INFLATION:

BEYOND SLOW-ROLL, 60 E-FOLDS, SMALL PERTURBATIONS AND INFLATION ITSELF

CAMBRIDGE, 17.12.2007

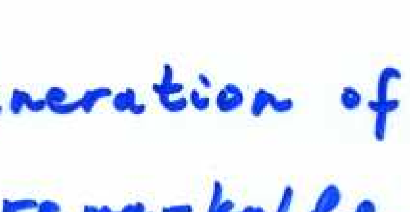
1. Present observational status of slow-roll
2. Exact solutions vs. slow-roll expansion ("beyond slow-roll")
3. $N > 60$ evolution ("beyond 60 e-folds")
4. Stochastic inflation ("beyond small perturbations")
5. Observational determination of local initial conditions for inflation ("beyond inflation itself")

Main positive prediction of inflation:

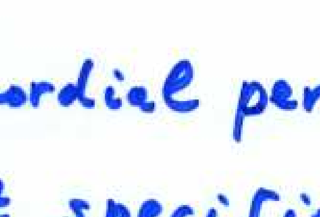
types, spectrum and statistics of primordial perturbations

Absence of spatial curvature may be considered as a particular case of this generic prediction → for the monopole, $l = 0$ mode

Generation of primordial perturbations – remarkable, but specific application of the theory of particle creation in external gravitational fields (which is, in turn, some approximation to quantum gravity + quantum matter)



scalar pert.



grav. waves

Genuine quantum-gravitational effect (space-time metric has to be quantized in both cases)

Is inflation falcifiable?

Any concrete model is falcifiable, and many of them have been already falcified.

Fortunately, there remain many viable models, and the first 3 are among them.

1. "Geometrical inflation" (A.S., 1980)

$R + \frac{R^2}{6M^2}$ model

$\frac{M}{M_{Pl}} = 2.7\cdot 10^{-6}\cdot \frac{53}{N}$ $t_{reh} \simeq 10^{-26}\,s$

$T_{reh} \sim 10^{-9} M_{Pl} = 10^{10}\,GeV$

$n_s - 1 = -\frac{2}{N} = -0.04\cdot\frac{53}{N}$, $r = \frac{12}{N^2} \simeq 4\cdot 10^{-3}\left(\frac{53}{N}\right)^2$

2. Inflation without phase transition

(Linde, 1983 – though the mathematics was known from A.S., 1978 where this model was used for the bouncing scenario)

$V = \frac{m^2\varphi^2}{2}$, $\frac{m}{M_{Pl}} = 1.2\cdot 10^{-6}$

$n_s - 1 = \frac{2}{N} = -0.04\cdot\frac{50}{N}$, $r = \frac{8}{N} = 0.16\cdot\frac{50}{N}$

3. Inflation with phase transition

(Linde, Albrecht & Steinhardt – 1982)

$V = V_0 - \frac{\lambda\varphi^4}{4}$, $\lambda = 1.6\cdot 10^{-13}$

$n_s - 1 = \frac{3}{N} = -0.06\cdot\frac{50}{N}$, $r \ll 1$

Two small parameters:

1. $h \sim 10^{-5}$

2. $\frac{1}{N}$ with $N = \ln \frac{R_H}{\lambda_\gamma} - \frac{1}{2} \ln \frac{1}{H_f l_{Pl}} - \Delta N_{reh}$

$$N = 50-60$$

From observations:

$$\overline{|n_S(k) - 1|} < 0.1$$

Not assuming slow-roll both for background and perturbations, assuming only that $H(\varphi)$ may be Taylor-expanded inside $\Delta N \sim 10$ in the observable window:

$$\epsilon_H = \frac{m_{Pl}^2 H'^2}{4\pi H^2} = 0.006 \pm 0.005$$

$$\eta_H = \frac{m_{Pl}^2 H''}{4\pi H} = -0.006 \pm 0.020$$

$$\xi^2_H = \frac{m_{Pl}^4 H' H'''}{16\pi^2 H^2} = 0.010 \pm 0.008$$

J. Lesgourgues, A. S., W. Valkenburg JCAP (2007); arXiv: 0710.1630 [astro-ph]

EXACT LINEAR SOLUTIONS

$$3H^2 = \frac{8\pi G}{3}\left(\frac{\dot{\varphi}^2}{2} + V(\varphi)\right)$$

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0$$

$$\frac{d^2 u_k}{d\eta^2} + \left(k^2 - \frac{1}{z}\frac{d^2 z}{d\eta^2}\right) u_k = 0$$

$$u = Qa, \quad H = \frac{\dot{a}}{a}, \quad z = \frac{a\dot{\varphi}}{H}, \quad \eta = \int \frac{dt}{a(t)}$$

with the initial condition

$$u_k = \frac{e^{-ik\eta}}{\sqrt{2k}} \qquad k\eta \to -\infty$$

KNOWN EXACT SOLUTIONS

1. Power-law inflation (1984)

$$n_s = \text{const} < 1, \quad a(t) \propto t^q, \quad q = \frac{3-n_s}{1-n_s} > 1$$

$$V(\varphi) \propto H^2(\varphi) \propto \exp\left(\pm\sqrt{\frac{16\pi G}{q}}\,\varphi\right)$$

2. $n_s = 1$ $\qquad (r \neq 0\,!)$ $\qquad$ Slow-roll: $V(\varphi) \propto \varphi^{-2}$ only

$$y = \frac{\sqrt{4\pi G}\,H(\varphi)}{B} = \exp\left(\frac{x^2}{2}\right)\left(\int_x \exp\left(-\frac{\tilde{x}^2}{2}\right) d\tilde{x} + C\right)$$

$$x = \sqrt{4\pi G}\,\varphi\,; \quad \langle k^3 \zeta^2(\vec{k})\rangle = \frac{B^2}{2}$$

$$V(\varphi) = \frac{3B^2}{32\pi^2 G^2}\left(y^2(x) - \frac{1}{3}y'^2(x)\right)$$

A.S., JETP Lett. 82, 169 (2005)
astro-ph/0507193

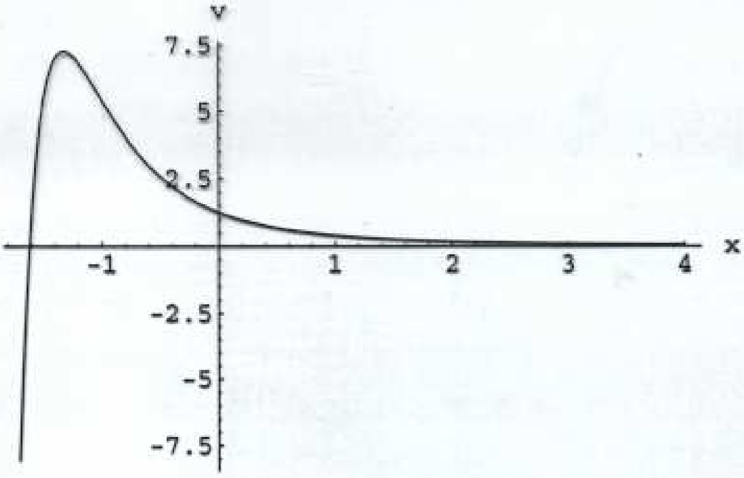


Figure 1: The dimensionless potential $v(x)$ for the $C = 0$ case.

$t \to t_0$, so this singularity is a strong one. The same refers to all initially expanding ($y > 0$) solutions with $C \neq 0$ and $C > -\sqrt{2\pi}$ – they all begin from such a singularity.

By taking $C < 0$ and very small, it becomes possible to construct a solution with a long but finite inflationary stage. Namely, if $C = -\sqrt{3}x_1^{-2}\exp(-x_1^2/2)$ with $x_1 \gg 1$, then $v(x)$ becomes zero at $x = x_1$ (y still remains $\sim x_1^{-1}$). In this case inflation ends (ϵ, $|\tilde{\eta}| \sim 1$) at $x = x_1 - \mathcal{O}(x_1^{-1})$. The total number of e-folds is $N_{tot} = 2\pi G\phi_1^2 = x_1^2/2$. Thus, $C \sim \exp(-N_{tot})$ that is in agreement with the general principle that terms not caught by an arbitrary order of a WKB-type expansion are exponentially small. For $x \geq x_1$, one may put $v \equiv 0$. Then the kinetic dominated phase $a(t) \propto t^{1/3}$ follows the inflationary stage. Or, we may assume that v has a local minimum $v = \frac{1}{2}\mu^2(x - x_1)^2$ around this point. It results in oscillations in ϕ and the matter-dominated post-inflationary stage $a(t) \propto t^{2/3}$.

Finally, note that the spectrum of gravitational waves (GW) is not flat for this model: for $1 \ll x \ll x_1$, the tensor-scalar ratio and the slope of the GW initial power spectrum $r = -8n_T = 16/x^2 = 8/N$ where N is the number of e-folds from the *beginning* of inflation. The present upper observational bound $r < 0.36$ [29] requires $N > 22$ for the comoving scale crossing the Hubble radius at present. So, N_{tot} should exceed ~ 70 in this model.

The research was partially supported by the Russian Foundation for Fundamental Research, grant No. 05-02-17450, by the Research Programme "Elementary Particles" of the Russian Academy of Sciences and by the scientific school grant No. 2338.2003.2 of the Russian Ministry of Education and Science.

References

[1] V.F. Mukhanov and G.V. Chibisov, Pis'ma v ZhETF **33**, 549 (1981) [JETP Lett. **33**, 532 (1981)].

[2] A.A. Starobinsky, Phys. Lett. B **91**, 99 (1980).

[3] S.W. Hawking, Phys. Lett. B **115**, 295 (1982).

3. Inflation near a jump in $V''(\varphi)$

$|[V''(\varphi)]| \ll H^2$

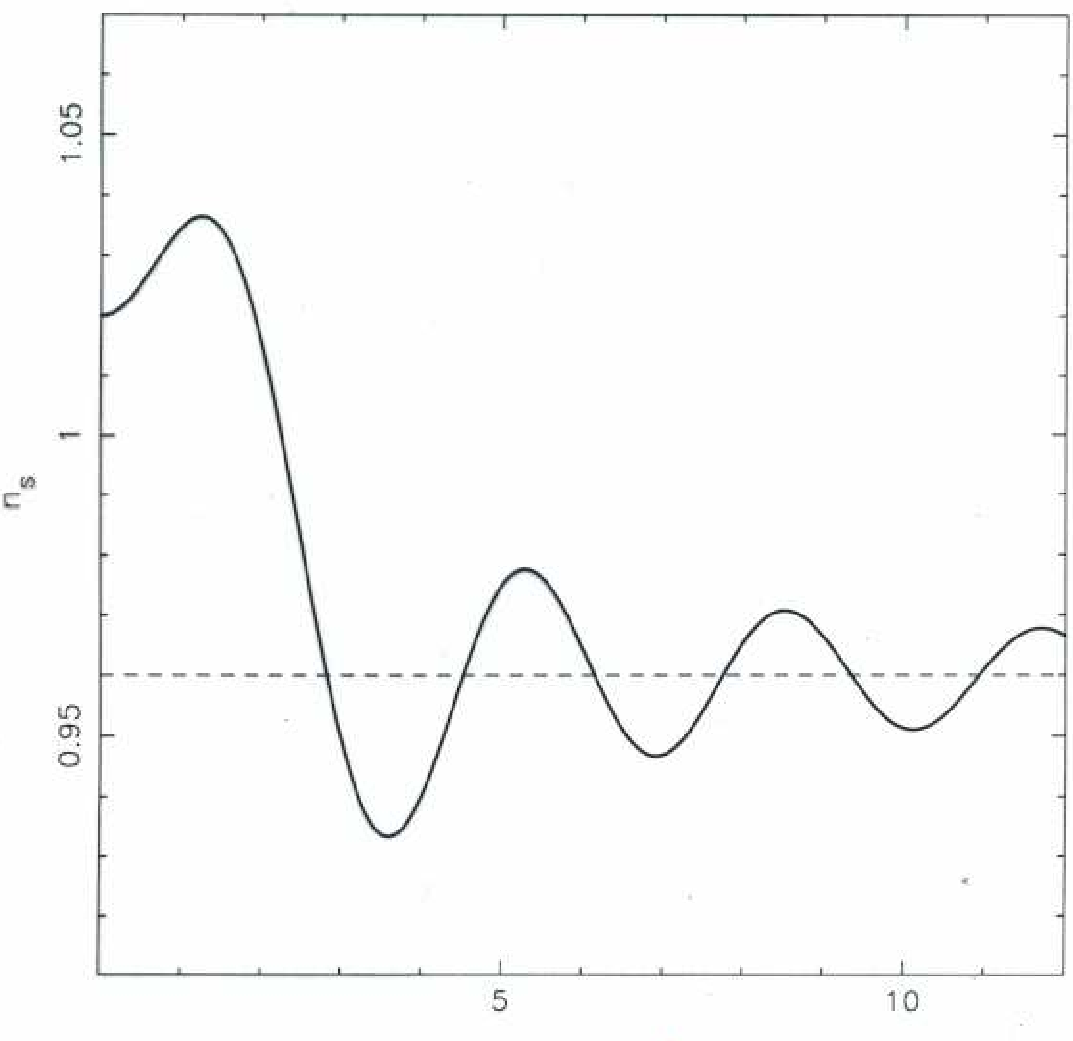
Mechanism: fast <u>second</u> order phase transition during inflation in another scalar field weakly coupled to the inflaton

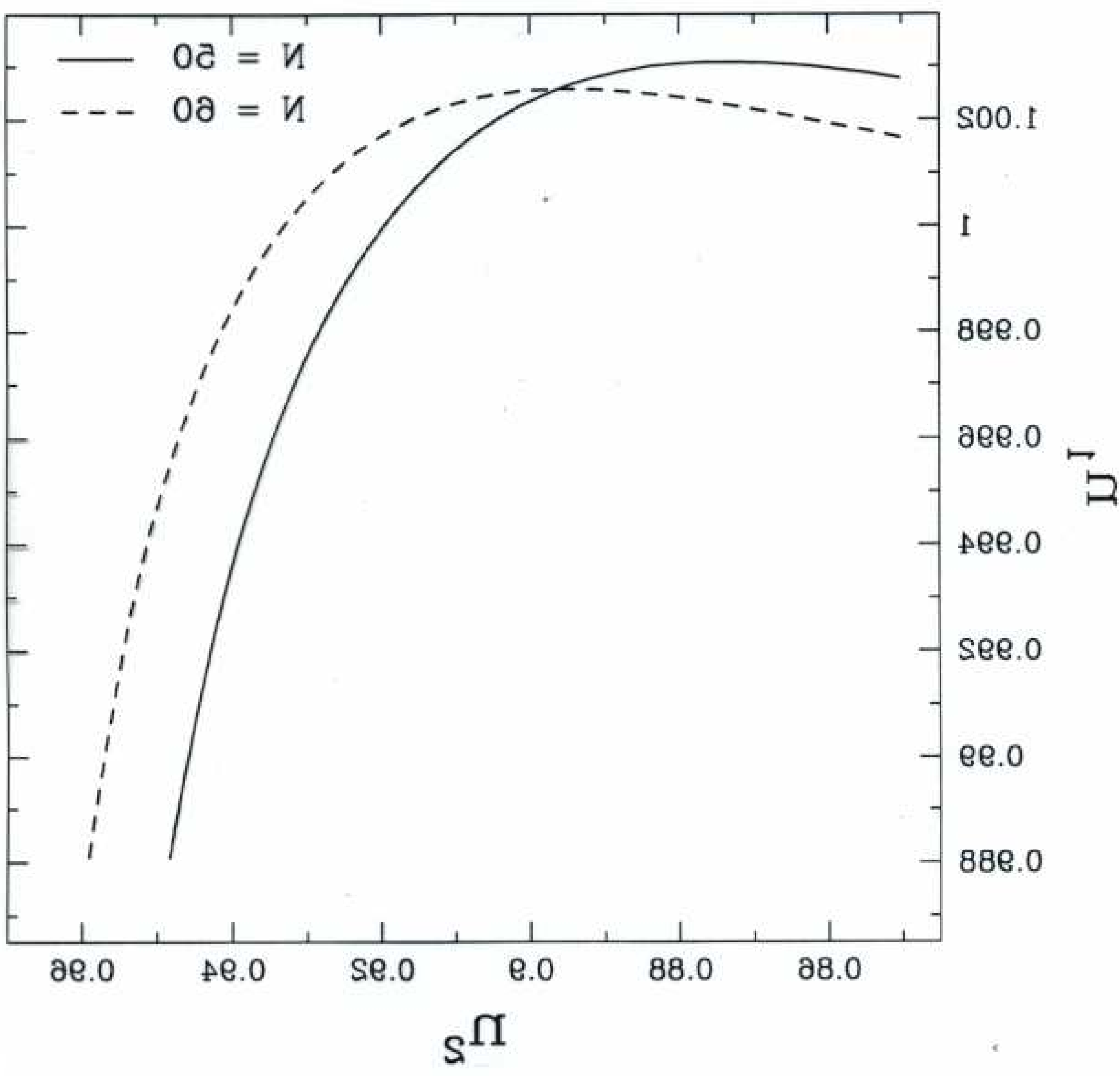
The same model as that used to end inflation in the hybrid inflationary model may be used (though with different parameter values)

Resulting feature in the perturbation spectum:
fast change of the slope n_s modulated by oscillations

M. Joy, V. Sahni and A.S., PRD (2008); arXiv: 0711.1582 [astro-ph]
'Localized running'

The smoothest way to solve large running problem using a fast phase transition during inflation


1.05
1
0.95
n_s
5
10
$x = \frac{k}{k_0}$


N = 50
N = 60
1.002
1
0.998
0.996
0.994
0.992
0.99
0.988
U_1
0.96
0.94
0.92
0.9
0.88
0.86
U_2

INVERSE (RECONSTRUCTION) PROBLEM

$P(k) \longrightarrow H(\varphi)$

1. Mathematical part → for noiceless $P(k)$
2. Physical part → for noicy data

In the slow-roll approximation:

1. $P_s(k)$ is not sufficient for a unique reconstruction
2. $P_g(k)$ is sufficient

For the exact solution:

both $P_s(k)$ and $P_g(k)$ alone (or their ratio) are not enough.

For $P_s(k)$ and $P_g(k)$ the answer is not known.

BEYOND LAST 60 e-folds OF INFLATION TO THE PAST

1. From metric perturbations

Some information can be obtained, though restricted and non-decisive (some classes of models can be excluded only)

Example: intermediate inflation

$V(\varphi) \propto \varphi^{-\alpha}, \quad \varphi > M_{Pl}$

$$n_S - 1 = \frac{2-\alpha}{2N_{beg}}, \quad n_T = -\frac{r}{8} = -\frac{\alpha}{2N_{beg}} = \frac{\alpha}{\alpha-2}(n_S-1)$$

$n_T < n_S - 1 \longrightarrow$ characteristic for such models

2. To obtain decisive information about the previous history of the Universe new measuring devices are required

Light scalar fields

$$\frac{d\langle\varphi^2\rangle}{dt} = \frac{H^3}{4\pi^2} \qquad (1982)$$

Beyond small perturbations

For $N \gg 1$, $\langle k^2 \rangle$ becomes $\gtrsim 1$

Stochastic approach to inflation
("stochastic inflation")

$$\hat{R}_i^k - \frac{1}{2}\delta_i^k \hat{R} = 8\pi G\, \hat{T}_i^k$$

with $\hat{T}_i^k = \hat{T}_i^k(\hat{g}_{lm})$ (not $(\langle \hat{g}_{lm} \rangle)$!)

leads to QFT in a stochastic background

1. Can deal with arbitrary large global inhomogeneity
2. Takes backreaction into account
3. Goes beyond any finite order of loop corrections

Another 'time' variables

$$\tau^{(n)} = \int H^{n}(t, \vec{x})\, dt \qquad H^2 = \frac{8\pi G V(\Phi)}{3}$$

This is not a time reparametrization in GR $t \to f(t)$

$\tau^{(n)}$ describe different stochastic processes and even have different dimensionality

Different 'clocks':

$n=0$ phase of the wave function of a massive particle $(m \gg H)$

$n=1$ perturbations

$n=3$ rms value of a light scalar field generated during inflation

$$\langle \chi^2 \rangle = \frac{1}{4\pi^2} \langle \int H^3 dt \rangle = \frac{\langle \tau^{(3)} \rangle}{4\pi^2}$$

$$\frac{d\Phi}{d\tau} = -\frac{1}{3H^{n+1}} \frac{dV}{d\varphi} + f$$

$$\langle f(\tau_1) f(\tau_2) \rangle = \frac{H^{3-n}}{4\pi^2} \delta(\tau_1 - \tau_2)$$

Einstein-Smolukovsky (Fokker-Planck) equation

$$\frac{\partial \rho}{\partial \tau} = \frac{\partial}{\partial \varphi}\left(\frac{V'}{3H^{n+1}}\rho\right) + \frac{\partial^2}{\partial \varphi^2}\left(H^{3-n}\rho\right)\cdot\frac{1}{8\pi^2}$$

$\int \rho\, d\varphi = 1 \rightarrow$ probability conservation

Remarks

1. More generally: $\ldots + \frac{\partial}{\partial \varphi}\left(H^{(3-n)\alpha}\frac{\partial}{\partial \varphi}\left(H^{(3-n)(1-\alpha)}\rho\right)\right)\cdot\frac{1}{8\pi^2}$

   $0 \leq \alpha \leq 1$

   $\alpha = 0$ – Ito calculus

   $\alpha = \frac{1}{2}$ – Stratonovich calculus

   However, keeping terms depending on α exceeds the accuracy of the stochastic approach. Thus, α may be put zero.

2. Results are independent on the form of the cutoff in the momentum space as far as it occurs for $k \ll aH$ ($\varepsilon \ll 1$)

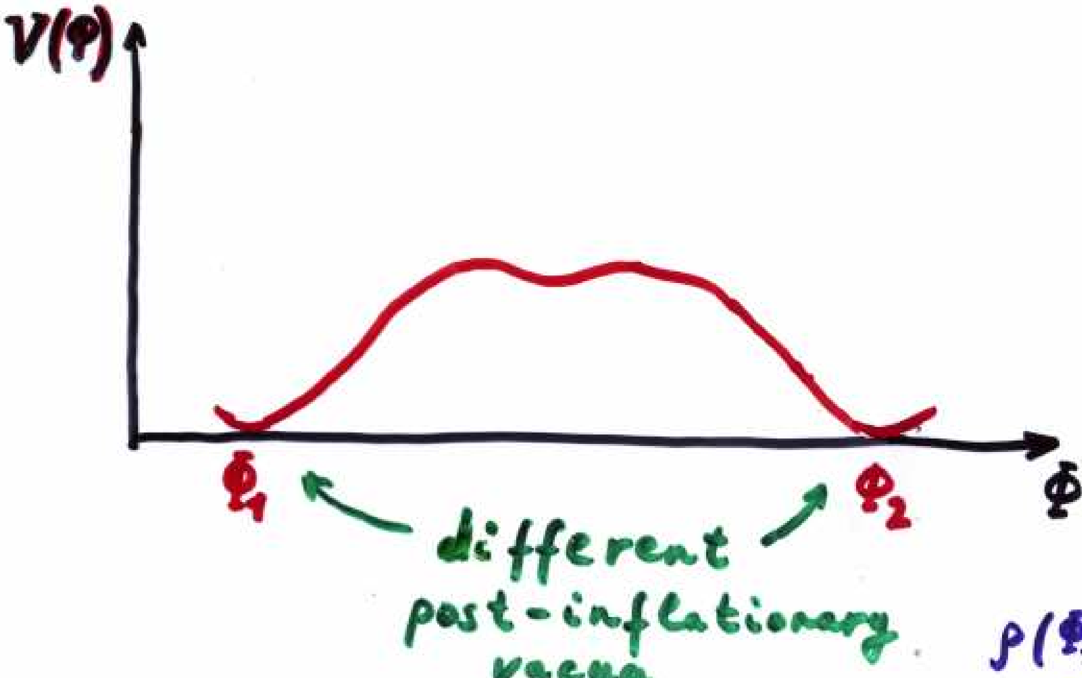
3. Backreaction is taken into account

   $$\delta T_\mu^\nu = \delta_\mu^\nu (V - V_{cl})$$

From $\rho(\varphi,\tau)$ during inflation to the distribution $w(\tau)$ over total local duration of inflation.

$$w(\tau) = \lim_{\varphi\to\varphi_{end}} j = \lim_{\varphi\to\varphi_{end}} \frac{|V'|}{3H^{n+1}}\rho(\varphi,\tau)$$

For a smooth transition to a post-inflationary epoch, the stochastic force must be much less than the classical one during last e-folds of inflation



$\rho(\Phi_1,\tau)=0$
$\rho(\Phi_2,\tau)=0$

Let $$Q_m(\Phi) = \int_{\tau_0}^{\infty}(\tau-\tau_0)^m \rho(\Phi,\tau)\,d\tau$$

$$Q_m(\Phi_1) = Q_m(\Phi_2) = 0$$

$\boxed{n=0}$

$$Q_0(\Phi) = \frac{8\pi^2}{H^{3-n}} \exp\left(\frac{\pi}{GH^2}\right) \int_{\varphi_1}^{\varphi} d\tilde{\varphi} \exp\left(-\frac{\pi}{G\tilde{H}^2}\right) \cdot$$

$$\cdot \left(C - \int_{\varphi_1}^{\tilde{\varphi}} p_0(\tilde{\tilde{\varphi}})\, d\tilde{\tilde{\varphi}} \right)$$

$$C = \frac{\int_{\varphi_1}^{\varphi_2} d\varphi \cdot \exp\left(-\frac{\pi}{GH^2}\right) \int_{\varphi_1}^{\varphi} p_0(\tilde{\varphi})\, d\tilde{\varphi}}{\int_{\varphi_1}^{\varphi_2} d\varphi \cdot \exp\left(-\frac{\pi}{GH^2}\right)}$$

$P_1 = C \quad , \quad P_2 = 1 - C$

probabilities of post-inflationary vacua

Do not depend on "n"! (branching ratios)

$\boxed{n=1}$

$$Q_1(\Phi) = \frac{8\pi^2}{H^{3-n}} \exp\left(\frac{\pi}{GH^2}\right) \int_{\varphi_1}^{\varphi} d\tilde{\varphi} \cdot \exp\left(-\frac{\pi}{G\tilde{H}^2}\right) \cdot$$

$$\cdot \left(C_1 - \int_{\varphi_1}^{\varphi} Q_0(\tilde{\varphi})\, d\tilde{\tilde{\varphi}} \right)$$

$$C_1 = \frac{\int_{\varphi_1}^{\varphi_2} d\varphi \cdot \exp\left(-\frac{\pi}{GH^2}\right) \int_{\varphi_1}^{\varphi} Q_0(\tilde{\varphi})\, d\tilde{\varphi}}{\int_{\varphi_1}^{\varphi_2} d\varphi \cdot \exp\left(-\frac{\pi}{GH^2}\right)}$$

$\langle \tau_1 \rangle = \frac{C_1}{C} \quad , \quad \langle \tau_2 \rangle = \frac{\tilde{C}_1}{1-C}$

$$\langle \tau \rangle_{tot} = C \langle \tau_1 \rangle + (1-C) \langle \tau_2 \rangle = \underline{\int_{\varphi_1}^{\varphi_2} Q_0(\Phi)\, d\Phi}$$

The main problem: choice of the initial condition $\rho_0(\varphi)$ at the local beginning of inflation.

1. Static solutions $\rightarrow$ non-normalizable
2. $\rho_0(\varphi) = \delta(\varphi - \varphi_0)$. why ?
3. "Eternal inflation as the initial condition": $\rho_0(\Phi) = \rho_{E_1}(\varphi)$ $(E_0 = 0)$

Not possible for the continuum spectrum case.
In the discrete spectrum case, generically $E_2 - E_1 \sim E_1 \rightarrow$ not enough time for relaxation.

'Eternal' inflation is not eternal enough to fix the initial condition uniquely.

Two different types of inflationary models

1. Exponentially decaying

$\int \frac{d\Phi}{H(\Phi)}$ converges

Discrete spectrum of eigenvalues

$\sqrt{-g} \propto a^{3-\lambda_1}$

$\langle \ln \frac{a_f}{a_0} \rangle$ finite

$\rho = \rho_0(\Phi) \cdot \left(\frac{a}{a_0}\right)^{-\lambda_1}$, $\lambda_1 \ll 1$ for $a \to \infty$

$w(a_f) \propto \left(\frac{a_f}{a_0}\right)^{-\lambda_1}$

$\boxed{\text{Quasi-stationary regime}}$

Examples: a) new inflation

b) chaotic inflation $V(\Phi) \propto \Phi^n$, $n > 2$

2. Non-exponentially decaying

$\int \frac{d\Phi}{H(\Phi)}$ diverges

Continuous spectrum of positive eigenvalues

$w \propto (\ln a_f)^{-3/2}$

$\langle \ln \frac{a_f}{a_0} \rangle$ infinite

$\sqrt{-g} \propto a^3/(\ln a)^\gamma$

Examples: a) chaotic inflation $V(\Phi) \propto \Phi^n$, $n \leq 2$

b) $R + R^2$ model

c) $V = V_0 - C\varphi^{-n}$

In both cases: $\langle \frac{a_f}{a_0} \rangle$ infinite

CONCLUSIONS REGARDING BEGINNING OF INFLATION

1. No problems of principle in predicting probability distributions during and after inflation in the original (probability conserving) stochastic approach to inflation, once the initial condition $\rho_0(\varphi)$ is given. No necessity to refer to other universes (they exist but outside our past light cone)
2. No satisfactory principle to fix $\rho_0(\varphi)$ uniquely.
3. Some dependence on $\rho_0(\varphi)$ remains in final answers – a possibility to get some knowledge on it from observations does not seem hopeless.

However, for almost all $\rho_0(\varphi)$ apart from $\rho_0(\varphi) \propto \exp\left(\frac{\pi}{GH^2(\varphi)}\right)$, the main contribution is from the maximum of $V(\varphi)$

No necessity in the tunneling $\rho_0(\varphi)$ specifically

CONCLUSIONS

1. In the zero order, inflationary predictions for scalar perturbations have been confirmed. Slow-roll approximation works.

2. In the next order, corrections remain small but the validity of the slow-roll approximation is not established.

3. New observational data with the accuracy $\lesssim 1\%$ are required.

   Expected discoveries (in order of decreasing probability).

   1. $n_s(k) - 1$ → guaranteed !
   2. r, GW
   3. local features, phase transitions
   4. some completely new physics

4. Going back into the past history of the Universe ($N > 60$, $N \gg 1$ and even before local beginning of inflation) is possible but requires measurement of some new 'light' scalars.