Non-Abelian Statistics

versus

The Witten Anomaly

John McGreevy, MIT

based on:

JM, Brian Swingle, 1006.0004

in 3D

Interactions between high energy (particle physics, cosmology, strings) physics and cond-mat have been very fruitful:
SSB, higgs mechanism, topological solitons ...
More recently: hopes for many practical uses for string theory.
e.g. controllable examples of non-Fermi liquid fixed points
(possible states of fermions at finite density other than Landau's nearly free effective field theory).

Today's Q: Is it possible to realize deconfined particles in 3+1 dimensions which exhibit non-abelian statistics?
There's a recent set of ideas, inspired by work in cond-mat, suggesting a route to doing this seemingly-impossible thing.
Its failure mode is interesting.

Particle statistics

In 3+1 dims particles are either bosons or fermions.

Why: boring topology of configuration space:

$\pi_0\,(\text{paths}) = \pi_1(\mathcal{C}_n^{3+1}) = S_n$

$\mathcal{C}_n^{d+1} \equiv \{\text{config space of } n \text{ particles}\} \backslash \{\text{close approaches}\}$

In 2+1: $\pi_1(\mathcal{C}_n^{2+1}) = \mathcal{B}_n$, braid group (infinite-dimensional) $\rightarrow$ anyons.

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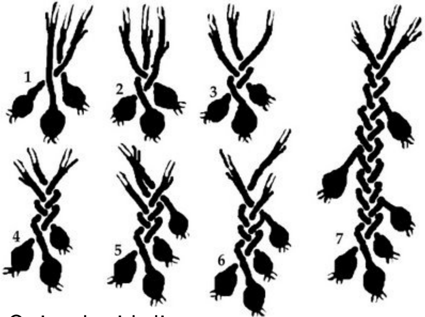


Figure: Onion braid diagram from [gypsymagicspells.blogspot.com]

Anyons

Abelian anyons: state of several anyons acquires a phase upon braiding.

Non-Abelian anyons: braiding acts by a unitary on degenerate statespace.

Abelian anyons exist and have been observed as quasiparticles in well-understood FQHE states.

∃ good evidence that non-Abelian anyons are also realized in FQHE states.

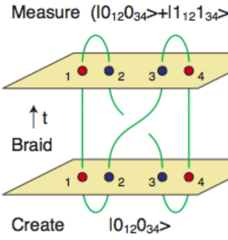
Non-Abelian anyons would make a great quantum computer [Kitaev, Freedman]

• Quantum state stored non-locally

protected from decoherence to (local) environment.

• Do computations

by adiabatically braiding anyons.



[Hasan-Kane]

Majorana solitons

A framework for realizing a class of non-abelian anyons:
Majorana zeromode localized on soliton

$$\gamma_i = \gamma_i^\dagger \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij} \quad i, j = 1..n$$

Hilbert space of groundstates of n solitons represents this algebra.

$$\Gamma_1 \equiv \gamma^1 + i\gamma^2, \ldots \quad \Gamma_1 |\downarrow\downarrow\rangle \equiv 0, \Gamma_1^\dagger |\downarrow\downarrow\rangle = |\uparrow\downarrow\rangle \ldots$$

n such 'Ising anyons' make a degenerate space of $\dim \mathcal{H}_n \sim \sqrt{2}^n$.
info about $\mathcal{H}_n$ not localized on particles (despite realization in local QFT).

Realizations in $2+1$d:
$\nu = \frac{5}{2}$ QH states [Moore-Read, Nayak-Wilczek], $p + ip$ superconductors [Ivanov, Read-Green], surface states of TI [Fu-Kane], solvable toy models [Kitaev]

Majorana solitons, an example in 2+1 d

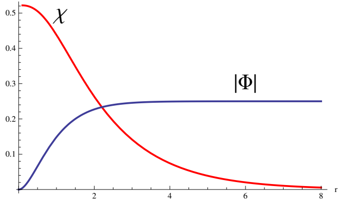
Fermionic quasiparticles in certain 2d superconductors:

$$\chi \equiv \begin{pmatrix} c_\uparrow \\ c_\downarrow \\ c_\uparrow^\dagger \\ c_\downarrow^\dagger \end{pmatrix} \qquad \mathcal{L}_{\text{fermions}} = i\chi^T \left(\sigma^i \partial_i + \Phi\Gamma^+ + \bar{\Phi}\Gamma^-\right)\chi$$

Vortex: $\Phi(r,\varphi) = e^{i\varphi}|\Phi(r)|$

[Jackiw-Rossi, Ivanov, Read-Green]

has a majorana zeromode.



Note: Ising anyons are a special case (not universal for quantum computation).

Lesson: All we need to do to realize non-Abelian (Ising) statistics is to find solitons with normalizable majorana zeromodes.

Majorana hedgehogs

Consider a 3+1d system with a *global* $SO(3)$ symmetry broken by an adjoint scalar vev

$$\langle \Phi^A \Phi^A \rangle = v^2 \qquad A = 1, 2, 3.$$

Couple to a real 8-component spinor
(two majorana doublets of $SU(2) \simeq SO(3)$):

$$H_{\text{fermions}} = i\chi^T \left(\gamma^i \partial_i + \lambda \Phi_A \Gamma^A \right) \chi$$

$\langle \Phi \rangle$ gaps fermions, $m_{\text{bulk}} \sim \lambda v$.

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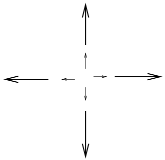

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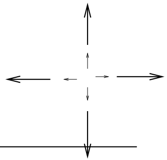
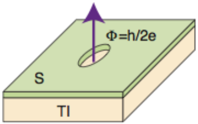


Aside on motivation from topological insulators with superconductors attached: [Fu-Kane 08, Teo-Kane 09, Wilczek, unpublished]

$\Phi^1 + i\Phi^2 =$ supercond. order parameter (zero at vortex)
$\Phi^3 =$ Dirac mass (changes sign at bdy of TI)



Problems of majorana hedgehogs

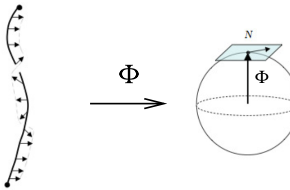
The hedgehogs are not quite particles: spatial var. of Φ is extra data.

Minimal data for topology:

preimage under Φ of north pole and nearby point

$\longrightarrow$ ribbon between hedgehog pairs.

[Freedman et al, 1005.0583]



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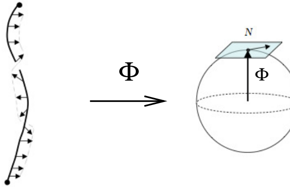


"projective ribbon statistics"

Observation: Variation of Φ costs energy.

Hedgehogs are not finite-energy excitations

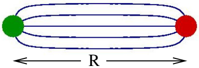
$$E = H[\Phi_{\text{hedgehog}}] \sim \int_0^L d^3x \left(\vec{\nabla}\Phi^A \cdot \vec{\nabla}\Phi_A + \ldots \right) \sim v^2 L$$

(like global SO(2) vortex in 2+1 dims).

Configurations with zero total hedgehog number have finite energy

But: $V_{\text{eff}}(R) \sim \int_0^R r^2 dr \cdot \left(\frac{\phi}{r}\right)^2 \sim Rv^2$.

linear confinement.



Not so good for adiabatic motion.

Deconfined majorana solitons in $3+1$ dims?

Two apparently-different routes to models with *deconfined* majorana particles:

- Gauge the SU(2) symmetry
- Disorder the $\langle \Phi \rangle$. (Zero stiffness, no gradient energy.)

Gauge the SU(2)

- $\text{SU}(2) \stackrel{\langle\Phi\rangle\in\text{adj}}{\rightarrow} U(1)$
- Sol'n with $\Phi^A = \hat{r}^A\phi(r) \rightarrow$ 't Hooft-Polyakov monopole:

$$\boxed{A_i^A = \epsilon_{ijA}\hat{r}^j A(r), \; A_0^A = 0}$$

$$\phi(r) \stackrel{r\to\infty}{\sim} v, \quad A(r) \stackrel{r\to\infty}{\sim} \frac{1}{r} \implies D_i\Phi \stackrel{r\to\infty}{\rightarrow} 0.$$

- carries magnetic charge = hedgehog #

$\implies$ magnetic coulomb force $F \sim \frac{q_m q_m'}{r^2}$ (falls off!)

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- $$\mathcal{L}_{\text{fermions}} = \chi^\dagger i\bar{\sigma}^\mu D_\mu \chi - \frac{1}{2}\lambda\chi^\vee \vec{\tau}\cdot\vec{\Phi}\chi + h.c.$$

$\chi_{\alpha a}$ Weyl $\in (1,2,2)$ of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(2)_{\text{gauge}}$

$$\chi^\vee \equiv \chi^T i\sigma^2 i\tau^2 \in (1,\bar{2},\bar{2})$$

- Two independent mass scales:

$m_W = gv$, and the mass of the fermion λv.

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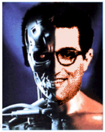


fig: [W. Siegel]

[• Not OK: Witten anomaly [Witten 1982]]

Majorana zeromode

Momentarily treat A, Φ as classical background fields:
Dirac equation

$$0 = \delta_{\bar{\chi}} S_{\text{fermion}} = -i\bar{\sigma}^\mu D_\mu \chi + \lambda^\dagger i\sigma^2 \Phi \cdot \tau i\tau^2 \chi^\star \ .$$

ansatz from [Jackiw-Rebbi, 1976], with reality conditions.

$\chi_{\alpha a} = i\tau^2_{\alpha a} g(r)$ (α: spin index, a: SU(2) doublet index).

$$(\partial_i + 2\hat{r}_i A)g + i\lambda\phi\hat{r}_i g^\star = 0.$$

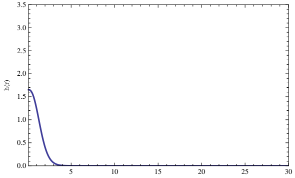
rephasing $\chi \implies \lambda > 0$ WLOG

$$g(r) = c e^{-\pi i/4} e^{-\int^r (\lambda\phi - 2A)}$$

c is a *real* constant.

phase of the normalizable

solution determined by normalizability at $r \to \infty$.



Witten anomaly

$$\int [D\chi] e^{iS_{\text{fermions}}[\chi, A, \Phi]} \equiv e^{i\Gamma[A,\Phi]} \times \text{non-universal stuff}$$

Fermion determinant represents $\pi_4(\text{SU}(2)) = \mathbb{Z}_2$:

$$(\star) \qquad e^{i\Gamma[A^g, \Phi^g]} = (-1)^{[g]} e^{i\Gamma[A,\Phi]} \quad .$$

But A, A^g are continuously connected:

$$\Longrightarrow \int [DAD\Phi] e^{i\Gamma[A,\Phi]} \times (\text{anything gauge invariant}) = 0$$

One argument for $(\star)$**:** Embed the theory in an SU(3) gauge theory with a perturbative gauge anomaly

[Witten:1983,Elitzur:1984,Klinkhamer:1990].

Calculate the variation of the fermion measure between $\mathbb{1}$ and g by integrating the SU(3) anomaly.

Claim: The addition of the adjoint scalar Φ doesn't change this.

Witten anomaly with adjoint scalar

SU(3) adjoint scalar $\tilde{\Phi}$, an SU(3) triplet of Weyl fermions $\tilde{\chi}$ and an SU(3) triplet of scalars Υ, with the coupling

$$L_{\mathrm{SU(3)}} \supset \tilde{\chi}_a^T i\sigma^2 \Upsilon_b \epsilon_{abc} \tilde{\Phi}_{cd} \tilde{\chi}_d,$$

$a = 1, 2, 3$ is a triplet index.
$\langle \Upsilon \rangle = \lambda$ breaks the SU(3) down to SU(2),
The form of the perturbative SU(3) anomaly is unaffected by the addition of scalars.

$\Gamma[A, \Phi]$ is a smooth functional for invertible Φ
(integrate out massive fermions)
Ineffable: naive $\Gamma_{WZW}[A, \Phi] = 0$ for SU(2).

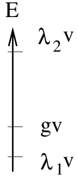
Canceling the Witten anomaly

$$S[\chi, \Phi, A] \to S[\chi, \Phi, A] + \Gamma[\Phi, A]$$

But: if $\Phi = 0$ anywhere, Γ is ill-defined. *e.g.* core of monopole. Requires UV completion.
Important point: presence of fermion zermodes is UV sensitive question.

$$L_{2 \text{ fermions}} = \chi^{I\dagger} i\bar{\sigma}^\mu D_\mu \chi_I - \lambda^{IJ} \chi_I^\vee \vec{\tau} \cdot \vec{\Phi} \chi_J - m^{IJ} \chi_I^\vee \chi_J + h.c.$$

$\chi_{I\alpha a}$ a pair of (left-handed) Weyl doublets of SU(2):
$I = 1, 2$ a flavor index, $\alpha = 1, 2$: spin, $a = 1, 2$ gauge. 2^3 complex fermions
Same spectrum as [Jackiw-Rebbi 76] but more general couplings.
Three mass scales:
the mass of the W-bosons, $m_W = gv$,
and the masses of the two Weyl fermions $\lambda_{1,2} v$



$$\text{For } \lambda_1 v \ll m_W \ll \lambda_2 v$$

large window of energies with same bulk spectrum as above.

Relation to Jackiw-Rebbi model

λ is symmetric, $\lambda^{IJ} = \lambda^{JI}$ by Fermi statistics.
By field redefinitions, can diagonalize λ with real eigenvalues $\lambda_{1,2}$.
Phase of m is physical.
$m = m^\dagger$, preserves a CP symmetry $\chi \mapsto i\sigma^2 i\tau^2 \chi^\star$.

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$m = m^\dagger$, preserves a CP symmetry $\chi \mapsto i\sigma^2 i\tau^2 \chi^\star$.
Jackiw-Rebbi case: $\lambda_1 = \lambda_2 \equiv \lambda_0 \implies$ extra U(1) symmetry:

$$\chi_1 \mapsto e^{i\theta}\chi_1, \quad \chi_2 \mapsto e^{-i\theta}\chi_2 \qquad \left(\text{in basis where } \lambda = \begin{pmatrix} 0 & \lambda_0 \\ \lambda_0 & 0 \end{pmatrix}\right)$$

$$\Psi \equiv \begin{pmatrix} \chi_1 \\ \chi_2^\star i\tau^2 i\sigma^2 \end{pmatrix}. \quad \lambda_0 \equiv \lambda_0^R + i\lambda_0^I$$

$$L_{\text{2fermions}} = \bar\Psi i \not{D} \Psi - \bar\Psi \left(\lambda_0^R + i\lambda_0^I \gamma^5\right) \vec\tau \cdot \vec\Phi \Psi + m\bar\Psi\Psi.$$

For $m = 0$, JR found in this model a *complex* zeromode of the monopole.
Quantizing this mode makes the monopole into a pair of *bosons* of charge $\pm e/2$ (under the 'extra' $U(1)$).

Fermion zeromodes in the two-doublet model

For $m_{\text{Dirac}} = 0$:

In the basis where λ is diagonal with real evals $\lambda_{1,2}$, zeromode equations for $\chi_{1,2}$ decouple. Two real solutions, like JR:

$$\chi_{I\alpha a}(r) = i\tau^2_{\alpha a} g_I, \quad g_I = c_I e^{-\pi i/4} e^{-\int^r (\lambda_I \phi - 2A)} \ .$$

For $m_{\text{Dirac}} \neq 0$:

Ansatz which decomposes $\chi \in (2,2)$ into irreps of the unbroken $\text{SU}(2) \subset \text{SU}(2)_{\text{gauge}} \times \text{SU}(2)_{\text{spin}}$:

$$\chi_{a\alpha I} = i\tau^2_{a\alpha} g_I + i \left(\tau^2 \tau^i\right)_{a\alpha} g^i_I .$$

Guess: $\vec{g} = \hat{r} g_r(r)$. Dirac equation becomes:

$$\begin{aligned} 0 &= i\vec{\nabla} g - 2i\hat{r} A g - \lambda^\dagger g^\star \phi \hat{r} - m^\dagger \vec{g}^\star \\ 0 &= i\vec{\nabla} \cdot \vec{g} + 2iA\vec{g} \cdot \hat{r} + \lambda^\dagger \vec{g}^\star \cdot \hat{r}\phi + m^\dagger g^\star \ . \end{aligned}$$

Conclusions about zeromodes

- First assume $m = m^\dagger$.

For $\sqrt{\lambda_1 \lambda_2} v < m$, *both* modes are non-normalizable.

(Else, both normalizable.)

Check: $\det M_{\text{bulk}} = \left(\lambda_1 \lambda_2 v^2 - m^2\right)^2 \to 0$ precisely at marginal normalizibility.

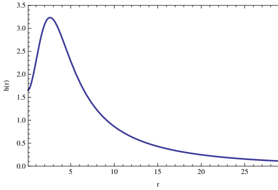
Sizes of zms can be varied independently by $\lambda_{1,2}$.

The zeromode wavefunctions involve products of exponentials of the form $e^{mr} e^{-\lambda v r}$, one might have thought (pantingly) that one zeromode would become non-normalizable, *e.g.* for

$\lambda_1 v < m < \lambda_2 v$.

This hope is not realized.

Remnant: sometimes zm profile is ring-like: $\longrightarrow$



- For $m \neq m^\dagger$ no zero-energy solutions.

No-go arguments

1. If we gauge away or disorder the Station Q ribbon, the configuration space has $\pi_1(\mathcal{C}_n) = S_n$.
2. Rough sketch of argument for inevitability of Witten anomaly:

Witten anomaly $\longleftrightarrow$ chiral anomaly mod two $\longleftrightarrow$ majorana number of monopole

In a Witten-anomalous theory, $(-1)^F = e^{i\pi j_0^{\text{axial}}} = e^{i\pi\tau^3}$ is a gauge symmetry. [Goldstone, 83]

$\Longrightarrow$ chiral anomaly mod two is a gauge anomaly.

(In a normal theory: $\psi_L \to \bar{\psi}_R$.

Here $\psi_L \to$vacuum.)

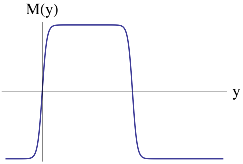
$$\text{ind}_{\mathbb{R}} \not{D}[\text{monopole}] = \oint_{S^2_\infty} \vec{\nabla} \cdot \vec{j}^{\text{axial}}$$

[Callias 78]: $\text{ind}_{\mathbb{C}} \not{D}$

[Santos-Nishida-Chamon-Mudry, 09]: real index for vortex in 2d

5d model

Some theories are only realizable as the boundary of a higher-dimensional model. [Nielsen-Ninomiya, Kaplan]
e.g.: domain-wall fermions in lattice QCD,
single dirac cones on surface of a topological insulator
Consider SU(2) gauge theory in 4+1 dimensions with a Dirac fermion doublet and adjoint Higgs.
On a circle: fourth spatial dimension $y \simeq y + 2\pi R$.

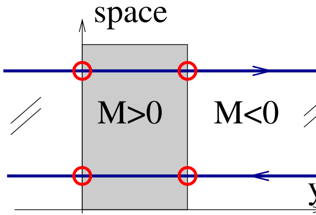


Kink of $M(r)$
supports a 4d massless Weyl fermion.

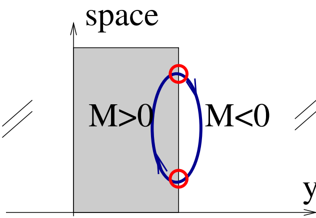
Bad features: 5d; needs UV completion (lattice, strings); kinks can annihilate.
Mass scales: M_W, R^{-1}, the Dirac mass m,
the inverse thickness of the kink, extreme UV cutoff
At energies $E \ll 1/R$, this model reduces to the two-doublet theory above.

Majorana monopole strings

$q \in \pi_2(S^2)$ supports monopole strings.
(3d particles when stretched along y).
Intersections between monopole strings and domain walls of 5d mass $\rightarrow$ localized Majorana zeromodes.



- the two Majorana modes need not pair up. For $m \gg R^{-1}$, their wavefunction overlap is exponentially small.
- low-energy braiding always exchanges majoranas in pairs
- dyon rotor $\rightarrow$ 1+1 XY model along string. decoherence to local basis or linear confinement from monopole string tension:



Disordering the SU(2)

Imagine a state with hedgehogs but zero stiffness (no LRO).
Effective field theories for such states are usefully studied using "slave particle" techniques
(successful in similar problem of spin liquid states).
Result: emergent U(1) gauge theory, under which the defects are magnetically charged.
Again requires UV completion.
One way to do it: SU(2) gauge theory with a Weyl doublet.

Another attempt [Freedman, Hastings, Nayak, Qi, to appear]: a lattice model
(majorana fermions with hopping amplitudes determined by a quantum dimer model configuration)
They argue for a majorana zeromode on the defect.
But: gapless bulk fermions.
Reduces to *two-doublet* model with $m = 0$, $\lambda_1 \neq 0$, $\lambda_2 = 0$.

Other possible loopholes?

- What if we just gauge the $U(1) \subset SU(2)$?
  Still linear tension.
- What if the 5th dimension ends?
  For some boundary conditions: gapless 4d mode [Station Q].
- Lorentz-breaking fermion kinetic terms [Tong]?
  Still anomalous?

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  Still linear tension.
- What if the 5th dimension ends?
  For some boundary conditions: gapless 4d mode [Station Q].
- Lorentz-breaking fermion kinetic terms [Tong]?
  Still anomalous?

Conclusion: It would be nice to tighten the no-go statement, and it will be interesting to see what other physics has to come in to save the world from non-Abelian statistics in 3+1 dimensions.

The end

Thanks for listening.