Gravitational waves from the electroweak phase transitions

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Introduction

The Universe has expanded and cooled down from a very hot initial state to (presently) $2.7^\circ$K. It seems likely that it underwent several phase transitions during its evolution of adiabatic expansion.
Inflation
Events

- **Inflation**
- **Pre-heating**, $T_i \simeq 10^{14}$ GeV, $t_i = 2.3 \text{ sec} \left( \frac{1 \text{ MeV}}{T_i} \right)^2 g_{\text{eff}}(T)^{-1/2} \simeq 10^{-32} \text{ sec}$,

  $$\eta_i = a(t_i) / H(t_i) = 2t_i / (1 + z_i) \simeq 10^{-10} \text{ sec},$$

  $$\omega_i \simeq (10^9 - 10^{12}) \text{ Hz}.$$
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Gravitational waves are sourced by fluctuations in the energy momentum tensor which have a non-vanishing spin-2 contribution.

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where \( h_{ij} \) is transverse and traceless.

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Here \( \Pi_{ij}(k) \) is the Fourier component of the tensors type (spin-2) anisotropic stress and

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During a first order phase transition anisotropic stresses can be generated by (Kamionkowski, Kosowsky, Turner, Watkins, 92-94)

- Colliding bubbles
- Inhomogeneities in the distribution of the order parameter field.
- Inhomogeneities in the cosmic fluid (e.g. turbulence) or other fields (e.g. magnetic field).
Because of causality, the correlator \( \langle \Pi_{ij}(\eta_1, x)\Pi_{lm}(\eta_2, y) \rangle = M_{ijlm}(\eta_1, \eta_2, x - y) \) is a function of compact support. For distances \(|x - y| > \max(\eta_1, \eta_2)\), \( M \equiv 0 \).
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We decompose $\Pi_{ij}$ into two helicity modes which we assume to be uncorrelated (parity),

$$
\Pi_{ij}(\eta, k) = e^+_{ij} \Pi^+(\eta, k) + e^-_{ij} \Pi^-(\eta, k)
$$

$$
\langle \Pi^+(\eta, k)\Pi^+(\eta', k') \rangle = \langle \Pi^-(\eta, k)\Pi^-(\eta', k') \rangle = (2\pi)^3 \delta^3(k - k') \rho_X^2 P(\eta, \eta', k)
$$

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\langle \Pi^+(\eta, k)\Pi^-(\eta', k') \rangle = 0.
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Here $\rho_X$ is the energy density of the component $X$ with anisotropic stress $\Pi$ which has been factorized in order to keep $k^3 P(\eta, \eta', k)$ dimensionless.
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\Pi_{ij}(\eta, k) = e_{ij}^+ \Pi_+(\eta, k) + e_{ij}^- \Pi_-(\eta, k)
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\langle \Pi_+(\eta, k)\Pi_+^{*}(\eta', k') \rangle = \langle \Pi_-(\eta, k)\Pi_-^{*}(\eta', k') \rangle = (2\pi)^3 \delta^3(k - k') \rho_X^2 P(\eta, \eta', k)
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Here \( \rho_X \) is the energy density of the component \( X \) with anisotropic stress \( \Pi \) which has been factorized in order to keep \( k^3 P(\eta, \eta', k) \) dimensionless.

Causality implies that the function \( P(\eta, \eta', k) \) is analytic in \( k \). We therefore expect it to start out as white noise and to decay beyond a certain correlation scale \( k_c(\eta, \eta') > \min(1/\eta, 1/\eta') \).
The spectrum
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If the gravitational wave source is active only for a short duration $\Delta \eta$ (less than one Hubble time), we can neglect the damping term $2\mathcal{H}$ in the equation of motion for $h$. The solution with vanishing initial conditions is then

\[
h(k, \eta) = \frac{8i\pi G a^3_*}{6ak} \left[ e^{-ik\eta} \int_{\eta_*}^{\eta_*+\Delta \eta} d\eta' e^{ik\eta'} \Pi(\eta', k) + e^{ik\eta} \int_{\eta_*}^{\eta_*+\Delta \eta} d\eta' e^{-ik\eta'} \Pi(\eta', k) \right]
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The gravitational wave energy density is given by

\[
\rho_{gw}(\eta, x) = \frac{1}{32\pi G a^2} \langle \partial_\eta h_{ij}(\eta, x) \partial_\eta h_{ij}^*(\eta, x) \rangle
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If the Universe is radiation dominated during the phase when the gravitational waves are generated, this gives on large scales, $k < k_c$

$$\frac{d\Omega_{gw}}{d \ln(k)}(\eta_0) = \frac{12 \Omega_{rad}(\eta_0)}{\pi^2} \left( \frac{\Omega_X(\eta_*)}{\Omega_{rad}(\eta_*)} \right)^2 \mathcal{H}_*^2 k^3 \text{Re}[P(k, k, k)] .$$
The spectrum

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Here

\[ P(\omega, \omega', k) \equiv \int_{-\infty}^{-\infty} d\eta \int_{-\infty}^{-\infty} d\eta' P(\eta, \eta', k) e^{i(\omega \eta - \omega' \eta')} . \]
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On large scales, \( k < k_c > \mathcal{H}_* \) the GW energy density from a 'causal' source always scales like \( k^3 \). This remains valid also for long duration sources. \( 1/k_c \) is the correlation scale which is smaller than the co-moving Hubble scale \( 1/\mathcal{H}_* = \eta_* \).
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In the following we consider a discontinuous \( (g_1) \), a continuous but not differentiable \( (g_2) \) and a once differentiable \( (g_3) \) time behavior.
\[ P(\eta, \eta', k) = \delta(\eta - \eta') F(k) g^2(\eta). \]

Hence
\[ P(k, k, k) = F(k) \int_{\eta_*}^{\eta_* + \Delta \eta_*} g^2(\eta) d\eta. \]

\[ k_c = \pi \beta / \nu \]

Caprini, RD, Konstandin and Servant, 2009
A coherent source

\[ P(\eta, \eta', k) = \sqrt{P(\eta, \eta, k)} \sqrt{P(\eta', \eta', k)} . \]

Hence

\[ P(k, k, k) = \left( \int_{\eta_*}^{\eta_* + \Delta \eta_*} \sqrt{P(\eta, \eta, k)} d\eta \right)^2 . \]

\[ \nu = 1 \]

\[ \nu = 0.01 \]

\[ k_c = \frac{\pi \beta}{\nu} \]

\[ \omega_c = \pi \beta \]

Caprini RD, Konstantin and Servant, 2009
For a totally incoherent source the peak position of the GW spectrum is determined by the peak of the spatial Fourier transform of the source.
Peak position

- For a totally incoherent source the peak position of the GW spectrum is determined by the peak of the **spatial** Fourier transform of the source.
- For a coherent source with $P(\eta, \eta, k) = g(\eta)^2 F(k)$, the peak position depends on the time structure of $g(\eta)$:
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- For a coherent source with $P(\eta, \eta, k) = g(\eta)^2 F(k)$, the peak position depends on the time structure of $g(\eta)$:
  - If $g(\eta)$ is discontinuous, hence $g(\omega) \propto \omega^{-1}$ beyond the peak, $P(k, k, k) \propto k^{-2} F(k)$, the peak position of the GW spectrum $\propto k^3 P(k, k, k) \propto k F(k)$ is again determined by the peak of the spatial Fourier transform of the source.
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2. If $g(\eta)$ is continuous but not differentiable, hence $g(\omega) \propto \omega^{-2}$ beyond the peak, $P(k, k, k) \propto k^{-4} F(k)$, the peak position of the GW spectrum $\propto k^3 P(k, k, k) \propto k^{-1} F(k)$ is determined by the peak of the temporal Fourier transform of the source.
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3. Idem if $g(\eta)$ is once continuously differentiable, $g(\omega) \propto \omega^{-3}$, $P(k, k, k) \propto k^{-3} F(k)$. 

Ruth Durrer (Université de Genève) Gravitational waves from the ew phase transition Cambridge 2009 12 / 24
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Caprini, RD, Konstandin and Servant, 2009
$$k_c = \frac{\pi \beta}{\nu}, \quad \nu = 0.01$$

Caprini, RD, Konstandin and Servant, 2009

$$g_1 \not\in C^0, \quad g_2 \in C^0 \text{ but } g_2 \not\in C^1, \quad g_3 \in C^1 \text{ but } g_3 \not\in C^2.$$
According to the standard model, the electroweak transition is not even second order, but only a cross-over. Then, this transition does not lead to the formation of gravitational waves.
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The spectrum is supposed to peak at the correlation scale $k_c = \beta \simeq 100/\eta_\ast \sim 10^{-3}\text{Hz}$, which is close to the frequency of the peak sensitivity for the space born gravitational wave antenna LISA, proposed for launch in 2018, a ESA cosmic vision project.
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The spectrum is such that

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The spectrum goes like \( \frac{d\Omega_{GW}}{d\ln(k)} \propto k^3 \), \( k < k_c \simeq \pi \beta \) and \( \frac{d\Omega_{GW}}{d\ln(k)} \propto k^{-1} \), \( k > k_c \). The peak sensitivity of LISA is supposed to be about \( h^2 \frac{d\Omega_{GW}}{d\ln(k)} \bigg|_{k=k_p} \simeq 10^{-12} \), \( k_p \sim 10^{-3} \text{Hz} \).
The electroweak phase transition: GW’s from bubble collisions

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Caprini, RD, Konstandin, Servant, 2009
Huber & Konstandin 2008

$\Omega_{GW}$ from colliding bubbles, numerical results.
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The electroweak phase transition: GW’s from turbulence and magnetic fields

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- Because both, the vorticity and the magnetic field are divergence free, causality requires that both, $P_v(k)$ and $P_B(k) \propto k^2$ for small $k$.

  $$\langle \mathbf{v}(k) \mathbf{v}(k') \rangle = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') (\hat{k}_j \hat{k}_i - \delta_{ij}) P_v(k),$$

  $$\langle \mathbf{B}(k) \mathbf{B}(k') \rangle = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') (\hat{k}_j \hat{k}_i - \delta_{ij}) P_B(k)$$

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- Because both, the vorticity and the magnetic field are divergence free, causality requires that both, $P_v(k)$ and $P_B(k) \propto k^2$ for small $k$.
- The behavior of the spectrum on scaler smaller than the correlations scale $k > k_c$ is expected to be a Kolmogorov spectrum for the vorticity field, $P_v \propto k^{-11/3}$ and an Iroshnikov–Kraichnan spectrum for the magnetic field, $P_B \propto k^{-7/2}$.
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\[
\langle v_i(k)v_j(k') \rangle = (2\pi)^3 \delta^3(k-k')(\hat{k}_j \hat{k}_i - \delta_{ij})P_v(k),
\]

\[
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- For the induced GW spectrum this yields

\[
\frac{d\Omega_{GW\bullet}(k, \eta_0)}{d \ln(k)} \simeq \Omega_{\text{rad}}(\eta_0) \left( \frac{\Omega_\bullet(\eta_*)}{\Omega_{\text{rad}}(\eta_*)} \right)^2 \times \begin{cases} 
(k/k_c)^3 & \text{for } k < k_c \\
(k/k_c)^{-\alpha} & \text{for } k > k_c 
\end{cases}
\]

For $\bullet = v$ we have $\alpha = 11/3 - 1 = 8/3$ and for $\bullet = B$ we have $\alpha = 7/2 - 1 = 5/2$.

(See Caprini & RD, 2006)
The electroweak phase transition:
GW’s from turbulence and magnetic fields

\[ \Omega_{GW} \text{ from magnetic fields (red) and turbulence (blue), total (black). Modelling the time-decorrelation of the source (Kraichnan decorrelation) by a 'top-hat' in Fourier space. Sensitivity curves from A. Buonanno 2003.} \]
We also consider a phase transition at $T = 5 \times 10^6$ GeV with $\beta/H = 50$. 

Caprini, RD, Servant, in preparation
It is difficult to estimate $\Omega_B(\eta_*)$ or $\Omega_V(\eta_*)$, but since causality requires the spectra to be so blue, \( \frac{d\Omega_B(k,\eta_*)}{d\ln(k)} \propto k^5 \), the limit on gravitational waves (which comes from small scales \( k \simeq k_c \) yields very strong limits on primordial magnetic fields on large scales already from the simple *nucleosynthesis constraint*, \( \Omega_{GW} \lesssim 0.1\Omega_{\text{rad}} \).
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E.g. for $k = (0.1\text{Mpc})^{-1}$ we obtain $k^{3/2}B(k) < 10^{-30}\text{Gauss}$.

![Graph](image_url)

Caprini & RD., 2001
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Such helical magnetic fields lead to T-B and E-B correlations in the CMB, and they also generate gravitational waves with non-vanishing helicity (Caprini, Kahniashvili, RD. 2004).
Contrary to a non-helical magnetic field, helicity conservation for a helical field does lead to an inverse cascade in the evolution of the magnetic field:

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This can move power from small to larger scales. However, this is not quite sufficient to present a way out for the electroweak phase transition, but it can work for the QCD phase transition (Caprini, RD, Fenu 2009).
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\frac{\epsilon_B(k,t)}{\epsilon_B(0)} \frac{\xi_B(0)}{\xi_B(k,t)} \propto k_B^{-6.3} (k_B^{-1} - 10^9) \]

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In this case, the GW background would not be parity symmetric. There would be more GW’s of one helicity than of the other.
First order phase transitions stir the relativistic cosmic plasma sufficiently to lead to the generation of a stochastic gravitational wave background.
Conclusions

- First order phase transitions stir the relativistic cosmic plasma sufficiently to lead to the generation of a stochastic gravitational wave background.
- Observing such a background would open a new window to the early Universe and to high energy physics!
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Generically, the density parameter of the GW background is of the order of

$$\Omega_{GW}(t_0) \sim \Omega_{rad}(t_0) \left( \frac{\Omega_X(t_*)}{\Omega_{rad}(t_*)} \right)^2$$
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- The spectrum grows like \( \frac{d\Omega_{GW}(k,t_0)}{d \ln(k)} \propto k^3 \) on large scales and decays on scales smaller than the correlations scale \( k_c \sim 1/\eta_* \). The decay law depends of the physics of the source.
If the SM holds, the electroweak phase transition is not of first order and does (probably) not generate an appreciable gravitational wave background. However, simple deviations from the SM can make it first order (like adding a Higgs singlet (Ashoorioon & Konstandin 2009).
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In this case we also expect a parity violating gravitational wave background, $|h_+(k)|^2 \neq |h_-(k)|^2$. 