A systematic approach to model building

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PASCOS — 5^{th} July 2011

Talk based on arXiv:1105.4604

Overview

Want to systematically find all the symmetries of an action,
$\rightarrow$ even if symmetry is spontaneously broken,
$\rightarrow$ also derive parameter relationships that give enhanced symmetries.

Overview:

- The Lie point symmetry (LPS) method.
- Example with 2 scalars.
- Automation.
- N interacting scalars.
- Spin-1 plus N scalars.
- Spontaneous symmetry breaking.
- The standard model.

The Lie point symmetry method

The Lie point symmetry method consists of finding the determining equations, whose solutions describe infinitesimal symmetries, and then solving these equations.

Point: transformations depend only on the coordinates and fields themselves, and not the derivatives of the fields.

1. Derive the determining equations of the system.
   Equations for infinitesimal variations of the coordinates and fields.
2. Solve the determining equations, or at least reduce to standard form. Solution can branch depending on parameter values.
3. Compute the rank (number of generators) of the symmetry set(s).
4. Optional: compute the action of the symmetries.

Variation of the action

Lie point symmetries: $\eta^\mu(x,\phi)$, $\chi_i(x,\phi)$.

$$x^\mu \to x^\mu + \eta^\mu$$
$$\phi_i \to \phi_i + \chi_i$$

$S \to S + \delta S$ should be unchanged.

Solve for the fields $\to$ Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) = 0\,.$$

Solve for the infinitesimals $\to$ master determining equation:

$$\boxed{\mathcal{L}\frac{\mathrm{d}\eta^\mu}{\mathrm{d}x^\mu} + \frac{\partial \mathcal{L}}{\partial x^\mu}\eta^\mu + \frac{\partial \mathcal{L}}{\partial \phi_i}\chi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\left(\frac{\mathrm{d}\chi_i}{\mathrm{d}x^\mu} - \frac{\partial \phi_i}{\partial x^\nu}\frac{\mathrm{d}\eta^\nu}{\mathrm{d}x^\mu}\right) = 0}$$

Total derivative: $\frac{\mathrm{d}}{\mathrm{d}x^\mu} \equiv \frac{\partial}{\partial x^\mu} + \frac{\partial \phi_i}{\partial x^\mu}\frac{\partial}{\partial \phi_i}$.

Example: two scalars

No coordinates symmetries, $\phi_i \to \phi_i + \chi_i(\phi_i)$.
Master determining equation:

$$\frac{\partial \mathcal{L}}{\partial \phi_i}\chi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\frac{\partial \phi_j}{\partial x^\mu}\frac{\partial \chi_i}{\partial \phi_j} = 0\,.$$

Apply to Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi_1\partial_\mu\phi_1 + \frac{1}{2}\partial^\mu\phi_2\partial_\mu\phi_2 - \frac{1}{2}m_1^2\phi_1^2 - \frac{1}{2}m_2^2\phi_2^2\,.$$

Determining equation is

$$\begin{aligned} &- m_1^2\phi_1\chi_1 - m_2^2\phi_2\chi_2 + \partial^\mu\phi_1\partial_\mu\phi_1\frac{\partial\chi_1}{\partial\phi_1} \\ &\quad + \partial^\mu\phi_1\partial_\mu\phi_2\frac{\partial\chi_1}{\partial\phi_2} + \partial^\mu\phi_2\partial_\mu\phi_1\frac{\partial\chi_2}{\partial\phi_1} + \partial^\mu\phi_2\partial_\mu\phi_2\frac{\partial\chi_2}{\partial\phi_2} = 0\,. \end{aligned}$$

Equate independent terms to zero:

$$-m_1^2\phi_1\chi_1 - m_2^2\phi_2\chi_2 = 0\,, \quad \frac{\partial\chi_1}{\partial\phi_1} = 0\,, \quad \frac{\partial\chi_1}{\partial\phi_2} + \frac{\partial\chi_2}{\partial\phi_1} = 0\,, \quad \frac{\partial\chi_2}{\partial\phi_2} = 0\,.$$

Example: two scalars

Determining equations:

$$-m_1^2\phi_1\chi_1 - m_2^2\phi_2\chi_2 = 0\,, \quad \frac{\partial\chi_1}{\partial\phi_1} = 0\,, \quad \frac{\partial\chi_1}{\partial\phi_2} + \frac{\partial\chi_2}{\partial\phi_1} = 0\,, \quad \frac{\partial\chi_2}{\partial\phi_2} = 0\,.$$

General solution to last three equations:

$$\chi_1(\phi_2) = \alpha_1 + \beta\phi_2\,, \qquad \chi_2(\phi_1) = \alpha_2 - \beta\phi_1\,.$$

Maximum rank $R = 3$. Symmetries:

- α_1: shift of ϕ_1.
- α_2: shift of ϕ_2.
- β: rotation between ϕ_1 and ϕ_2.

Final determining equation is

$$\alpha_1 m_1^2\phi_1 + \alpha_2 m_2^2\phi_2 + \beta(m_1^2 - m_2^2)\phi_1\phi_2 = 0\,.$$

$\longrightarrow$ *the model parameters dictate the symmetries.*

Automation of LPS method

Two (massive) scalars have algebraic determining equation

$$\alpha_1 m_1^2 \phi_1 + \alpha_2 m_2^2 \phi_2 + \beta(m_1^2 - m_2^2)\phi_1\phi_2 = 0\,.$$

Gaussian elimination (with branching) to find null space of

$$\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_1^2 - m_2^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta \end{pmatrix} = 0\,.$$

Differential equations $\rightarrow$ generalised Gaussian elimination.

Define ordering on η^μ and χ_i. Sort terms. Arrange as rows.

Perform "row reduction" to "diagonal" form.

$$c_1(\lambda_i)\,\partial_i f + X_1(f) = 0\,,$$
$$c_2(\lambda_i)\,\partial_{i+j} f + X_2(f) = 0\,.$$

- $c_1(\lambda_i) = 0$: remove $\partial_i f$ term.
- $c_1(\lambda_i) \neq 0$: use $\partial_i f$ to eliminate $\partial_{i+j} f$.

N interacting scalar fields

Symmetries dictated by structure of interactions between fields.

General Lagrangian for N spin-0 fields

$$\mathcal{L} = T_{ij}\partial^\mu\phi_i\partial_\mu\phi_j - V(\phi)\,.$$

Determining equations are

$$V\partial_\mu\eta^\mu + \frac{\partial V}{\partial\phi_i}\chi_i = 0\,,$$

$$\partial^\mu\chi_i - V\frac{\partial\eta^\mu}{\partial\phi_i} = 0 \qquad \forall\mu\;\forall i\,,$$

$$\partial^\mu\eta^\nu + \partial^\nu\eta^\mu = 0 \qquad \forall\mu\;\forall\nu,\; \mu\neq\nu\,,$$

$$\frac{\partial\chi_i}{\partial\phi_j} + \frac{\partial\chi_j}{\partial\phi_i} = 0 \qquad \forall i\;\forall j,\; i\neq j\,,$$

$$\frac{1}{2}\partial_\sigma\eta^\sigma - \partial_{\bar{\mu}}\eta^{\bar{\mu}} + \frac{\partial\chi_{\bar{i}}}{\partial\phi_{\bar{i}}} = 0 \qquad \forall\bar{\mu}\;\forall\bar{i}\,,$$

$$\frac{\partial\eta^\mu}{\partial\phi_i} = 0 \qquad \forall\mu\;\forall i\,.$$

N interacting scalar fields

Recall general Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi_i \partial_\mu \phi_i - V(\phi)\,.$$

For $D \neq 2$ the general coordinate symmetries are ($b^{\mu\nu}$ anti-symm)

$$\eta^\mu(x) = a^\mu + b^\mu{}_\nu x^\nu - \frac{2\gamma}{D-2}x^\mu\,.$$

General field symmetries are (β_{ij} anti-symm)

$$\chi_i(\phi) = \alpha_i + \beta_{ij}\phi_j + \gamma\phi_i\,.$$

Remaining determining equation is

$$-\frac{2D}{D-2}\gamma V + \frac{\partial V}{\partial \phi_i}\left(\alpha_i + \beta_{ij}\phi_j + \gamma\phi_i\right) = 0\,.$$

Form of V $\leftrightarrow$ allowed symmetries.

Spin-1 plus N scalars

$$\mathcal{L} = -\frac{1}{2}\partial^\mu \phi_i \partial_\mu \phi_i - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J_i A^\mu \partial_\mu \phi_i + K_{ij} A^\mu \phi_i \partial_\mu \phi_j - V(\phi, A^2)$$

Point symmetries (in addition to coordinates):

$$\phi_i \to \phi_i + \chi_i(x, \phi, A)\,, \qquad A^\mu \to A^\mu + \xi^\mu(x, \phi, A)$$

General solution:

$$\begin{aligned} \chi_i(x,\phi) &= \alpha_i(x) + \beta_{ij}(x)\phi_j + \gamma(x)\phi_i \\ \xi^\mu(x,\phi) &= \partial^\mu \Lambda(x) + \pi^\mu{}_\nu(x) A^\nu + \gamma(x) A^\mu \end{aligned}$$

Massive $\mathrm{U}(1)$: when solving rest of determining equations, demand:

- gauge symmetry: $\Lambda(x)$ is arbitrary,
- massive vector: $\frac{\partial V}{\partial A^\mu} = m^2 A_\mu + \ldots$.

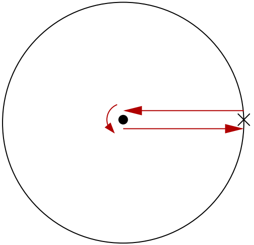
$\to$ derive allowed form of $\mathcal{L}$ and relations between parameters.

1 field: Stückelberg ($J = m$), 2 fields: Higgs.

Spontaneously broken scale symmetry:

$V = \lambda\phi^4$ has scale symmetry.

$V = \lambda(\phi + v)^4$ has shift-scale-shift symmetry.



$V = \lambda(\phi_1^2 + \phi_2^2 - v^2)^2$ has $\mathrm{U}(1)$.

Define $\phi_2 = v + \varphi$.

$V = \lambda(\phi_1^2 + \varphi^2 + 2v\varphi)^2$ has shift-$\mathrm{U}(1)$-shift.

LPS method will find symmetry, no matter how broken/hidden it may be.

For example, solve for relationships between c_i in

$$V = c_1 + c_2\phi_1 + c_3\phi_2 + c_4\phi_1^2 + c_5\phi_1\phi_2 + c_6\phi_2^2 + c_7\phi_1^3 + c_8\phi_1^2\phi_2 + c_9\phi_1\phi_2^2 + c_{10}\phi_2^3 + c_{11}\phi_1^4 + c_{12}\phi_1^3\phi_2 + c_{13}\phi_1^2\phi_2^2 + c_{14}\phi_1\phi_2^3 + c_{15}\phi_2^4\,.$$

The standard model

Schematic structure of the standard model:

$$\mathcal{L}_{\mathsf{SM}} \sim (\partial\phi)^2 + \phi^2\partial\phi + \phi^2 + \phi^4 + \psi\partial\psi + \phi\psi^2 \,.$$

- $N = 244$ real degrees of freedom (with RH neutrinos and Higgs).
- About 10^7 terms in $\mathcal{L}_{\mathsf{SM}}$.
- Maximum number of determining equations: 2.5×10^6 (but many are duplicated, and many are single term).

Apply the LPS method:

- Rediscover Higgs mechanism.
- Find all symmetries and prove that there are no more.
- Add new degrees of freedom looking for new symmetries (e.g. GUT).

Conclusions

Coordinate variation η^μ, field variation χ_i.
Master determining equation:

$$\mathcal{L}\frac{\mathrm{d}\eta^\mu}{\mathrm{d}x^\mu} + \frac{\partial\mathcal{L}}{\partial x^\mu}\eta^\mu + \frac{\partial\mathcal{L}}{\partial\phi_i}\chi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\left(\frac{\mathrm{d}\chi_i}{\mathrm{d}x^\mu} - \frac{\partial\phi_i}{\partial x^\nu}\frac{\mathrm{d}\eta^\nu}{\mathrm{d}x^\mu}\right) = 0$$

The Lie point symmetry method:

- Counterpart to the Euler-Lagrange equations.
- Finds all possible symmetries.
- Finds all interesting relationships between parameters.
- Works even for spontaneously broken symmetries.
- Can be automated; crucial for large systems.

Future work:

- Find all symmetries of the standard model.
- Adapt to include surface terms.
- Extend to supersymmetry [Grundland, Hariton, Snobl (2008)].
- Allow for discrete symmetries [Hydon (1998)].

References

Text book:

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LPS method and computation:

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Previous work using LPS for field theories:

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Symmetries of one scalar

Specialise to $N = 1$:

$$-d\gamma V + \frac{\mathrm{d}V}{\mathrm{d}\phi}(\alpha + \gamma\phi) = 0\,.$$

Four distinct cases:

$V = 0$: α and γ free. Independent shift and scale symmetries.
Rank associated with field is $R_\chi = (2)$.

$V = \text{const}$: $\gamma = 0$ but α is free.
Field rank $R_\chi = (1)$.

$V = \lambda(\phi + v)^d$: Solve above differential equation.
Given v, relationship between shift and scale symmetry is fixed by $v = \alpha/\gamma$.
Field rank $R_\chi = (1)$.

V arbitrary: $\alpha = \gamma = 0$. No shift or scale symmetry.
Field rank $R_\chi = (0)$.

Symmetries of two scalars

$$-d\gamma V + \frac{\partial V}{\partial \phi_1}\left(\alpha_1 + \beta\phi_2 + \gamma\phi_1\right) + \frac{\partial V}{\partial \phi_2}\left(\alpha_2 - \beta\phi_1 + \gamma\phi_2\right) = 0\,.$$

Go to polar field variables, $\phi_1 = r\cos\theta$, $\phi_2 = r\sin\theta$:

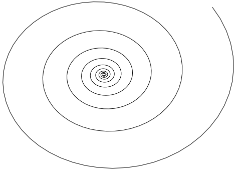
$$\mathcal{L} = \frac{1}{2}\partial^\mu r\partial_\mu r + r^2\frac{1}{2}\partial^\mu\theta\partial_\mu\theta - V(r,\theta)\,.$$

Determining equation is

$$-d\gamma V + \frac{\partial V}{\partial r}\left(\alpha_1\cos\theta + \alpha_2\sin\theta + \gamma r\right) - \frac{\partial V}{\partial\theta}\left(\alpha_1\frac{\sin\theta}{r} - \alpha_2\frac{\cos\theta}{r} + \beta\right) = 0.$$

A solution:

$$V(r,\theta) = \lambda\left(r^k - v\mathrm{e}^{l\theta}\right)^m\,.$$



k and m related by $mk = d$. Relationship between scale and rotation symmetry fixed by $k\gamma = l\beta$. Action of the symmetry is $r \to \mathrm{e}^\gamma r$, $\theta \to \theta - k\gamma/l$ and $x^\mu \to \mathrm{e}^{-d\gamma/D}x^\mu$.

Non-linear symmetries

Field (no coordinate) symmetries of

$$\mathcal{L} = \phi^m (\partial^\mu \phi \partial_\mu \phi)^n \,.$$

m and $n \neq 0$ are constant exponents.

Determining equation

$$m\phi^{m-1}\chi + 2n\phi^m \frac{\mathrm{d}\chi}{\mathrm{d}\phi} = 0\,.$$

Solve for χ:

$$\chi = a\phi^{-m/2n} \qquad a \text{ is integration constant}\,.$$

Non-linear symmetry acts by $\bar{\phi}' = a\bar{\phi}^{-m/2n}$, solution

$$\phi \to (\phi^p + pa\epsilon)^{1/p} \qquad \text{with} \quad p = 1 + m/2n\,.$$

The action versus the equations of motion

Distinction between the symmetries of action and symmetries of corresponding equations of motion.

G a symmetry of an action $\implies$ G also a symmetry of the Euler-Lagrange equations. Converse not necessarily true.

Denote the system by $\Delta_j(x^\mu, \phi_i, \partial\phi_i) = 0$.

1. Construct the prolonged symmetry operator $\mathrm{pr}^{(k)}\,\boldsymbol{\alpha}$.

$$\boldsymbol{\alpha} = \eta^\mu \frac{\partial}{\partial x^\mu} + \chi_i \frac{\partial}{\partial \phi_i}\,.$$

   Prolongation extends $\boldsymbol{\alpha}$ to include all possible combinations of derivatives of ϕ, to order k.

2. Apply $\mathrm{pr}^{(k)}\,\boldsymbol{\alpha}$ to the system: $(\mathrm{pr}^{(k)}\,\boldsymbol{\alpha} \cdot \Delta)|_{\Delta=0} = 0$.
3. Equate all independent coefficients to zero $\rightarrow$ determining equations.

Equations of motion example

System defined by Euler-Lagrange equation $\ddot{\phi} - \phi'' + m^2\phi = 0$.

What are its symmetries?

- $m = 0$ has

$$\begin{aligned} \eta^t(t,x) &= F_+(t+x) + F_-(t-x) \,, \\ \eta^x(t,x) &= F_+(t+x) - F_-(t-x) + f \,, \\ \chi(t,x,\phi) &= G_+(t+x) + G_-(t-x) + g\,\phi(t,x) \,. \end{aligned}$$

- $m \neq 0$ has

$$\begin{aligned} \eta^t(x) &= a^t + bx \,, \\ \eta^x(t) &= a^x + bt \,, \\ \chi(t,x,\phi) &= \int_{-\infty}^{+\infty} dk \left[H_+(k)\,\mathrm{e}^{i(\omega t + kx)} + H_-(k)\,\mathrm{e}^{i(\omega t - kx)} \right] + g\,\phi(t,x) \,, \end{aligned}$$

where $\omega = \sqrt{k^2 + m^2}$.

Degrees of freedom in the standard model

$N = 244$ real degrees of freedom (with RH neutrinos):

- gauge $=$ 4 real components $\times$ (1 hyp $+$ 3 weak $+$ 8 strong) $=$ 48,
- leptons $=$ 8 real components $\times$ 3 gens $\times$ (ν $+$ e) $=$ 48,
- quarks $=$ 8 real components $\times$ 3 gens $\times$ 3 cols $\times$ (u $+$ d) $=$ 144,
- and Higgs $=$ 2 real components $\times$ weak-doublet $=$ 4.