# Generalized perturbations in dark energy and modified gravity

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# Motivation

How to parameterize deviations from GR to "understand" dark energy and/or dark matter

1. **Write down explicit theory**

   e.g. $\Lambda$, Quintessence, $F(R)$, TeVeS, Einstein-æther,...

2. **Parameterize an unknown theory**

   e.g. modify equations unique to GR

$$k^2\Phi = -4\pi G Q a^2 \rho\Delta, \quad \Psi - R\Phi = -12\pi G Q a^2 \rho(1+w)\sigma$$

if $R \neq 1$ and/or $Q \neq 1$ then deviation from GR found.

Want to do this for ***specified field content***

# Framework – dark sector

Einstein-Hilbert action + matter + dark sector

$$S = \int \mathrm{d}^4x \sqrt{-g}\left[R - 16\pi G\mathcal{L}_{\mathrm{m}} + \mathcal{L}_{\mathrm{d}}\right]$$

Field equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + U_{\mu\nu},$$

where dark energy-momentum tensor defined via

$$U_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g}\mathcal{L}_{\mathrm{d}}.$$

Background & perturbed level:

$$\bar{G}_{\mu\nu} = 8\pi G\bar{T}_{\mu\nu} + \bar{U}_{\mu\nu}, \qquad \delta G_{\mu\nu} = 8\pi G\delta T_{\mu\nu} + \delta U_{\mu\nu}.$$

Use for *any* background: FRW, $\Lambda$CDM, $F(R)$, DGP, Brans-Dicke, ...

# Framework – (3+1) split

Define time-like vector $u_\mu$, and orthogonal "space-only" tensor $\gamma_{\mu\nu}$:

$$u^\mu u_\mu = -1, \qquad u^\mu \gamma_{\mu\nu} = 0, \qquad g_{\mu\nu} = \gamma_{\mu\nu} - u_\mu u_\nu.$$

e.g. rank-2 symmetric tensor, $T_{\mu\nu} = T_{\nu\mu}$, only way to write is:

$$T_{\mu\nu} = \rho u_\mu u_\nu + P\gamma_{\mu\nu},$$

so time-part and space-part found via projections

$$\rho = u^\mu u^\nu T_{\mu\nu}, \qquad P = \frac{1}{3}\gamma^{\mu\nu} T_{\mu\nu}.$$

# Varying the action

Everything constructed from action of the form

$$S = \int \mathrm{d}^4x \sqrt{-g}\, \mathcal{L}.$$

Variations:

$$\delta S = \int \mathrm{d}^4x \sqrt{-g}\, \Diamond\mathcal{L}, \qquad \delta^2 S = \int \mathrm{d}^4x \sqrt{-g}\, \Diamond^2\mathcal{L},$$

where

$$\Diamond^n\mathcal{L} \equiv \frac{1}{\sqrt{-g}}\delta^n\big(\sqrt{-g}\,\mathcal{L}\big).$$

e.g. to quadratic order in perturbations

$$\Diamond\mathcal{L} = \delta\mathcal{L} - \frac{1}{2}\mathcal{L}g_{\mu\nu}\delta g^{\mu\nu},$$

$$\Diamond^2\mathcal{L} = \delta^2\mathcal{L} - (\delta\mathcal{L})g_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{4}\mathcal{L}\bigg(g_{\mu\nu}g_{\alpha\beta} + 2g_{\mu(\alpha}g_{\beta)\nu}\bigg)\delta g^{\mu\nu}\delta g^{\alpha\beta}.$$

# The perturbations

Interested in linear perturbations, $\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} + \delta U_{\mu\nu}$.

- KNOW field content – metric, scalar field, vector field
- KNOW how to construct equations of motion (Euler-Lagrange equations)
- KNOW how to construct constraint equations (Bianchi identity)
- DONT KNOW action of dark sector $\rightarrow$ parameterize this

Dark energy-momentum tensor: rank-2 symmetric

$$U_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \sqrt{-g}\mathcal{L}$$

Dont need to know field content to decompose:

$$U_{\mu\nu} = \rho u_\mu u_\nu + P\gamma_{\mu\nu} \quad \Rightarrow \quad 2 \text{ parameters}$$

# No extra fields: $\mathcal{L} = \mathcal{L}(g_{\mu\nu})$

To quadratic order in perturbations, action density describing perturbations is given completely by rank-4 tensor:

$$\diamondsuit^2 \mathcal{L} = \frac{1}{4}\mathcal{W}_{\mu\nu\alpha\beta}\delta g^{\mu\nu}\delta g^{\alpha\beta}.$$

Piece in field equation $\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} + \delta U_{\mu\nu}$. One finds:

$$\delta U_{\mu\nu} = \frac{1}{2}\left\{\mathcal{W}_{\alpha\beta\mu\nu} + g_{\alpha\beta}U_{\mu\nu}\right\}\delta g^{\alpha\beta}.$$

Symmetries:

$$\mathcal{W}_{\mu\nu\alpha\beta} = \mathcal{W}_{(\mu\nu)(\alpha\beta)} = \mathcal{W}_{\alpha\beta\mu\nu}.$$

# No extra fields: $\mathcal{L} = \mathcal{L}(g_{\mu\nu})$

Decompose $\mathcal{W}_{\mu\nu\alpha\beta}$ into $\{u_\mu, \gamma_{\mu\nu}\}$ with the above symmetries:

$$\begin{aligned}\mathcal{W}_{\mu\nu\alpha\beta} &= A_{\mathcal{W}} u_\mu u_\nu u_\alpha u_\beta + B_{\mathcal{W}} \left( \gamma_{\mu\nu} u_\alpha u_\beta + \gamma_{\alpha\beta} u_\mu u_\nu \right) \\ &\quad + 2C_{\mathcal{W}} \left( \gamma_{\mu(\alpha} u_{\beta)} u_\nu + \gamma_{\nu(\alpha} u_{\beta)} u_\mu \right) + \mathcal{E}_{\mu\nu\alpha\beta}, \\ \mathcal{E}_{\mu\nu\alpha\beta} &\equiv D_{\mathcal{W}} \gamma_{\mu\nu} \gamma_{\alpha\beta} + 2E_{\mathcal{W}} \gamma_{\mu(\alpha} \gamma_{\beta)\nu}\end{aligned}$$

Note: $u^\mu \mathcal{E}_{\mu\nu\alpha\beta} = 0 \Rightarrow$ entirely spatial "elasticity tensor".

FIVE parameters completely specify dark sector perturbations

$$\left\{ A_{\mathcal{W}}, B_{\mathcal{W}}, C_{\mathcal{W}}, D_{\mathcal{W}}, E_{\mathcal{W}} \right\}.$$

Some theories have these being equal, or have concrete physical interpretation.

# Dark fluids: $\mathcal{L} = \mathcal{L}(\phi, g_{\mu\nu}, \nabla_\mu \phi)$

Perturbation action density to quadratic order in perturbations $\{\delta\phi, \delta g_{\mu\nu}, \nabla_\mu \delta\phi\}$:

$$\begin{aligned} \diamondsuit^2 \mathcal{L} &= \mathcal{A}(\delta\phi)^2 + \mathcal{B}^\mu \delta\phi \nabla_\mu \delta\phi + \mathcal{C}^{\mu\nu} \nabla_\mu \delta\phi \nabla_\nu \delta\phi \\ &\quad + \frac{1}{4}\Big[\mathcal{Y}_{\alpha\mu\nu} \nabla^\alpha \delta\phi \delta g^{\mu\nu} + \mathcal{V}_{\mu\nu} \delta\phi \delta g^{\mu\nu} + \mathcal{W}_{\alpha\beta\mu\nu} \delta g^{\mu\nu} \delta g^{\alpha\beta}\Big]. \end{aligned}$$

Perturbed dark energy-momentum tensor:

$$\delta U_{\mu\nu} = \frac{1}{2}\Big\{\mathcal{V}_{\mu\nu}\delta\phi + \mathcal{Y}_{\alpha\mu\nu}\nabla^\alpha \delta\phi\Big\} + \frac{1}{2}\Big\{\mathcal{W}_{\alpha\beta\mu\nu} + g_{\alpha\beta} U_{\mu\nu}\Big\}\delta g^{\alpha\beta}.$$

Symmetries: (must also take into account $\delta(\nabla^\mu U_{\mu\nu}) = 0$)

$$\mathcal{V}_{\mu\nu} = \mathcal{V}_{(\mu\nu)}, \qquad \mathcal{Y}_{\alpha\mu\nu} = \mathcal{Y}_{\mu\nu\alpha} = \mathcal{Y}_{\nu\alpha\mu}$$

# Dark fluids: $\mathcal{L} = \mathcal{L}(\phi, g_{\mu\nu}, \nabla_\mu \phi)$

Decompose into $\{u_\mu, \gamma_{\mu\nu}\}$:

$$\mathcal{V}_{\mu\nu} = A_{\mathcal{V}} u_\mu u_\nu + B_{\mathcal{V}} \gamma_{\mu\nu},$$

$$\mathcal{Y}_{\alpha\mu\nu} = A_{\mathcal{Y}} u_\alpha u_\mu u_\nu + B_{\mathcal{Y}} \left( u_\alpha \gamma_{\mu\nu} + 2\gamma_{\alpha(\mu} u_{\nu)} \right).$$

NINE parameters completely specify dark sector perturbations

$$\left\{ A_{\mathcal{W}}, \cdots, E_{\mathcal{W}}, A_{\mathcal{V}}, B_{\mathcal{V}}, A_{\mathcal{Y}}, B_{\mathcal{Y}} \right\}.$$

# Kinetic fluid: $\mathcal{L} = \mathcal{L}(\phi, \mathcal{X})$

Introduce kinetic term, $\mathcal{X} = -\frac{1}{2}\nabla^{\mu}\phi\nabla_{\mu}\phi$. One finds

$$A_{\mathcal{V}} = 2\big(\mathcal{L}_{,\mathcal{X}\phi}\dot{\phi}^2 - \mathcal{L}_{,\phi}\big), \qquad B_{\mathcal{V}} = 2\mathcal{L}_{,\phi},$$

$$A_{\mathcal{Y}} = 2\big(\mathcal{L}_{,\mathcal{X}\mathcal{X}}\dot{\phi}^3 - 3\dot{\phi}\mathcal{L}_{,\mathcal{X}}\big), \quad B_{\mathcal{Y}} = 2\mathcal{L}_{,\mathcal{X}}\dot{\phi},$$

$$A_{\mathcal{W}} = \mathcal{L}_{,\mathcal{X}\mathcal{X}}\dot{\phi}^4 - 2\rho + p, \qquad B_{\mathcal{W}} = \rho, \qquad C_{\mathcal{W}} = -p, \qquad D_{\mathcal{W}} = E_{\mathcal{W}} = p$$

$\Rightarrow$ cant just "switch off" coefficients (necessarily)

# Operator expansion

Generalize & use arbitrary order of derivative. Write

$$\delta U_{\mu\nu} = \hat{\mathbb{Y}}_{\mu\nu}\delta\phi + \hat{\mathbb{W}}_{\mu\nu\alpha\beta}\delta g^{\alpha\beta},$$

where

$$\hat{\mathbb{Y}}_{\mu\nu} = \mathbb{L}_{\mu\nu} + \mathbb{M}^{\alpha}{}_{\mu\nu}\nabla_\alpha + \mathbb{N}^{\alpha\beta}{}_{\mu\nu}\nabla_\alpha\nabla_\beta + \cdots$$

$$\hat{\mathbb{W}}_{\mu\nu\alpha\beta} = \mathbb{A}_{\mu\nu\alpha\beta} + \mathbb{B}^{\rho}{}_{\mu\nu\alpha\beta}\nabla_\rho + \mathbb{C}^{\rho\sigma}{}_{\mu\nu\alpha\beta}\nabla_\rho\nabla_\sigma + \cdots$$

High-order theories fit here: $f(R), f(G), \ldots$

e.g. $F(R)$ has $\hat{\mathbb{Y}}_{\mu\nu} = 0$ and uses up to 4$^{\text{th}}$-order derivatives in $\hat{\mathbb{W}}_{\mu\nu\alpha\beta}$

# Useful identities

$$\delta R_{\mu\nu} = -g_{\pi(\alpha}g_{\beta)\gamma}C^{\rho\sigma\pi\gamma}{}_{\mu\nu}\nabla_\rho\nabla_\sigma\delta g^{\alpha\beta},$$

$$\delta\Gamma^{\lambda}{}_{\alpha\beta} = B^{\lambda\xi\pi\rho}{}_{\alpha\beta}\nabla_\xi\delta g_{\pi\rho},$$

$$C^{\rho\sigma\pi\gamma}{}_{\mu\nu} \equiv g^{\alpha\beta}\left[\delta^{\rho}_{\alpha}\delta^{\gamma}_{\beta}\delta^{\sigma}_{(\mu}\delta^{\pi}_{\nu)} - \delta^{\sigma}_{(\mu}\delta^{\pi}_{\alpha)}\delta^{\gamma}_{\beta}\delta^{\rho}_{\nu} + \delta^{\rho}_{[\nu}\delta^{\gamma}_{\alpha]}\delta^{\pi}_{\mu}\delta^{\sigma}_{\beta}\right],$$

$$B^{\xi\alpha\pi\rho}{}_{\mu\nu} \equiv g^{\xi\sigma}\left[\delta^{\alpha}_{(\nu}\delta^{(\rho}_{\sigma)}\delta^{\pi)}_{\mu} - \frac{1}{2}\delta^{\alpha}_{\sigma}\delta^{(\pi}_{\mu}\delta^{\rho)}_{\nu}\right].$$

# Summary

- Way to completely parametrize dark sector for a *given field content* – small number of parameters required
- Important extra concepts:

  $\Rightarrow$ ***Bianchi identities***

$$\nabla^\mu U_{\mu\nu} = 0, \qquad \delta(\nabla^\mu U_{\mu\nu}) = 0$$

  $\Rightarrow$ ***Equations of motion***

$$\partial_\mu\left(\frac{\Diamond\mathcal{L}}{\delta\partial_\mu X}\right) - \frac{\Diamond\mathcal{L}}{\delta X} = 0, \qquad \partial_\mu\left(\frac{\Diamond^2\mathcal{L}}{\delta\partial_\mu X}\right) - \frac{\Diamond^2\mathcal{L}}{\delta X} = 0.$$

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