APPLICATIONS OF MODAL METHODS TO CMB NON-GAUSSIANITY

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[ArXiv: 0812.3413, 0912.5516, 1004.2915, 1006.1642, 1012.6039, 1105.2791, 1105.xxxx]
INTRODUCTION

Non-Gaussianity is a good discriminator for models of the early universe.

The CMB is currently the best dataset we have for searching for non-Gaussianities.

Higher order correlators as the best means for finding non-Gaussianity in the CMB.
Higher order correlators can be written as

\[ \langle a_{\varphi} \rangle \equiv \langle a_{l_1m_1} a_{l_2m_2} \ldots a_{l_pm_p} \rangle \]

\[ c_{\varphi \varphi}^{-1} \equiv C_{l_1m_1,l_1'm_1}^{-1} \ldots C_{l_pm_p,l_p'm_p}^{-1} \]

Where \( \varphi \) represents the \( \varphi = \{l_1, m_1, l_2, m_2, \ldots, l_p, m_p\} \) degrees of freedom
The estimator for a general polyspectrum is then defined as

\[ \bar{\mathcal{E}} \equiv \frac{\sum_{\varphi \varphi'} \langle a_{\varphi} \rangle \mathcal{C}_{\varphi \varphi'}^{-1} (a_{\varphi} - a_{\varphi}^{lin})}{\sum_{\varphi \varphi'} \langle a_{\varphi} \rangle \mathcal{C}_{\varphi \varphi'}^{-1} \langle a_{\varphi} \rangle} \]

where \( a_{\varphi}^{lin} \) is the appropriate linear term.
We will now go one step further by defining the weighted vectors (and matrix)

\[ A_\varphi = \frac{\langle a_\varphi \rangle}{\sqrt{C_{l_1}C_{l_2}...C_{l_p}}} , \quad B_\varphi = \frac{a_\varphi - a^\text{lin}_\varphi}{\sqrt{C_{l_1}C_{l_2}...C_{l_p}}} , \quad C_{\varphi\varphi'} = \frac{e_{\varphi\varphi'}}{\sqrt{C_{l_1}C'_{l_1}...C_{l_p}C'_{l_p}}} , \]

And we can then write the estimator in matrix form as

\[ \bar{\mathcal{E}} = \frac{A^T C^{-1} B}{A^T C^{-1} A} \]
BASIS

If we then suppose the existence of an orthonormal basis

$$\sum_{n} R_{n\varphi} R_{n'\varphi} = \delta_{nn'} \quad (RR^T = I)$$

built from some separable functions $R = \lambda Q$

$$R_{n\varphi} = \lambda_{ns} Q_{s\varphi} = \lambda_{ns} (q_{s_1 l_1 m_1} \cdots q_{s_n l_n m_n} + \text{symm})$$
BASIS

Then we can decompose our theory representing it as a set of modal coefficients

$$A_\varphi = \sum_n \alpha_n R_{n\varphi} \quad (A = R^T \alpha)$$

$$\alpha = RA$$

We will truncate our basis at some \( n_{\text{max}} \) so we can also define a projection operator

$$\mathcal{P} = R^T R$$

And we take our theory to be completely described by this basis

$$\mathcal{P} A = A$$
BASIS

\[ \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \ldots \]
BASIS

We can perform the same modal decomposition on the data and the covariance

\[
\alpha = \mathcal{R}A \\
\beta = \mathcal{R}B \rightarrow \mathcal{P}B = \mathcal{R}^T \beta \\
\zeta = \mathcal{R}C\mathcal{R}^T
\]

\[
E \equiv \frac{\alpha^T \zeta^{-1} \beta}{\alpha^T \zeta^{-1} \alpha}
\]

\[
= \frac{(\mathcal{R}A)^T \mathcal{R}C^{-1} \mathcal{R}^T \mathcal{R}B}{\mathcal{R}A^T \mathcal{R}C^{-1} \mathcal{R}^T \mathcal{R}A} = \frac{\mathcal{A}^T \mathcal{P}C^{-1} \mathcal{P}B}{\mathcal{A}^T \mathcal{P}C^{-1} \mathcal{P}A}
\]
We can understand the effect of the projection by considering

\[
\mathbf{A} = \begin{bmatrix} \mathbf{A}_\parallel \\ 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_\parallel \\ \mathbf{B}_\perp \end{bmatrix} \quad \mathbf{C}^{-1} = \begin{bmatrix} \mathbf{C}_\parallel^{-1} & \mathbf{C}_\times^{-1} \\ \mathbf{C}_\parallel^{-1} \mathbf{T} & \mathbf{C}_\perp^{-1} \end{bmatrix}
\]

\[
\mathbf{x}_\parallel \equiv \mathbf{P} \mathbf{x} \\
\mathbf{x}_\perp \equiv (\mathbf{I} - \mathbf{P}) \mathbf{x} \\
\mathbf{M}_\parallel \equiv \mathbf{P} \mathbf{M} \mathbf{P} \\
\mathbf{M}_\perp \equiv (\mathbf{I} - \mathbf{P}) \mathbf{M} (\mathbf{I} - \mathbf{P}) \\
\mathbf{M}_\times \equiv \mathbf{P} \mathbf{M} (\mathbf{I} - \mathbf{P})
\]
BASIS

We can understand the effect of the projection by considering

\[ \bar{E} = \frac{A_\| \left( C^{-1}_\| B_\| + C^{-1}_\times B_\perp \right)}{A^T C^{-1}_\| A_\|} \]

\[ E = \frac{A_\| C^{-1}_\| B_\|}{A^T C^{-1}_\| A_\|} \]

The difference is the projection of contamination from the orthogonal space into the subspace \( \| \).
INVERSE COVARIANCE

Can we even calculate the covariance in the modal space?

Yes!

\[ \zeta = \frac{1}{6} \left\langle \beta \beta^T \right\rangle \]

\[
\langle n_n' \rangle = \sum_{l_1 \ell_1 \ell_2 \ell_3} \left( \frac{g_{l_1 l_2 l_3}}{v_1 v_2 v_3} a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} - 3 C_{l_1 m_1, l_2 m_2} a_{l_3 m_3} R_{n l_1 l_2 l_3} \right) 
\times \left( \frac{g_{l_1' l_2' l_3'}}{v_1' v_2' v_3'} a_{l_1' m_1'} a_{l_2' m_2'} a_{l_3' m_3'} - 3 C_{l_1' m_1', l_2' m_2'} a_{l_3' m_3'} R_{n l_1' l_2' l_3'} \right) 
\]

\[
= \sum_{l_1 \ell_1 \ell_2 \ell_3} \frac{g_{l_1 l_2 l_3}}{v_1 v_2 v_3} g_{l_1' l_2' l_3'} R_{n l_1 l_2 l_3} \left[ 6 \left\langle a_{l_1 m_1} a_{l_1' m_1'} \right\rangle \left\langle a_{l_2 m_2} a_{l_2' m_2'} \right\rangle \left\langle a_{l_3 m_3} a_{l_3' m_3'} \right\rangle 
+ 9 \left\langle a_{l_1 m_1} a_{l_2 m_2} \right\rangle \left\langle a_{l_1' m_1'} a_{l_2' m_2'} \right\rangle \left\langle a_{l_3 m_3} a_{l_3' m_3'} \right\rangle - 9 C_{l_1 m_1, l_2 m_2} \left\langle a_{l_1' m_1'} a_{l_2' m_2'} \right\rangle \left\langle a_{l_3 m_3} a_{l_3' m_3'} \right\rangle 
- 9 \left\langle a_{l_1 m_1} a_{l_2 m_2} \right\rangle C_{l_1 m_1, l_2 m_2} \left\langle a_{l_1' m_1'} a_{l_2' m_2'} \right\rangle \left\langle a_{l_3 m_3} a_{l_3' m_3'} \right\rangle + 9 C_{l_1 m_1, l_2 m_2} C_{l_1' m_1', l_2' m_2'} \left\langle a_{l_3 m_3} a_{l_3' m_3'} \right\rangle \right] R_{n l_1' l_2' l_3'} 
\]

\[
= 6 \sum_{l_1 \ell_1 \ell_2 \ell_3} \frac{g_{l_1 l_2 l_3}}{v_1 v_2 v_3} g_{l_1' l_2' l_3'} R_{n l_1 l_2 l_3} \frac{C_{l_1 m_1, l_1' m_1'} C_{l_2 m_2, l_2' m_2'} C_{l_3 m_3, l_3' m_3'}}{\sqrt{C_{l_1 l_2 l_3} C_{l_1' l_2' l_3'}}} R_{n l_1' l_2' l_3'} 
\]
INVERSE COVARIANCE

Also as all covariance matrices are symmetric positive definite they have a Cholesky decomposition

\[ \zeta = \tilde{\lambda} \tilde{\lambda}^T \]

And we can absorb the covariance into our modes. This amounts to a re-orthogonalisation to an uncorrelated orthonormal basis

\[ \alpha' = \tilde{\lambda}^{-1} \alpha \quad \beta' = \tilde{\lambda}^{-1} \beta \]

\[ \mathcal{E} = \frac{\alpha'^T \beta'}{\alpha'^T \alpha'} \quad \zeta' = I \]
ESTIMATION

We have used these methods to constrain all scale invariant models and an oscillatory model for a selection of parameter space via the bispectrum

<table>
<thead>
<tr>
<th>Model</th>
<th>$F_{NL}$</th>
<th>$(f_{NL})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>35.1 ± 27.4</td>
<td>(149.4 ± 116.8)</td>
</tr>
<tr>
<td>DBI</td>
<td>26.7 ± 26.5</td>
<td>(146.0 ± 144.5)</td>
</tr>
<tr>
<td>Equilateral</td>
<td>25.1 ± 26.4</td>
<td>(143.5 ± 151.2)</td>
</tr>
<tr>
<td>Flat (Smoothed)</td>
<td>35.4 ± 29.2</td>
<td>(18.1 ± 14.9)</td>
</tr>
<tr>
<td>Ghost</td>
<td>22.0 ± 26.3</td>
<td>(138.7 ± 165.4)</td>
</tr>
<tr>
<td>Local</td>
<td>54.4 ± 29.4</td>
<td>(54.4 ± 29.4)</td>
</tr>
<tr>
<td>Orthogonal</td>
<td>−16.3 ± 27.3</td>
<td>(−79.4 ± 133.3)</td>
</tr>
<tr>
<td>Single</td>
<td>28.8 ± 26.6</td>
<td>(142.1 ± 131.3)</td>
</tr>
<tr>
<td>Warm</td>
<td>24.2 ± 27.3</td>
<td>(94.7 ± 106.8)</td>
</tr>
</tbody>
</table>

\[
S^{feat}(k_1, k_2, k_3) = \frac{1}{N} \sin \left( 2\pi \frac{k_1 + k_2 + k_3}{3k^*} + \Phi \right)
\]

Tuesday, 5 July 2011
This method can be used to simulate maps with a given bispectrum and trispectrum

\[
a_{lm} = a_{lm}^G + \frac{1}{6} F_{NL} a_{lm}^B + \frac{1}{24} G_{NL} a_{lm}^T
\]

\[
a_{lm}^B = \sum_{l_1 m_1} \int Y_{l_1 m_1} Y_{l_2 m_2} Y_{l_3 m_3} b_{l_1 l_2 l_3} \frac{a_{l_2 m_2}^G}{C_{l_2}} \frac{a_{l_3 m_3}^G}{C_{l_3}}
\]

\[
a_{lm}^T = \sum_{l_1 m_1} \int Y_{l_1 m_1} Y_{l_2 m_2} Y_{l_3 m_3} Y_{l_4 m_4} t_{l_1 l_2 l_3 l_4} \frac{a_{l_2 m_2}^G}{C_{l_2}} \frac{a_{l_3 m_3}^G}{C_{l_3}} \frac{a_{l_4 m_4}^G}{C_{l_4}}
\]
Using the expansion the non-Gaussian contributions can be easily calculated

\[ a^{B}_{lm} = \sum_{n} \alpha_n^{Q} \frac{q^i_n}{v_l \sqrt{C_l}} \int d^2 \hat{n} Y_{lm}(\hat{n}) M^j(\hat{n}) M^k(\hat{n}) \]

\[ M^i(\hat{n}) = \sum_{lm} \frac{q^i_l Y_{lm}(\hat{n}) a^{G}_{lm}}{v_l \sqrt{C_l}} \]
We have $\langle \beta \rangle = \alpha$ so can reconstruct the best fit bispectrum to the data by using the $\beta$ as our $\alpha$. If we have constructed a primordial basis as well then we can use the decomposition of projected primordial modes to find the best fit primordial bispectrum.
In addition to constraining particular models we can perform a blind search

\[ F_{NL}^2 = \frac{\beta' T \beta'}{\alpha' T \alpha'} \]
CONTAMINANTS

As we expect the covariance matrix to be the identity we can use principle component analysis to identify the shape of contaminants.

We first calculate the covariance matrix for beta from simulations

\[ V \zeta V^T = D \]

And then find the rotation which diagonalises it. This is equivalent to performing an eigen decomposition. The result is that you obtain a new orthonormal basis but now your modes are uncorrelated and ordered from greatest to least variance.
CONTAMINANTS

WMAP Mask + Inhomogeneous noise
CONTAMINANTS

Point sources
CONCLUSIONS

**PRIMORDIAL**

- Shape mode expansion
  \[ S(k, k, k) = \sum_n \alpha_n Q_n \]

- Map-making
  - validation
  - variance estimate

- Primordial estimator
  - transfer functions
  - map mode filtering

**CMB**

- Direct CMB bispectrum calculation
  \[ S(k, k, k) \rightarrow B_{\text{|||}} \]

- CMB mode expansion
  \[ B_{\text{|||}} = \sum_n \bar{\alpha}_n \bar{Q}_n \]

- Map-making
  - validation
  - variance estimate

- CMB estimator
  - map mode filtering

**Bispectrum measures**

- Primordial estimator
  \[ f_{NL} = \sum_n \alpha_n \beta_n \]

- CMB estimator
  \[ f_{NL} = \sum_n \bar{\alpha}_n \bar{\beta}_n \]

**Cosmic microwave background background maps**

- \[ \frac{\Delta T}{T} \]

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Figure 1: Flow chart for the two general estimator methodologies described and implemented in this article using complete separable mode expansions. Note the overall redundancy which assists estimator validation and the independence of the extraction of expansion coefficients from theory \( \alpha_n \) (cycle 1) and data \( \beta_n \) (cycle 2). Explanations for the schematic equations can be found in the main text.

Present measurements of this local \( f_{NL} \) are equivocal with the WMAP team reporting \[ f_{NL} = r_n \pm s_m \] and with other teams obtaining higher \[ f_{NL} = p_u \pm q_m \] while with improved WMAP noise analysis a lower value was found \[ f_{NL} = p_u \pm q_m \] but at a similar \( \sigma \) significance. The Planck satellite experiment is expected to markedly improve precision measurements with \[ \Delta f_{NL} = r_u \] or better. Further motivation for the study of the bispectrum comes from the prospect of distinguishing alternative more complex models of inflation which can produce non-Gaussianity with potentially observable amplitudes \( f_{NL} \) but also in a variety of different bispectrum shapes that is consistent with the non-Gaussian signal peaked for different triangle configurations of wavevectors. To date only special separable bispectrum shapes have been constrained by CMB data, that is, those that can be expressed schematically in the form

\[ B_{k_1, k_2, k_3} = \sum \alpha_{k_1} \beta_{k_2} \gamma_{k_3} \] or else can be accurately approximated in this manner. All CMB analyses such as those quoted above for the local shape \( B_{|||} \) exploits this separability to reduce the dimensionality of the required integrations and summations to bring them to a tractable form. The separable approach reduces the problem from one of \( O(d_{l_5}^5 \max) \) operations to a manageable \( O(d_{l_3}^3 \max) \).

Other examples of meaningful constraints on separable bispectrum shapes using WMAP data include those for the equilateral shape \( B_{|||} \) and another shape 'orthogonal' to both equilateral and local. Despite these three shapes being a good approximation to non-Gaussianity from a number of classes of inflation models, they are not exhaustive in their coverage of known primordial models nor other types of late-time non-Gaussianity, such as that from cosmic strings. They cannot be expected to be, given the functional degrees of freedom available.

Bringing observations to bear on this much broader class of cosmological models, therefore, is the primary motivation for this paper.

In a previous paper, we described a general approach to the estimation of non-is separable CMB bispectrum. The method has developed out of the first direct calculations of the reduced CMB bispectrum \( B_{l_1, l_2, l_3} \) which surveyed a wide variety of non-is separable primordial models, revealing smooth coherent patterns of non-Gaussianity.