsic spin* $\mathbf{s}$ of magnitude $|\mathbf{s}| = \frac{\tilde{}}{2}$, and an intrinsic *magnetic moment* μ, so that immersed in a magnetic field $\mathbf{B}$ the spin changes as

$$\frac{d\mathbf{s}}{dt} = \mu \times \mathbf{B}. \tag{455}$$

In fact Goudsmit and Uhlenbeck argued on the grounds of atomic spectra that

$$\mathbf{s} = g\frac{e}{2m}\mathbf{s}, \tag{456}$$

with the *gyromagnetic ratio* $g = 2$. Thus the spin precesses according to the equation

$$\boxed{\frac{d\mathbf{s}}{dt} = g\frac{e}{2m}\mathbf{s} \times \mathbf{B}\,.} \tag{457}$$

The reason for the apparently odd normalization is that for ordinary orbital motion for which the spin coincides with the orbital angular momentum, $\mathbf{s} = \mathbf{L}$, g takes the value 1. In fact, a little later in 1927? Paul Adrien Maurice Dirac (1902-1984), the 15th Lucasian professor) proposed that relativistic electrons satisfy what we now call the *Dirac equation*, rather than it's non-relativistic approximation the *Schrödinger equation*. Dirac showed that the value $g = 2$ follows naturally from his equation. His work was recognized by the ward of the Nobel prize in (1933).

Later, in the 1940's advances in radio engineering allowed more precise measurements in atomic spectral lines, and revealed a level of *hyper-fine structure* beyond that predicted from the Dirac equation. In particular, there is shift or splitting in the lowest lines of hydrogen, due to a difference in the energy between an orbiting electron spinning up or spinning down, relative to the direction of the orbital angular momentum. The difference in energy, called the *Lamb shift* after the man who measured it is extremely small and a transition between the two levels gives rise to radio waves of 21cm wavelength. It was suggested in 1944 by the Dutch astronomer Henk van der Hulst that radiation of this wavelength should be emitted by interstellar clouds of neutral Hydrogen and its detection was achieved by various groups in 1951. Today radio-astronomy using the 21cm line allows is an important area of research, not least because the precise frequency allows the measurement of the velocity of clouds of neutral Hydrogen using the Doppler effect.

To account for the Lamb shift it is necessary to assign an *anomalous gyromagnetic moment* to the electron, in other words $g-2 \neq= 0$. The value of $g-2$ can be calculated using the relativistic quantum mechanical theory of photons interacting with electrons known as *quantum electrodynamics* QED . At present the agreement between theory and experiment is better than

To measure $g-2$ one accelerates electrons in known electromagnetic fields and measures the precession of the spin. A relativistically covariant set of equations describing this, properly taking into account the effects of Thomas precession, was set up by Bargmann-Michel and Telegdi, then at Princeton.

Basically one needs a covariant expression for the torque which will reproduce (**??**) in the rest frame of the electron. One's first guess might be

$$H = g\frac{e}{2m}Fs, \qquad H_\mu = g\frac{e}{2m}F_{\mu\nu}s^\nu ? \tag{458}$$

but this does not satisfy

$$H_\mu u^\mu = 0. \tag{459}$$

In order to remedy this defect we introduce a *projection operator*

$$h^\mu_\nu = \delta^\mu_\nu - u^\mu u_\nu . \tag{460}$$

which thought of as an endomorphism h projects an arbitrary vector orthogonal to u. One may also think of $h_{\mu\nu} = h_{\nu\mu}$ as the restriction of the spacetime metric $\eta_{\mu\nu}$ to a local 3-plane orthogonal to the tangent of the world line C. The projection operator satisfies

$$h^2 = h\,, \qquad h^\mu_\lambda h^\lambda_\nu = h^\mu_\nu \qquad h^\mu_\nu u^\nu = 0. \tag{461}$$

Now using the projection operator we are able to propose

$$\boxed{H = g\frac{e}{2m}hFs = g\frac{e}{2m}\big(Fs - u(u\cdot Fs)\big)} \tag{462}$$

or

$$\boxed{H_\mu = g\frac{e}{2m}h^\lambda_\mu F_{\lambda\tau}s^\tau = g\frac{e}{2m}\big(F_{\mu\tau}s^\tau - u_\mu u^\alpha F_{\alpha\beta}s^\beta\big)\,.} \tag{463}$$

Note that $s\cdot H = -(u\cdot s)(u\cdot Fs)$ and so if $u\cdot s = 0$, the length $|s|$ of the spin-vector s is constant. In the presence of an electromagnetic field one has

$$\dot{u} = \frac{e}{m}Fu\,, \tag{464}$$

thus

$$\boxed{\frac{ds}{d\tau} = g\frac{e}{2m}Fs + (2-g)\frac{e}{2m}u(u\cdot Fs)\,.} \tag{465}$$

Clearly the case $g=2$ is very special. The spin vector s and the 4-velocity u obey the same equation, and thus they move rigidly together. By contrast, if $g \neq 2$, this is not the case, the spin precesses in the moving frame, allowing a measurement to be made.

References

[1] O Roemer, *Phil Trans* **12** (1677) 893

[2] A A Michelson *Studies in Optics*(1927) University of Chicago Press

[3] A A Michelson and Morley *Phil Mag* **24** (1887) 449

[4] Bradley *Phil Trans* **35**(1728)637

[5] A Einstein et al. *The Principle of Relativity* (Dover)

[6] H Fizeau, *Compte Rendu Aacadamie des Sciences de Paris* **29** (1849) 90

[7] M L Foucault*Compte Rendu Aacadamie des Sciences de Paris* **30** (1850) 551 *Compte Rendu Aacadamie des Sciences de Paris* **55** (1862) 501 *Compte Rendu Aacadamie des Sciences de Paris* **55** (1862) 792

[8] D F Comstock *Phys Rev* **30** (1920) 267

[9] W De-Sitter,*Proc Amsterdam Acad***16** (1913) 395

[10] H E Ives and CR Stillwell *J opt Soc Amer***28**(1939 215;**31**(1941)369

[11] A H Joy and R F Sanford *Ap J* **64** (1926) 250

[12] D Sadeh *Phys Rev Lett* Experimental Evidence for the constancy of the velocity of of gamma rays using annihilation in flight**10** (1963) 271-273

[13] W Bertozzi Amer J Phys **55**(1964)1

[14] Alväger, Farley, Kjellamn and Wallin *Phys Lett* 12(1964) 260

[15] L Marder , *Time and the space traveller* (1971) George Allen and Unwin

[16] J C Hafele and R E Keating *Science* **177** (1972) 166-168,168-170

[17] M J Longo, *Phys Rev***36D** (1987) 3276-3277

[18] J Terrell *Phys Rev***116**(1959) 1041

[19] R Penrose, *Proc Camb Phil Soc* **55** (1959) 137

[20] A H Bucherer *Ann Physik* **28**(1909) 513

[21] H Jeffreys *Cartesian tensors*

[22] M J Rees, *Nature* **211** (1966) 468- 470

[23] L Dalrymple Henderson *The Fourth Dimension in Non-Euclidean geometry in Modern Art* Princeton University Press

[24] K Brecher *Phys Rev Lett* **39**(1977) 1051

[25] W Pauli *Theory of Relativity* Pergamon

[26] C Doppler *Über das farbige Licht der Doppelsterne* (1842)

[27] Y Z Zhang *Special Relativity and its Experimental Foundations* World Scientific (1997)

[28] M Lachièze-Rey and E Gunzig *The Cosmological Background Radiation* CUP

[29] W Zurhellen *Astr. Nachr.* **198**(1914) 1

[30] G Strömberg,*Pubs Astr. Socs of the Pacific* **43** (1931) 266

[31] G Van Biesebroeck, *Ap J* **75** (1932) 64

[32] O Heckmann *Ann d'Astrophysique* **23** (1960) 410

[33] H J Hay, J P Schiffer, T E Cranshaw and P A Egelstaff *Phys Rev Lett* **4** (1960) 165

[34] J B Hartle *Gravity*

[35] G Sagnac, *Compte Rendue de 'Academie des Sciences de Paris* **157**(1913) 708

[36] B Rossi and D B Hall *Phys Rev* **59**(1941) 223

[37]

[38]

[39] L H Thomas *Phil Mag***3**(1927)1-22

[40] R Bonola *Non-Euclidean Geometry* Dover re-print of 1912 translation of 1911 original. Contains translations of the original papers by Bolyai and Lobachevksy

[41] B A Rosenfeld *A History of Non-Euclidean geometry* Springer

[42] R Toretti, *Philosophy of Geometry from Riemann to Poincaré* D Reidel

[43] A Calinon *Revue Philosophique* **32**(1891) 368-375

[44] J C F Zollner, *Transcendental Physics*

[45] R Goldblatt *Orthogonality and Spacetime Geometry* (Springer)

[46] A Brillet and J L Hall *Phys Rev* **42** (1979) 549-552

[47] H Helmholtz *Mind* **1** (1876) 301-321

[48] G B Airy *Proc Roy Soc***20**(1871-1872) 35-39

[49] *The Electrician* May 5th (1899) 40-41

[50] E Mach, *The Science of Mechanics*

[51] M Jammer, *Concepts of Space*, Dover

[52] L Lange *Ber der köngl. sächs. Ges. der Wis.* (1885)335-351

[53] J Zeeman, *J Math Phys***4** (1964) 490-493

[54] A D Alexandrov *Can J Math* 1119-1128

[55] L-L Hua, *Proc Roy Scoc* bf A 380 (1982)487-488

Index