Holographic Rényi entropy

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Work in progress with R. Myers, M. Smolkin, A. Yale

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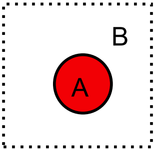
Outline

Reduced density matrix

- Reduced density Matrix

$$\rho = |\Psi\rangle\langle\Psi|, \qquad \rho_A = \mathrm{tr}_B \rho \tag{1}$$



Reduced density matrix

- *Modular* Hamiltonian R. Haag (1992)

$$\rho_A = e^{-H} \tag{2}$$

- symmetry

$$U(s) = \rho_A^{is} = e^{-iHs}\,, \qquad \mathrm{tr}(\rho U(s)\mathcal{O}U(-s)) = \mathrm{tr}(\rho\mathcal{O}) \tag{3}$$

- periodicity in imaginary time–thermal problem with inverse temperature $\beta = 1$.

$$\mathrm{tr}(\rho_A U(i)\mathcal{O}_1 U(-i)\mathcal{O}_2) = \mathrm{tr}(\rho\mathcal{O}_2\mathcal{O}_1) \tag{4}$$

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- local H? eg Rindler Wedge $\mathcal{W}$. $A = x^1 > 0$
- CFT$_d$: $A = B^{d-1}, \partial A = S^{d-2}$. Causal diamond $\mathcal{D} \to \mathcal{W}$. Casini, Huerta, Myers 1102.0440

Outline

Holographic Rényi entropy

- Another convenient choice of conformal frame: $\mathcal{D} \to \mathcal{W} \to H_{d-1} \times R$
- AdS/CFT: Hyperbolic AdS$_{d+1}$ black hole- constant r slicing $H_{d-1} \times R$

$$ds^2 = \frac{dr^2}{1 - r^2/L^2 f(r)} + L^2(r^2/L^2 f(r) - 1)dt^2 + r^2 d\Sigma^2_{H_{d-1}\times R} \tag{6}$$

- e.g. Einstein Gravity $f(r) = 1 - \frac{\omega^d}{r^d}$
- black hole inverse temperature $\beta = n$
- Baez 1102.2098

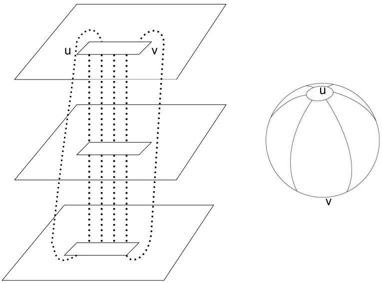
$$S_n = \frac{1}{1-n}(\ln \mathrm{tr} e^{-nH} - n \ln \mathrm{tr} e^{-H}) = -\frac{F(\beta_n) - F(\beta = 1)}{T_n - 1} \tag{7}$$

Rényi entropy at $d = 2$

- Cardy, Calabrese e.g. hep-th/0405152

$$\mathrm{tr}\rho_A^n = \langle \sigma_n(u)\sigma_{-n}(v)\rangle = \frac{s_n}{(u-v)^{4\Delta_n}}\,, \qquad \Delta_n = \frac{c}{24}\left(n - \frac{1}{n}\right) \tag{8}$$

- uniformization map $\left(\frac{\omega-u}{\omega-v}\right)^{1/n} = z$



Rényi entropy at $d = 2$

Hyperbolic blackhole = BTZ (without orbifolding)

$$\begin{aligned} ds^2 &= \frac{L^2 dr^2}{r^2 - r_0^2} + (r^2 - r_0^2) dt^2 + r^2 dx^2 && (9) \\ r_0 &= 2\pi L T \,, \qquad \beta \equiv T^{-1} = n \,, && (10) \\ t &\sim t + 2\pi n \,, \qquad \sigma = x + it && (11) \end{aligned}$$

Using (7),

$$S_n = \frac{c}{6}(1 + \frac{1}{n}) \ln[\frac{R}{\delta}] + .. \qquad (12)$$

- Bulk is entirely regular. Where are conical singularities introduced by the twist operator?

Higher dimensions

- Higher dimensions: (Einstein Gravity)

$$S_n = \frac{\pi n}{1-n}(L/l_p)^{d-1}(x^d + x^{d-2} - 2)V_\sigma\,, \tag{13}$$

$$V_\sigma = \frac{A_{d-2}}{\delta^{d-2}} + \ldots \tag{14}$$

$$x = \frac{1}{nd}(1 + \sqrt{1 - 2dn^2 + d^2n^2}) \tag{15}$$

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- Satisfy various well known inequalities for Rényi entropy.

$$\frac{\partial S_n}{\partial n} \leq 0 \qquad \frac{\partial}{\partial n}\frac{n-1}{n}S_n \geq 0$$

$$\frac{\partial}{\partial n}(1-n)S_n \leq 0 \qquad \frac{\partial^2}{\partial n^2}(1-n)S_n \geq 0$$

Outline

Future directions

- twist operators in higher dimensions? e.g. $\langle T \rangle_n$,..
  e.g. Kapustin hep-th/0501015, Gomis,Okuda 0906.3011
- relevant/irrelevant perturbations? e.g. Cardy, Calabrese 1002.4353 n-dependent phase transition?