Ghosts in Massive Gravity

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with A. Chamseddine and V. Mukhanov, arXiv:1011.0183

Linearized Massive gravity

The linear action for metric perturbations $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[-R + \underbrace{\frac{m_g^2}{4}\left(h^2 - h_{\mu\nu}h^{\mu\nu}\right)}_{\text{Fierz-Pauli mass term (1939)}}\right]$$

- is ghost-free
- propagates 5 massive degrees of freedom

PROBLEMS

- breaks the diffeomorphism invariance of general relativity
- leads to unacceptable observational consequences (vDVZ discontinuity 1970)

SOLUTIONS

- introduction of four Higgs (Stückelberg) fields
- non-linear modifications of Einstein gravity (Vainshtein mechanism 1972)

Higgs massive gravity

Any theory of massive gravity can be represented as Einstein gravity interacting with four scalar Higgs fields ϕ^A with $A = 0, 1, 2, 3$:

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[-R + \frac{m_g^2}{4} \mathcal{L}_{FP} \left(\phi^A, g^{\mu\nu} \right) \right]$$

The interaction term $\mathcal{L}_{FP} \left(\phi^A, g^{\mu\nu} \right)$ is a function of a diffeomorphism invariant composite scalar

$$\bar{h}^A_B = \eta_{BC} g^{\mu\nu} \partial_\mu \phi^A \partial_\nu \phi^C - \delta^A_B$$

such that after expanding the fields around the backgrounds

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}, \quad \phi^A = x^\mu \delta^A_\mu + \chi^A$$

at the linear level the action reduces to the Fierz-Pauli mass term.

The minimal such diffeomorphism invariant generalization of the Fierz-Pauli action is

$$\mathcal{L}_{FP} = \bar{h}^2 - \bar{h}^A_B \bar{h}^B_A$$

PROBLEMS

- in general propagates 6 degrees of freedom
- suffers from ghost instability (Boulware and Deser 1972)

Nonlinear extensions of massive gravity

POSSIBLE SOLUTION: Add higher order terms in $\bar{h}^A_B$!

$\Rightarrow$ There is a unique two-parameter family of potentially consistent non-linear actions which have no ghosts in the limit $m_g \to 0$, $M_{Pl} \to \infty$, $m_g^2 M_{Pl} = \mathrm{const}$

$$S_\phi = \frac{m_g^2}{8} \int d^4x \sqrt{-g} \Big[\bar{h}^2 - \bar{h}^2_{AB} + \frac{1}{2}\left(\bar{h}^3_{AB} - \bar{h}\bar{h}^2_{AB}\right) - \frac{5}{16}\bar{h}^4_{AB} + \frac{1}{4}\bar{h}\bar{h}^3_{AB} + \frac{1}{16}\left(\bar{h}^2_{AB}\right)^2 + \\ + c_3\left(2\bar{h}^3_{AB} - 3\bar{h}\bar{h}^2_{AB} + \bar{h}^3 + \frac{3}{4}\left(2\bar{h}^3_{AB}\bar{h} - 2\bar{h}^4_{AB} + \left(\bar{h}^2_{AB}\right)^2 - \bar{h}^2_{AB}\bar{h}^2\right)\right) \\ + d_5\left(6\bar{h}^4_{AB} - 8\bar{h}^3_{AB}\bar{h} - 3\left(\bar{h}^2_{AB}\right)^2 + 6\bar{h}^2_{AB}\bar{h}^2 - \bar{h}^4\right)\Big]$$

de Rham, Gabadadze (2010)

HOWEVER: It is shown by explicit calculation that away from decoupling limit it contains a negative energy mode in the 4^{th} order in perturbations independently of the choice of c_3 and d_5!

Tracing the dynamics of χ^0 mode

- Consider the matter and scalar field perturbations:

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}, \quad \phi^A = x^\mu \delta^A_\mu + \chi^A$$

CLAIM: The χ^0 mode induces the propagation of the BD ghost

- The only terms in the action which contribute to the $(\dot{\chi}^0)^2$ are

$$\delta_{3+4} S_\phi = \frac{m_g^2}{16} \int d^4x \left[\left(\dot{\chi}^i + h^{0i} + \chi^0_{,i} \right)^2 \left(\dot{\chi}^0 \right)^2 + F(\delta g, \chi^A) \dot{\chi}^0 + \ldots \right]$$

- By using the constraint equation relating χ^0 to the spatial trace of the scalar metric perturbations $\psi = h^i_i/6$

$$\chi^0 = -\frac{2\Delta + 3m_g^2}{m_g^2 \Delta} \dot{\psi}$$

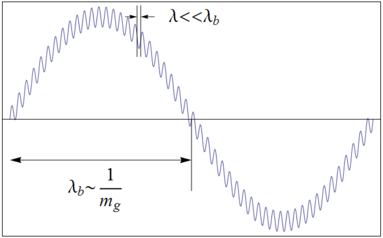
one obtains

$$\delta S = -3 \int d^4x \left[\psi \left(\partial_t^2 - \Delta + m_g^2 \right) \psi - \frac{1}{12 m_g^2} \left({}^{(T)}\dot{\chi}^i + h^{0i} - \frac{6}{\Delta} \dot{\psi}_{,i} \right)^2 \left(\ddot{\psi} \right)^2 + \ldots \right]$$

Ghost on locally non-trivial background

Consider small perturbations around the background fields

$$\psi = \psi_b + \delta\psi, \quad {}^{(T)}\chi^i = {}^{(T)}\chi_b^i + \delta {}^{(T)}\chi^i$$



Then

$$\delta S \approx -\frac{3}{m_{Gh}^2} \int d^4x \delta\psi \left(\partial_t^2 + \ldots\right) \left(\partial_t^2 + m_{Gh}^2 + \ldots\right) \delta\psi$$

with

$$m_{Gh}^2 \approx -12 m_g^2 \left({}^{(T)}\dot{\chi}_b^i + h^{0i} - \frac{6}{\Delta}\dot{\psi}_{b,i}\right)^{-2}$$

Ghost on locally non-trivial background

The perturbation propagator is then

$$\frac{m_{Gh}^2}{\partial^2\left(\partial^2+m_{Gh}^2\right)} \simeq \left(\frac{1}{\partial^2}-\frac{1}{\partial^2+m_{Gh}^2}\right)$$

$\Rightarrow$ two scalar degrees of freedom:

- scalar mode of the massive graviton
- Boulware-Deser ghost with mass

$$m_{Gh} \sim \frac{m_g}{\psi_b}$$

which

- becomes infinitely heavy on flat background
- is trustable only on energy scales $m_{Gh} < \Lambda = m_g^{8/11} \sim 10^{-17} eV$

ADM formalism of the non-linear massive gravity

$3+1$ dimensional parametrization of the gravitational field $g_{\mu\nu}$:

$$\gamma_{ij} = g_{ij}, \quad N = \sqrt{-g^{00}}, \quad N_i = g_{0i}$$

Arnowitt, Deser, Misner (1959)

allows to rewrite the action of general relativity in the first order form

$$\mathcal{L}_E = M_{Pl}^2 \sqrt{-g} R = M_{Pl}^2 \left[-\gamma_{ij} \partial_t \pi^{ij} - N R^0 - N_i R^i \right]$$

$\Rightarrow$ the Einstein action is linear in the lapse N and shift N_i $\Rightarrow$ 2 degrees of freedom!

Meanwhile, the interaction term of massive gravity in the unitary gauge can be written in a closed form as

$$\mathcal{L}_{FP} = 2 m_g^2 \sqrt{\gamma} N \left[\mathrm{tr} \sqrt{g^{-1}\eta} - 3 \right]$$

Hassan, Rosen (2011)

$\Rightarrow$ the massive gravity action is non-linear in N, N_i $\Rightarrow$ 6 degrees of freedom!

CLAIM: After an appropriate field redefinition $N^i = \left(\delta^i_j + N D^i_j \right) n^j$
the Lagrangian is linear in N and gives rise to a secondary constraint.

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Where did the ghost go?

Around flat background the lapse can be expanded in terms of metric perturbations as

$$N = 1 + \delta N, \quad \delta N = -\frac{1}{2}h^{00} + \frac{3}{8}\left(h^{00}\right)^2 + \mathcal{O}\left[\left(h^{00}\right)^3\right]$$

The ghost reappears after reintroducing the scalar fields by coordinate transformation

$x^\mu \to x^\mu - \chi^A \delta^\mu_A$ under which

$$\begin{aligned} h^{00} &\to h^{00} + 2\dot{\chi}^0 + 2\chi^0\ddot{\chi}^0 + \left(\dot{\chi}^0\right)^2 + \dots \\ \delta N &\to \delta N - \dot{\chi}^0 + \left(\dot{\chi}^0\right)^2 - \chi^0\ddot{\chi}^0 + \dots \end{aligned}$$

$\Rightarrow$ The field redefinition $N^i = \left[\delta^i_j + (1+\delta N)D^i_j\right] n^j$ involves 2^{nd} order time derivatives!

Conclusions

- The non-linear BD ghost reappears in the full non-linear theory in the 4^{th} order in perturbations even for the potentially ghost-free extensions of massive gravity.
- The apparent resolution of the ghost problem by field redefinitions involves 2^{nd} order time derivatives and therefore trivially eliminates a degree of freedom.

Thank you for your attention!

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