Inflationary Correlation Functions without Infrared Divergences

Mischa Gerstenlauer

with G. Tasinato, A. Hebecker, C. Byrnes, S. Nurmi

JCAP 1008:006,2010 *and* JCAP 1106:021,2011

IR divergences in δN - formalism

Starobinsky '85, Sasaki/Stewart '95, Wands/Malik/Lyth/Liddle '00
Lyth/Malik/Sasaki '04, Lyth/Rodriguez '05

- Consider some late constant energy-density surface (reheating surface)

$$ds^2 = e^{2\zeta(x)} dx\, dx$$

- The curvature perturbation ζ can be interpreted as perturbation in the *local* number of e-foldings $N(\,\phi(x)\,)$ $\qquad (\ldots)_\phi = \frac{d}{d\phi}\ldots$

$$\begin{aligned}\zeta(x) = \delta N(x) &= N(\,\phi_0 + \delta\phi(x)\,) \;-\; N(\,\phi_0\,) \\ &= N_\phi\, \delta\phi(x) + \frac{1}{2} N_{\phi\phi}\, \delta\phi(x)^2 + \ldots\end{aligned}$$

- Yields 2-point correlator in Fourier space:

$$\langle\, \zeta_k\, \zeta_p\, \rangle \sim N_\phi^2 \langle\, \delta\phi_k\, \delta\phi_p\, \rangle + \frac{1}{4} N_{\phi\phi}^2 \langle\, (\delta\phi^2)_k\, (\delta\phi^2)_p\, \rangle + N_\phi N_{\phi\phi\phi} \ldots$$

Infrared Divergences in Inflation

- Focus on the second term

$$N_{\phi\phi}^2 \langle\, (\delta\phi^2)_k \; (\delta\phi^2)_p \,\rangle \sim N_{\phi\phi}^2 \int_{q,l} \langle\, \delta\phi_q \; \delta\phi_{k-q} \; \delta\phi_l \; \delta\phi_{p-l} \,\rangle \quad .$$

- use

$$\delta\phi_k \sim \frac{H}{k^{3/2}} \; a_k$$

- to yield leading-log contribution from $q, l \ll k, p$:

$$N_{\phi\phi}^2 H^4 \int \frac{d^3q}{q^3} \sim N_{\phi\phi}^2 H^4 \; \ln(kL) \quad ,$$

with cut-off $1/L$.

Fluctuations of the Hubble scale H

Byrnes, MG, Hebecker, Tasinato, Nurmi '10

- Origin of IR effects is the dependence of $N(\,\phi(x)\,)$ on $\delta\phi_q$ with $q \ll k$
- A similar dependence appears in the Hubble function $H(\,\phi(x)\,)$.
- The Hubble scale $H(\,\phi(x)\,)$ should be modified analogously!
- This has not been taken into account in (higher-order) δN-calculations.

Fluctuations of the Hubble scale H

- Define local background of the scalar field:

$$\delta\bar{\phi}(x) = \int_{q \ll k} e^{-iqx} \delta\phi_q$$

- Hubble function should be evaluated at horizon exit of mode k. Local scalar field value at horizon exit: $\phi_0 + \delta\bar{\phi}(x)$

$$\delta\phi(x) \sim \int_k \frac{e^{-ikx}}{k^{3/2}} \, H(\, \phi_0 + \delta\bar{\phi}(x) \,) \; a_k$$

- Use this modified $\delta\phi$ in $\zeta = N(\phi_0 + \delta\phi) - N(\phi_0)$ and expand in both, $\delta\phi$ and $\delta\bar{\phi}$.

Geometry of the Reheating Surface

- This yields: $(H^2 \ln kL \sim \langle\, \delta\bar{\phi}^2 \,\rangle$ and $\mathcal{P}_\zeta^{(0)} \sim N_\phi^2 H^2)$

$$\mathcal{P}_\zeta(k) = \mathcal{P}_\zeta^{(0)}(k) + \frac{1}{2}\langle\, \delta\bar{\phi}^2 \,\rangle \, \frac{d^2}{d\phi^2}\mathcal{P}_\zeta^{(0)} \ldots$$

- Replace $\delta\bar{\phi}$ by $\bar{\zeta} \sim \int_{q\ll k} e^{-iqx}\,\zeta_q$ and use $\frac{d}{d\phi} = N_\phi \frac{d}{d\ln k}$ to write this as

$$\mathcal{P}_\zeta(k) = \left(1 - \langle\, \bar{\zeta} \,\rangle \frac{d}{d\ln k} + \frac{1}{2}\langle\, \bar{\zeta}^2 \,\rangle \frac{d^2}{(d\ln k)^2}\right) \mathcal{P}_\zeta^{(0)}(k)$$

see also Giddings/Sloth '10

- Obviously, these are the first terms of the Taylor expansion of

$$\mathcal{P}_\zeta(k) = \langle\, \mathcal{P}_\zeta^{(0)}(ke^{-\bar{\zeta}}) \,\rangle$$

where $\langle \ldots \rangle$ is the average in $\bar{\zeta}$ over a box of size L

Infrared-safe Correlator

MG, Hebecker, Tasinato '11

- According to its definition

$$\langle\, \zeta_k \; \zeta_p \,\rangle \sim \frac{\delta(k+p)}{k^3}\, \mathcal{P}_\zeta(k)$$

the power spectrum may be written as

$$\mathcal{P}_\zeta(k) \sim k^3 \int_y e^{iky} \,\langle\, \zeta(x)\; \zeta(x+y) \,\rangle$$

- The average $\langle \dots \rangle$ is over pairs of points seperated by a coordinate vector y .
- In other words, we are averaging over the location x of such pairs.

Infrared-safe Correlator

- At each location there is a local background due to long-wavelength modes

$$\bar{\zeta}(x) = \int_{q \ll k \sim 1/y} dq \;\; e^{-iqx} \zeta_q$$

- The physical separation $z = e^{\bar{\zeta}(x)} y$ of the pairs is x-dependent and therefore different for each pair

- Moreover, this mismatch between y and the true distance z grows with L. More precise: $\langle \, \bar{\zeta}^2 \, \rangle \sim \mathcal{P}_\zeta^{(0)} \ln(kL)$.

$\Rightarrow$ Select pairs separated by same physical distance z $\Leftarrow$

IR-safe Correlator

- The correlator

$$\langle\, \zeta(x)\; \zeta(x + e^{-\bar{\zeta}(x)} z\,)\,\rangle$$

averages over pairs of points ALL separated by the same physical distance z .
(Note: Now the coordinate vector $y(x) = e^{-\bar{\zeta}(x)} z$ is different for each pair.)

- The z-dependence of this correlator is then a background-independent and, hence, IR-safe object.
- Consequently, the Fourier transform is the desired IR-safe power spectrum

$$\mathcal{P}_{\zeta}^{(0)}(k) \sim k^3 \int_z e^{ikz}\, \langle\, \zeta(x)\; \zeta(x + z e^{-\bar{\zeta}})\,\rangle$$

related to Urakawa/Tanaka '10 ?

see also Giddings/Sloth '11

IR-safe Power Spectrum

- The conventional IR-sensitive power spectrum is obtained via

$$
\begin{aligned}
\mathcal{P}_\zeta(k) &\sim k^3 \int_y e^{iky} \, \langle \, \zeta(x) \, \zeta(x+y) \, \rangle \\
&\sim k^3 \int_y e^{iky} \, \langle \, \zeta(x) \, \zeta(x + (ye^{\bar{\zeta}})e^{-\bar{\zeta}}) \, \rangle \\
&\sim \langle \, (ke^{-\bar{\zeta}})^3 \int_z \exp(ike^{-\bar{\zeta}}z) \;\; \zeta(x) \, \zeta(x + ze^{-\bar{\zeta}}) \, \rangle \\
&\sim \langle \, \mathcal{P}_\zeta^{(0)}(ke^{-\bar{\zeta}}) \, \rangle
\end{aligned}
$$

  i.e. in a resummed, all-orders form.

- This is in absolute agreement with the previous result!!!

Tensor modes γ

- Include tensor modes in the metric (γ: transverse, traceless)

$$ds^2 = e^{2\zeta(x)} \left(e^{\gamma(x)} \right)_{ij} dx^i dx^j$$

- Generalization yields ($\hat{k}$: unit vector in k-direction)

$$\mathcal{P}_\zeta(k) = \langle\, (e^{-\bar{\gamma}/2}\hat{k})^{-3}\, \mathcal{P}_\zeta^{(0)}(e^{-\bar{\zeta}-\bar{\gamma}/2}k)\,\rangle$$

- Expanding to first non-trivial order yields:

$$\mathcal{P}_\zeta(k) = \left(1 - \frac{1}{20}\langle\, \mathrm{tr}\,\bar{\gamma}^2\,\rangle \frac{d}{d\ln k} + \frac{1}{2}\langle\, \bar{\zeta}^2\,\rangle \frac{d^2}{(d\ln k)^2} \right) \mathcal{P}_\zeta^{(0)}(k)$$

in agreement with Giddings/Sloth '10

Conclusions

- A wide class of inflationary IR divergences comes from long-wavelength background modes.
- This can be seen in an (appropriately modified) δN formalism as well as from the 'geometry of the reheating surface'.
- It is possible to define IR-safe correlators.
- Conventional correlators (with their IR-dependence) can be easily calculated.
- Inclusion of tensor modes is straightforward.

Beyond this talk: n-point correlators, all-orders evaluation, convergence of perturbative expansion