Dependence of the Power Spectrum on the Initial State

Sonia Paban
Work in progress, in collaboration with D. Carney, W. Fischler, N. Sivanandam
(University of Texas at Austin)

PASCOS 2011
July 6, 2011

Problem

- Experimental Agreement

$$< \zeta_k \, \zeta_{k'} >= (2\pi)^3 \, \delta(k + k') \, P_\zeta(k)$$

$$P_\zeta = \frac{1}{k^{3+(1-n_s)}}, \qquad n_s = 0.963 \pm 0.012$$

- Assumptions
  - Slow roll inflation that lasts for enough e-foldings
  - Bunch-Davis vacuum
- Challenges
  - Inflation started at a definite time t_I. Why assume BD-vacuum as initial condition when inflation has yet to start?
  - Maybe the BD is not the initial state but it is an attractor, and if so in what sense?
- History: Anderson, Eaker, Habib, Molina-Paris and Mottola (2000); Kaloper, Kleban, Lawrence, Shenker, and Susskind (2002); Brandenberger, Martin (2004); Danielsson...

Our Method

- Use the Schrödinger picture

$$\Psi(\zeta, t) = U(t, t_0)\, \Psi(\zeta, t_0)$$

- Choose $\Psi(\zeta, t_0)$, constraints: renormalizability of $T_{\mu\nu}$.
- Choose a cosmology that includes a period of inflation where we can solve for $U(t, t_0)$
- Compute $\Psi(\zeta, t)$
- Compute $< \zeta_k \, \zeta_{k'} >$

Cosmology

$$ds^2 = dt^2 - a^2(t)\, d\mathbf{x}^2$$

$$S = \frac{1}{2}\int d^4x\,\sqrt{-g}\,\frac{\dot\phi^2}{H^2}\left[\dot\zeta^2 - \frac{1}{a^2}(\partial_i\zeta)^2\right]$$

ϕ is the inflaton background

$$\zeta_{\mathbf{k}} = \int d^3\mathbf{x}\,\zeta(\mathbf{x})e^{-i\mathbf{x}\cdot\mathbf{k}}$$

$$\mathcal{H} = \frac{1}{2m(t)}\,\Pi^2_{\zeta_k} + \frac{1}{2}m(t)\,\omega^2(t)\zeta_k^2$$

$$m(t) \equiv \frac{a^3(t)\dot\phi^2(t)}{H^2(t)} \quad \text{and} \quad \omega(t) = \frac{k}{a(t)}$$

Cosmology

$$\frac{1}{2}\dot{\phi}^2 + V(\phi) = \begin{cases} \frac{1}{2}\dot{\phi}^2 & \phi < \phi_{BI} & \text{region } I & P = \rho \\ \Lambda^4 - \alpha\,\phi & \phi_{BI} < \phi < \phi_{EI} & \text{region } II & P = -\rho \\ \frac{1}{2}\dot{\phi}^2 & \phi > \phi_{EI} & \text{region } III & P = \rho \end{cases}$$

$a(t)$ and $H(t)$ have to be continuous.

The equation of motion for the inflaton is

$$\ddot{\phi} + 3\,H(t)\,\dot{\phi} + V'(\phi) = 0$$

- In regions I and III this implies that

$$\frac{d}{dt}(a^3\dot{\phi}) = 0 \qquad H^2 = \frac{1}{6}\dot{\phi}^2$$

- In region II, assuming slow roll,

$$\dot{\phi} = \frac{\alpha}{3H} \qquad \frac{\dot{\phi}}{H} \ll 1$$

Unitary operator that implements the time evolution

$$U(t,0) = \exp\left[c_1(t)J_+\right]\exp\left[c_2(t)J_0\right]\exp\left[c_3(t)J_-\right]$$

$$\begin{aligned}
J_+ &= \frac{1}{2}\zeta^2 \\
J_0 &= \frac{i}{4}(\zeta\Pi_\zeta + \Pi_\zeta\zeta) \\
J_- &= \frac{1}{2}\Pi_\zeta^2
\end{aligned}$$

$$\begin{aligned}
c_1(t) &= i\,m(t)\frac{\partial}{\partial t}\ln G(t), \qquad c_1(0) = 0 \\
c_2(t) &= -2\ln\frac{G(t)}{G(0)} \\
c_3(t) &= -i\,G^2(0)\int_0^t \frac{du}{m(u)G^2(u)}
\end{aligned}$$

$$\frac{d^2G(t)}{dt^2} + \xi(t)\frac{dG(t)}{dt} + \omega^2(t)G(t) = 0, \qquad \xi(t) = \frac{\partial}{\partial t}\ln m(t)$$

$$m(t) \equiv \frac{a^3(t)\dot{\phi}^2(t)}{H^2(t)} \qquad \text{and} \qquad \omega(t) = \frac{k}{a(t)}$$

- This operator does not contain any arbitrary constants, it is completely fixed by the requirement that

$$U(0,0) = 1$$

- It can be computed for each of the regions in the chosen cosmology, the only freedom is in the choice of the initial state.

Constraints on the Initial State

- Consider the simplest case of a gaussian wave function. Unitary evolution transforms a gaussian into another gaussian.

$$\Psi(\zeta, t) = N(t)\, \exp\left(-\frac{1}{2}\int \frac{d\mathbf{k}}{(2\pi)^3}\, \zeta(\mathbf{k})\, F(k,t)\, \zeta(-\mathbf{k})\right)$$

$$F(k,t) = -c_1(t,t_0) + \frac{e^{c_2(t,t_0)}\, F(k,t_0)}{1 - c_3(t,t_0) F(k,t_0)}$$

- In general

$$F = \ \mathrm{Re}F + i\ \ \mathrm{Im}F$$

- To compute

$$< \zeta_k\, \zeta_{k'} >= \frac{1}{\mathrm{Re}F(k,t)}$$

Region I

$$c_1(w) = i\, f(a,H,\dot\phi)\, k\, \frac{J_1(w/2)Y_1(w_I/2) - J_1(w_I/2)Y_1(w/2)}{J_1(w_I/2)Y_0(w/2) - J_0(w/2)Y_1(w_I/2)}$$

$$e^{c_2(w)} = \left(\frac{J_1(w_I/2)Y_0(w/2) - J_0(w/2)Y_1(w_I/2)}{J_1(w_I/2)Y_0(w_I/2) - J_0(w_I/2)Y_1(w_I/2)}\right)^{-2}$$

$$c_3(w) = \frac{-i}{k} g(a,H,\dot\phi)\, \frac{J_0(w_I/2)Y_0(w/2) - J_0(w/2)Y_0(w_I/2)}{J_1(w_I/2)Y_0(w/2) - J_0(w/2)Y_1(w_I/2)}$$

$$w \equiv \frac{k}{a(t)H(t)}, \qquad w_I \equiv \frac{k}{a(t_I)H(t_I)}$$

Region II

$$
\begin{aligned}
c_1(w) &= i\frac{k^3\alpha^2\sin(w-w_{BI})}{9H^2 w(w\cos(w-w_{BI})-\sin(w-w_{BI}))}\\
e^{c_2(w)} &= \left(\frac{w\cos(w-w_{BI})-\sin(w-w_{BI})}{w_{BI}}\right)^{-2}\\
c_3(w) &= -i\frac{9H^6 w_{BI}^2}{k^3\alpha^3}\left(\frac{1}{w_{BI}}-\frac{\cos(w-w_{BI})+w\sin(w-w_{BI})}{w\cos(w-w_{BI})-\sin(w-w_{BI})}\right)
\end{aligned}
$$

$$
w \equiv \frac{k}{a(t)H(t)}, \qquad w_{BI} \equiv \frac{k}{a(t_{BI})H(t_{BI})}
$$

- Regardless of the initial wave function, as the wave exits the horizon:

$$w \to 0, \qquad F(k,t) \sim \text{Constant}$$

  when assuming $w_{BI} \gg 1$
- Although the k^3 behavior is correlated with an inflationary expansion, it is not inescapable, it depends on the the k-dependence of $F(k, t_{BI})$

Constraints on the Initial State

$$< \Psi | T_{\mu\nu} | \Psi >_{\text{div}} = c_1 \, g_{\mu\nu} + c_2 \, G_{\mu\nu} + c_3 \, H_{\mu\nu} + c_4 \, {}^{(1)}H_{\mu\nu} + c_5 \, {}^{(2)}H_{\mu\nu}$$

$$\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\
H_{\mu\nu} &= \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \int d^{d+1}x \, \sqrt{-g} \, R^{\alpha\beta\gamma\sigma} R_{\alpha\beta\gamma\sigma} \\
{}^{(1)}H_{\mu\nu} &= \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \int d^{d+1}x \, \sqrt{-g} \, R^2 \\
{}^{(2)}H_{\mu\nu} &= \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \int d^{d+1}x \, \sqrt{-g} \, R^{\alpha\beta} R_{\alpha\beta}
\end{aligned}$$

Initial State

For modes $\frac{k}{a(t)} \sim M$, where M is the cut-off of the effective theory

$$
\begin{aligned}
Re\, F &= m(t)\, \omega(t) \left\{ 1 + \sum_{n \geq 1} b_n \left(\frac{a(t) H(t)}{k} \right)^n \right\} \\
Im\, F &= m(t)\, H(t) \left\{ \sum_{n \geq 0} c_n \left(\frac{a(t) H(t)}{k} \right)^n \right\}
\end{aligned}
$$

Constraints on the Initial State

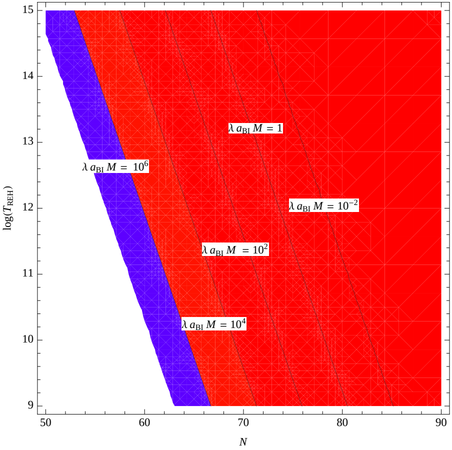
$$
\begin{aligned}
Re\, F &= m(t)\omega(t)\left\{1+\frac{1}{8}(d-1)(d\,w-1)\left(\frac{a(t)H(t)}{k}\right)^2\right.\\
&- \frac{1}{128}(d-1)(d\,w-1)(-5+(d(3+2w))(-5+d(2+3w))\\
&\left.\left(\frac{a(t)H(t)}{k}\right)^4+\cdots\right\}\\
Im\, F &= m(t)\,H(t)\,\frac{(d-1)}{2}\,\left\{1-\frac{1}{8}(d\,w-1)(-2+d(1+w))\right.\\
&\left(\frac{a(t)H(t)}{k}\right)^2+\frac{1}{32}(-1+dw)(-2+d+dw)\\
&(13-13d(1+w)+d^2(3+7w+3w^2)))\left(\frac{a(t)H(t)}{k}\right)^4+\cdots
\end{aligned}
$$

Large momentum behavior in de Sitter

$$
\begin{aligned}
Re\, F &= m(t)\,\omega(t)\left\{1-\left(\frac{a(t)H}{k}\right)^2+\left(\frac{a(t)H}{k}\right)^4+\cdots\right\}\\
Im\, F &= m(t)\,H\left\{1-\left(\frac{a(t)H}{k}\right)^2+\left(\frac{a(t)H}{k}\right)^4+\cdots\right\}
\end{aligned}
$$

This should be compared with the Bunch-Davis vacuum that corresponds to

$$
\begin{aligned}
Re\, F(t) &= m(t)\,\omega(t)\,\frac{(k/a(t)H)^2}{1+(k/a(t)H)^2}\\
Im\, F(t) &= m(t)\,H\,\frac{(k/a(t)H)^2}{1+(k/a(t)H)^2}
\end{aligned}
$$



$\lambda\, a_{BI}\, M = 1$
$\lambda\, a_{BI}\, M = 10^6$
$\lambda\, a_{BI}\, M = 10^{-2}$
$\lambda\, a_{BI}\, M = 10^2$
$\lambda\, a_{BI}\, M = 10^4$
15
14
13
12
11
10
9
50
60
70
80
90
$\log(T_{REH})$
N

Results

- If observed physical wavelength $\lambda a(t) < 1/M$ after onset inflation

$$P_\zeta(k) = \frac{1}{2\mathrm{Re}F(k,t)} = \left(\frac{H}{\dot{\phi}}\right)^2 H^2 \frac{1}{k^3}$$

- If $\lambda a(t) < 1/M$ before onset of inflation: compare the frequency associated to transition, ω_T, from one equation of state to other, with $k/a(t)$
  Adiabatic approximation: $k/a(t) \gg \omega_T$

$$P_\zeta(k) = \frac{1}{2\mathrm{Re}F(k,t)} = \left(\frac{H}{\dot{\phi}}\right)^2 H^2 \frac{1}{k^3}$$

Results

Sudden Approximation $k/a(t) \ll \omega_T \ll M$

$$P_\zeta(k) = \frac{1}{2\mathrm{Re}F(k,t)} = \left(\frac{H}{\dot{\phi}}\right)^4 H^2 \frac{1}{k^3} G(w_{BI})$$

$$G(w_{BI}) = \cos^2 w_{BI} - \frac{1}{w_{BI}} \sin 2w_{BI} + \frac{1}{w_{BI}^2} sin^2 w_{BI} \rightarrow \frac{1}{2}$$

$$w_{BI} = \frac{k}{a_{BI} H}$$

Assuming $F(t_{BI})$ to be what it was at the end of $w = 1$ phase.

Conclusions

- The $1/k^3$ behavior is the result of inflation + renormalizability conditions. The first without second wouldn't produce this behavior.
- The normalization is the most sensitive part to a prehistory. Modes will have a different normalization depending on when they became sub-effective cut-off and how the transition to slow-roll period happened.
- If the behavior of high momenta ($ka(t) \sim M$) is fixed by the condition of renormalizability of $T_{\mu\nu}$, all that is required to erase previous memory is to have many e-foldings ($N > 80$).
- Note that the renormalizability condition also erases trans-planckian information