Effects of *Nonlinear Dispersion Relations* on Non-Gaussianities

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Introduction

- Primordial non-Gaussianities is among the most promising probes of our early universe.

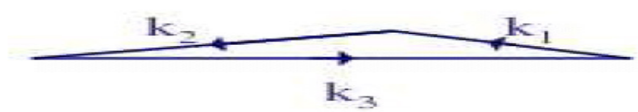
- Any observed departure from a Gaussian spectrum will likely point us to some interesting microphysics.

- Even within the context of single-field inflation, non-canonical kinetic terms can generate large bispectra (3-point function) of the equilateral shapes.

X. Chen, et. al. (2006)

- Or a a non Bunch-Davies initial state of inflation can boost the folded/flattened momentum configurations.

X. Chen, et. al. (2006)
Tolley & Holman (2007)
Ashoorioon & Shiu (2010)



$$k_1 + k_2 = k_3$$

- Features in the potential can result in a large bispectrum with oscillatory running.

X. Chen, R. Easther, E. A. Lim (2006)

- In this talk, I will investigate the effect of non-linear dispersion relations on the bispectrum.

Modified Dispersion Relation

$$u''_{\vec{k}} + \left(\omega(\eta,\vec{k})^2 - \frac{z''}{z}\right) u_{\vec{k}} = 0$$

$u_{\vec{k}}(\eta)$: Mukhanov-Sasaki variable

$$u_{\vec{k}}(\eta) = z\,\zeta_{\vec{k}}(\eta) \qquad z = \frac{a\dot{\phi}}{H}$$

- In the standard QFT formalism, $\omega(\eta,\vec{k}) = k \equiv |\vec{k}|$

- The dispersion relations were motivated by some condensed matter expectation from QG, that were used to study the trans-Planckian issue in Black hole physics.

$$\omega^2 = F^2(p)$$

Unruh (1996)
Jacobson, Corley (1996,1997)

- To implement this in an expanding background:

$$k^2 \Rightarrow a^2(\eta) F^2\left(\frac{k}{a(\eta)}\right)$$

Brandenberger & Martin (2001)
Brandenberger (2002)

we will focus on Corley-Jacobson dispersion relation with positive quartic correction

$$k^2 \Rightarrow a^2(\eta) F^2\left(\frac{k}{a(\eta)}\right) = k^2 + b_1 \frac{k^4}{p_c^2 a^2(\eta)}$$

Modified Dispersion Relation from Effective Field Theory of Inflation

C. Cheung, P. Creminelli, A. L. Fitzpatrick, J. Kaplan and Leonardo Senatore (2008)

Considering fluctuations around a FRW background, using a unitary gauge in which $\delta\phi(\vec{x},t)=0$, the most general form of the action that respects the remaining spatial diffeomorphism

$$S=\int d^4x\,\sqrt{-g}\Big[\frac{1}{2}M_{\rm Pl}^2R+M_{\rm Pl}^2\dot{H}g^{00}-M_{\rm Pl}^2(3H^2+\dot{H})+\frac{1}{2!}M_2(t)^4(g^{00}+1)^2+\frac{1}{3!}M_3(t)^4(g^{00}+1)^3+$$
$$-\frac{\bar{M}_1(t)^3}{2}(g^{00}+1)\delta K^\mu{}_\mu-\frac{\bar{M}_2(t)^2}{2}\delta K^\mu{}_\mu{}^2-\frac{\bar{M}_3(t)^2}{2}\delta K^\mu{}_\nu\delta K^\nu{}_\mu+\ldots\Big]. \qquad (10)$$

Time diffeomorphism is broken in the unitary gauge. The Goldstone boson, π that realizes this symmetry could be restored using the Stueckelberg procedure.

$$\mathcal{L}_2=a^3\Bigg[M_{\rm Pl}^2\dot{H}(\partial_\mu\pi)^2+2M_2^4\dot{\pi}^2-\bar{M}_1^3H\left(3\dot{\pi}^2-\frac{(\partial_i\pi)^2}{2a^2}\right)-\frac{\bar{M}_2^2}{2}\Big(9H^2\dot{\pi}^2$$
$$-3H^2\frac{(\partial_i\pi)^2}{a^2}+\frac{1}{a^4}(\partial_i^2\pi)^2\Big)-\frac{\bar{M}_3^2}{2}\left(3H^2\dot{\pi}^2-H^2\frac{(\partial_i\pi)^2}{a^2}+\frac{1}{a^4}(\partial_j^2\pi)^2\right)\Bigg].$$

Nicola Bartolo, Matteo Fasiello, Sabino Matarrese and A. Riotto (2010)

For $c_s \ll 1$ the terms proportional to $\frac{1}{a^4}(\partial_i^2\pi)^2$ becomes comparable with the quadratic terms. Thus in terms of $u(\vec{k},\eta)=a(\eta)\,\pi(\vec{k},\eta)$

$$u''-\frac{2}{\tau^2}u+\alpha_0k^2u+\beta_0k^4\tau^2u=0,$$

EOM for the perturbations in de-Sitter space

$$u_k''(\eta) + (k^2 + \epsilon^2 k^4 \eta^2 - \frac{2}{\eta^2})u_k = 0 \qquad \epsilon \equiv \frac{b_1^{1/2} H}{p_c}$$

Solution

$$\begin{aligned} u_k &= \frac{C_1}{\sqrt{-\eta}} \mathrm{WW}\left(\frac{i}{4\epsilon}, \frac{3}{4}, -i\epsilon k^2 \eta^2\right) + \frac{C_2}{\sqrt{-\eta}} \mathrm{WW}^*\left(\frac{i}{4\epsilon}, \frac{3}{4}, -i\epsilon k^2 \eta^2\right) \\ &= \frac{C_1}{\sqrt{-\eta}} \mathrm{WW}\left(\frac{i}{4\epsilon}, \frac{3}{4}, -i\epsilon k^2 \eta^2\right) + \frac{C_2}{\sqrt{-\eta}} \mathrm{WW}\left(\frac{-i}{4\epsilon}, \frac{3}{4}, i\epsilon k^2 \eta^2\right) \end{aligned}$$

Requiring the mode behaves like positive frequency WKB mode

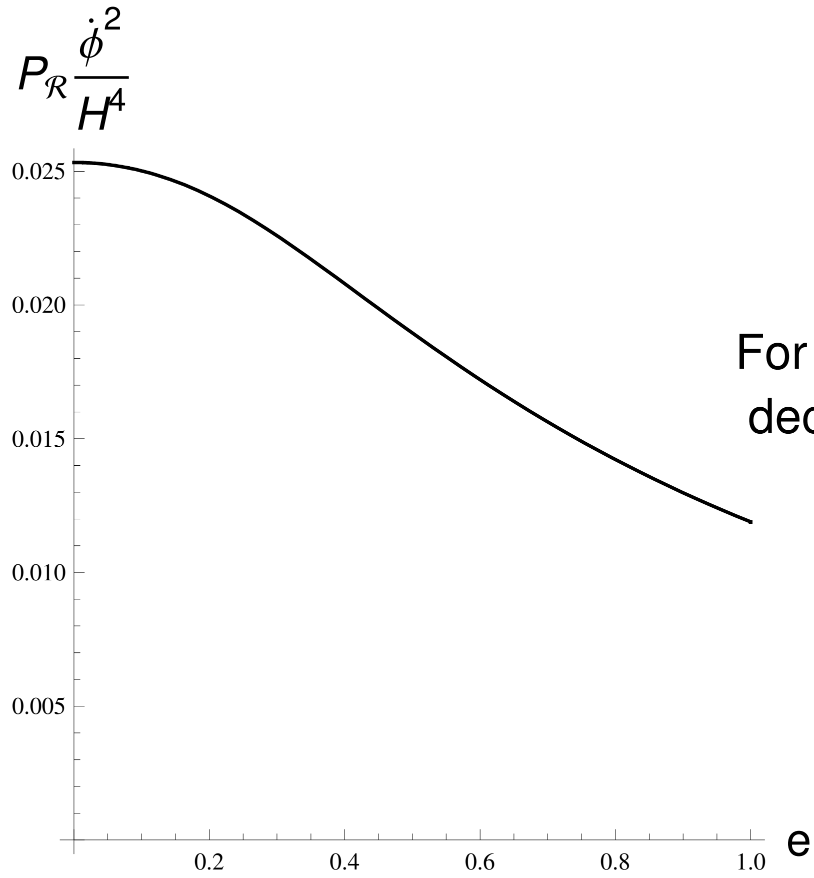
$$\begin{aligned} u_k(\eta) &\simeq \frac{1}{\sqrt{2\omega(\eta)}} \exp(-i \int^{\eta} \omega(\eta) d\eta') \\ &= \frac{1}{k\sqrt{-2\epsilon\eta}} \exp(\frac{i\epsilon k^2 \eta^2}{2}) \end{aligned} \quad \Longrightarrow \quad C_2 = 0$$

$$u(\eta)u'^*(\eta) - u^*(\eta)u'(\eta) = i \quad \Longrightarrow \quad C_1 = \frac{\exp(\frac{-\pi}{8\epsilon})}{\sqrt{2\epsilon k}}$$

$$P_{\mathcal{R}}(\epsilon) = \frac{H^4}{\dot{\phi}^2} \frac{\exp(-\frac{\pi}{4\epsilon})}{16\pi\epsilon^{3/2}\Gamma(\frac{5}{4} - \frac{i}{4\epsilon})\Gamma(\frac{5}{4} + \frac{i}{4\epsilon})}$$

The leading order correction to the power spectrum is proportional to ϵ^2

$$P_{\mathcal{R}}(\epsilon) \simeq \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2 \left(1 - \frac{5}{4}\epsilon^2\right)$$



For $\epsilon >> 1$ the power spectrum decreases like $\epsilon^{-3/2}$

Three-point function

One can calculate the Wightman function:

$$G_k(\eta,\eta') = \frac{\exp(-\frac{\pi}{4\epsilon})\sqrt{\eta\eta'}}{2ek^2}\frac{H^4}{\dot{\phi}^2}\mathrm{WW}\left(\frac{i}{4\epsilon},\frac{3}{4},-i\epsilon k^2\eta^2\right)\mathrm{WW}\left(\frac{-i}{4\epsilon},\frac{3}{4},i\epsilon k^2\eta'^2\right)$$

Recall that

$$\langle\zeta_{\vec{k_1}}\zeta_{\vec{k_2}}\zeta_{\vec{k_3}}\rangle = -i(2\pi)^3\delta^3\left(\sum\vec{k_i}\right)\left(\frac{\dot{\phi}}{H}\right)^4 M_P^{-2}H\int_{\eta_0}^0 d\eta\frac{1}{k_3^2}(a(\eta)\partial_\eta G^>_{k_1}(0,\eta))(a(\eta)\partial_\eta G^>_{k_2}(0,\eta))(a(\eta)\partial_\eta G^>_{k_3}(0,\eta))$$

$$a(\eta)\partial_\eta G^>_k(0,\eta)) = -\frac{(-1)^{\frac{1}{8}}H^3\exp(-\frac{\pi}{4\epsilon})\sqrt{\pi}}{8\epsilon^{\frac{9}{4}}k^{\frac{5}{2}}\dot{\phi}^2(-\eta)^{\frac{3}{2}}\Gamma\left[\frac{5}{4}-\frac{i}{4\epsilon}\right]}\left[-4\epsilon\mathrm{WW}\left(1-\frac{i}{4\epsilon},\frac{3}{4},i\epsilon k^2\eta^2\right)+(i+\epsilon+2i\epsilon^2k^2\eta^2)\times\mathrm{WW}\left(\frac{-i}{4\epsilon},\frac{3}{4},i\epsilon k^2\eta^2\right)\right]$$

First we write WW functions in terms of KummerM. Then we expand KummerM functions in terms of Bessel-J function

$$\mathrm{M}(a,b,z) = \Gamma(b)\exp(hz)\sum_{n=0}^{\infty}C_n z^n(-az)^{\frac{1}{2}(1-b-n)}J_{b-1+n}(2\sqrt{-az})$$

$$C_{n+1} = \frac{1}{(n+1)}\left[((1-2h)n-bh)C_n+((1-2h)a-h(h-1)(b+n-1))C_{n-1}-h(h-1)aC_{n-2}\right],$$

$$C_0=1,\qquad C_1=-bh,\qquad C_2=-\frac{1}{2}(2h-1)a+\frac{b}{2}(b+1)h^2$$

Abramowitz & Stegun

We set $h = 0$ in the expansion:

$$a(\eta)\partial_\eta G_k^>(0,\eta)) \simeq \frac{H^3}{\dot\phi^2}\exp\left(\frac{-i\epsilon k^2\eta^2}{2}+ik\eta\right)\left[-\frac{1}{2k}-\epsilon\frac{ik\eta^2}{4}+\epsilon^2\left(\frac{5}{8k}-i\frac{5\eta}{8}-\frac{1}{8}k\eta^2-\frac{i}{12}k^2\eta^3+\frac{1}{16}k^3\eta^4\right)\right]$$

BD-vacuum
QFT result

$$\langle\zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\rangle_0 = -i(2\pi)^3\delta^3\left(\sum\vec{k}_i\right)\left(\frac{\dot\phi}{H}\right)^4 M_P^{-2}H\int_{-\infty}^{0} d\eta\frac{1}{k_3^2}\frac{-H^9}{8\dot\phi^6k_1k_2k_3}\exp\left[-i\eta(k_1+k_2+k_3)+\frac{i\epsilon\eta^2}{2}(k_1^2+k_2^2+k_3^2)\right]$$

$$+ \text{ c.c.} + \text{permutations}$$

$$\langle\zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\rangle_0 = 2\Re\left[\frac{H^6}{\dot\phi^2M_P^2}\frac{(-1)^{\frac{3}{4}}\pi^{\frac{7}{2}}\exp(-\frac{ik_t^2}{2\epsilon k_s^2})\mathrm{Erfi}\left[\frac{(1-i)(-k_t+\epsilon k_s^2\eta)}{2\sqrt{\epsilon k_s^2}}\right]}{k_1k_2k_3^3\sqrt{2\epsilon k_s^2}}\right]_{\eta=-\infty}^{\eta=0} + \text{permutations},$$

$$k_t \equiv k_1+k_2+k_3$$

$$k_s^2 \equiv k_1^2+k_2^2+k_3^2$$

In contrast to the Lorentzian dispersion relation, the integrand remains finite at $\eta = -\infty$ after the change of variable from $\eta \to \eta + i\epsilon|\eta|$

$$\langle\zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\rangle_0 \simeq \delta^3\left(\sum_i\vec{k}_i\right)\left[\frac{2H^6\pi^3}{\dot\phi^2k_1k_2k_3k_tM_p^2}-\epsilon^2\frac{6H^6k_s^4\pi^3}{\dot\phi^2k_1k_2k_3k_t^5M_p^2}\right]\left(\frac{1}{k_1^2}+\frac{1}{k_2^2}+\frac{1}{k_3^2}\right)$$

Contribution of the term proportional to ϵ and could be also calculated

$$\langle\zeta_{\vec{k_1}}\zeta_{\vec{k_2}}\zeta_{\vec{k_3}}\rangle_1 \simeq \delta^3\left(\sum_i \vec{k_i}\right)\epsilon^2 \frac{12H^6k_s^4\pi^3}{\dot{\phi}^2k_1k_2k_3k_t^5M_p^2}\left(\frac{1}{k_1^2}+\frac{1}{k_2^2}+\frac{1}{k_3^2}\right)$$

$$\langle\zeta_{\vec{k_1}}\zeta_{\vec{k_2}}\zeta_{\vec{k_3}}\rangle_2 \simeq \delta^3\left(\sum \vec{k_i}\right)\epsilon^2 \frac{\pi^3H^6(6k_s^4-2k_c^3k_t+k_s^2k_t^2+10k_t^4)}{k_1k_2k_3k_t^5M_P^2\dot{\phi}^2}\left(\frac{1}{k_1^2}+\frac{1}{k_2^2}+\frac{1}{k_3^2}\right)$$

$$\langle\zeta_{\vec{k}_1}(\eta)\zeta_{\vec{k}_2}(\eta)\zeta_{\vec{k}_3}(\eta)\rangle = \delta(\sum\vec{k}_i)(2\pi)^3F(\vec{k}_1,\vec{k}_2,\vec{k}_3,\eta), \qquad k_c^3 = k_1^3+k_2^3+k_3^3$$

F is a function of degree -6. Due to rotational invariance, it is a function of $x_2 \equiv k_2/k_1$ and $x_3 \equiv k_3/k_1$

$$x_3 \leq x_2 \qquad 1-x_2 \leq x_3 \longrightarrow \text{Triangle Inequality}$$

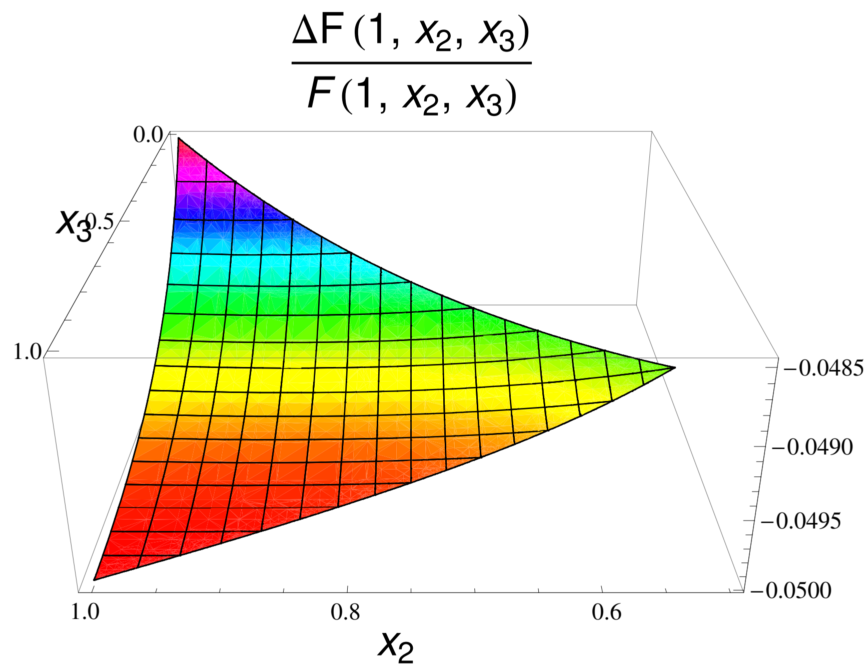
1) Local configuration: $x_3 \simeq 0 \text{ and } x_2 \simeq 1$

Models in which non-linearities are developed beyond the horizon, tend to produce a more local type of non-Gaussianity, like the Curvaton model [Lyth, Wands (2001)].

2) Equilateral configuration: $x_2 \simeq x_3 \simeq 1$

Correlation is among the modes with comparable wavelength that leave the horizon around the same time. These models tend to produce equilateral type of non-Gaussianities. Models like DBI inflation [Alishahiha, Silverstein, Tong (2004)] that involve derivative interactions.

$$\frac{\Delta F(\overrightarrow{k_1}, \overrightarrow{k_2}, \overrightarrow{k_3})}{F(\overrightarrow{k_1}, \overrightarrow{k_2}, \overrightarrow{k_3})} = (-5 + \frac{k_c^3}{k_t^3} - \frac{k_s^2}{2k_t^2})\epsilon^2$$



The quartic correction to the dispersion relation decreases the amplitude of NG.

The largest suppression occurs for the equilateral configurations.

Comparison with the gluing method:

$$(k|\eta|)^2 < 2 \qquad \text{region I}$$

$$2 < (k|\eta|)^2 < \epsilon^{-2} \qquad \text{region II}$$

$$(k|\eta|)^2 > \epsilon^{-2} \qquad \text{region III}$$

Martin & Brandenberger (2001)

- Knowing that the evolution in the region III leads to excited state as the initial condition for region II, one would expect to observe enhancement for the folded configurations!

- In fact,

$$u_k = \frac{\alpha_k}{\sqrt{2k}} e^{-ik\eta} + \frac{\beta_k}{\sqrt{2k}} e^{ik\eta} \qquad \begin{cases} \alpha_k = e^{-\frac{i}{2\epsilon}} \left(1 + i\frac{\epsilon}{4}\right) \\ \beta_k = -ie^{i\frac{3}{2\epsilon}} \left(\frac{\epsilon}{4}\right) \end{cases}$$

$$\frac{\Delta F}{F} \approx |C_4 k \eta_c| \approx 1 \qquad \eta_c = -\frac{1}{\epsilon k}$$

- The gluing method predicts enhancement to be of order one for the bispectrum!

- Calculating the power spectrum using the gluing method also shows that $\frac{\Delta P_s}{P_s} \approx \epsilon$!

- This specific example shows the incapability of the gluing method in capturing the correct modification.

Conclusion

- We calculated the effect of Jacobson-Corley nonlinear dispersion relation on the shape function for the non-Gaussianities

- The modification is small and slightly scale-dependent. Major modification occurs for the equilateral configurations.

- We showed that the gluing method overestimates the leading order correction to the spectrum and bispectrum by one and two orders, respectively, in H/p_c

- In the paper by Bartolo et. al. (2010), various bispectra, that are generated in effective field theory approach to single field inflation, is studied numerically. However they had assumed that the contribution of the contribution of the integral function at $\eta = -\infty$ has been set to zero, which is not really a correct thing to do!

$$S_3 = \int d^4x\sqrt{-g}\left[M_{\rm Pl}^2\dot{H}(\partial_\mu\pi)^2 + M_2(t)^4\left(2\dot{\pi}^2 - 2\dot{\pi}\frac{(\partial_i\pi)^2}{a^2}\right) - \frac{4}{3}M_3(t)^4\dot{\pi}^3\right.$$
$$- \frac{\bar{M}_1(t)^3}{2}\left(\frac{-2H(\partial_i\pi)^2}{a^2} + \frac{(\partial_i\pi)^2\partial_j^2\pi}{a^4}\right) - \frac{\bar{M}_2(t)^2}{2}\left(\frac{(\partial_i^2\pi)(\partial_j^2\pi) + H(\partial_i^2\pi)(\partial_j\pi)^2 + 2\dot{\pi}\partial_i^2\partial_j\pi\partial_j\pi}{a^4}\right)$$
$$- \frac{\bar{M}_3(t)^2}{2}\left(\frac{(\partial_i^2\pi)(\partial_j^2\pi) + 2H(\partial_i\pi)^2\partial_i^2\pi + 2\dot{\pi}\partial_{ij}^2\pi\partial_j\pi}{a^4}\right) - \frac{2}{3}\bar{M}_4(t)^3\frac{1}{a^2}\dot{\pi}^2\partial_i^2\pi + \frac{\bar{M}_5(t)^2}{3}\frac{\dot{\pi}}{a^4}(\partial_i^2\pi)^2$$
$$\left. + \frac{\bar{M}_6(t)^2}{3}\frac{\dot{\pi}}{a^4}(\partial_{ij}\pi)^2 - \frac{\bar{M}_7(t)}{3!}\frac{(\partial_i^2\pi)^3}{a^6} - \frac{\bar{M}_8(t)}{3!}\frac{\partial_i^2\pi}{a^6}(\partial_{jk}\pi)^2 - \frac{\bar{M}_9(t)}{3!}\frac{1}{a^6}\partial_{ij}\pi\partial_{jk}\pi\partial_{ki}\pi\right]. \quad (13)$$

Thank you!