PRIORS

G W Gibbons

DAMTP, Cambridge

Workshop Talk

The Very Early Universe 25 Years On

The title of this talk is meant to do double duty. Paul Shellard has asked for an historical introduction followed by the main substance of the talk.

Accordingly I will review some previous work on Cosmological Event horizons and the No Hair property of De-Sitter Spacetime and its significance for inflationary theories.

I will then go on to describe some current and ongoing work on the construction and use of *a priori* probabilities in fundamental physics and cosmology. As befits a workshop, this latter is a far from completed task.

Stephen Hawking's 1974 derivation * of black hole radiance for a free field theory on a fixed curved background using an ingeneous approximation for the relevant Bogoliubov coefficients raised many questions, among them

- What happens if interacting fields are considered? .
- What happens if the metric undergoes quantum fluctuations? .
- Is the effect limited to black holes or is it more general feature of gravitational fields?

*Black hole explosions. S.W. Hawking, *Nature* **248** 30-31,1974. Particle Creation by Black Holes. S.W. Hawking Commun.Math.Phys.43:199-220,1975, Erratum-ibid.46:206-206,1976.

An important insight was James Hartle and Stephen Hawking's (1975) * derivation of the Einstein A and B relations for the thermal emission of non-self-interactacting particles by black holes using a path integral method.

A key step in the derivation was the use of the periodicity in imaginary time of the Schwarzschild metric.

*Path Integral Derivation of Black Hole Radiance. J.B. Hartle, S.W. Hawking *Phys.Rev* **D13** 2188-2203,1976

This quickly lead to the realisation by myself and Malcom Perry * that finite temperature ‡ quantum field theoretic techniques could be used to show that the relation

$$\boxed{T = \frac{\kappa}{2\pi}} \tag{1}$$

is truly universal, continues to hold in the presence of interactions, and remains true even if gravity is quantised.

*Black Holes in Thermal Equilibrium. G.W. Gibbons, M.J. Perry, † **36** 985,1976; Black Holes and Thermal Green's Functions. G.W. Gibbons, M.J. Perry . *Proc.Roy.Soc.Lond* **.A358** 467-494,1978.

‡i.e. non-zero temperature

The first two questions having been answered, I began thinking about the third. My reading of Schrödinger's book *Expanding Universes*, an article in *Nature* * about the historical confusion about the redshift effect in de-Sitter spacetime, and Brandon Carter's Les Houches lectures lead to the realisation to the idea that

de-Sitter spacetime is just like a black hole turned inside out

*Carla and Franz Kahn, Letters from Einstein to de Sitter on the nature of the Universe *Nature* **257** (1975) 451-454

Unknown to me Bertand Russell had got there first *

> De Sitter's world is even odder than Einstein's, because time goes mad as well as space. I despair of explaining, in nonmathematical language, the particular form of lunacy with which time is afflicted, but some of its manifestations can be described. An observer in De Sitter's world, if he observes a number of clocks, each of which is perfectly accurate from its own point of view, will think that distant clocks are going slow as compared with those in his neighbourhood. They will seem to go slower and slower, until, at a distance of one quarter of the circumference of the universe, they will seem to have

*B. Russell, *The ABC of Relativity* London, Kegan Paul (1925) stopped altogether. That region will seem to our observer a sort of lotus-land, where nothing is ever done. He will not be able to have any cognizance of things further off, because no light-waves can get across the boundary. Not that there is any real boundary: the people who live in what our observer takes to be lotus-land live just as bustling a life as he does, but get the impression that he is eternally standing still. As a matter of fact, you would never become aware of the lotus-land, because it would take an infinite time for light to travel from it to you. You could become aware of places just short of it, but it would remain itself always just beyond your ken. There will not be the ghostly sums of Einstein's world, because light cannot travel so far.

One of the oddest things about this state of affairs is that empirical evidence for or against it is possible, and that there is actually some slight evidence in its favour. If all ‘clocks ’are slowed down at a great distance from the observer, this will apply to the periodic motions of atoms, and therefore to the light which they emit. Consequently all rays of light emitted by distant objects ought, when they reach us, to look rather more red or less violet than when they started. This can be tested by the spectroscope. We can compare a known line, as it appears in the spectrum of a spiral nebula, with the same line as it appears in a terrestrial laboratory. We find, as a matter of fact, that in a large majority of spiral nebulae there is a considerable displacement of spectral lines towards the red. The spiral nebulae are the most distant objects we can see: Eddington states that their distances” may perhaps be of the order of a million Iightyears.” (A light-year is the distance light travels in a year.) The usual interpretation of a shifting of spectral lines towards the red is that it is a "Doppler effect," due to the fact that the source of light is moving away from us. But one would expect to find the nebulae just as often moving towards us as moving away from us, if nothing operated but the law of chances. If the world is such as De Sitter says it is, the spectral lines of the spiral nebulae will be displaced towards the red owing to the slowing down of distant clocks, even if in fact they are not moving away from us. This, for what it is worth, is an argument in favour of De Sitter.

The same facts afford another argument in favour of De Sitter, for another reason. If, at a given moment, a body is at rest relatively to the observer, and at a distance from him, it will (in the absence of counteracting causes) not remain at rest from his point of view, but will begin to move away from him, and will continue to move away faster and faster; the further it is from him, the more its retreat will be accelerated. For bodies which are not too distant from each other, gravitation may overcome this tendency; but as this tendency increases with the distance, while gravitation diminishes, we should expect to find very distant bodies receding from us if De Sitter's theory is right.

This insight proved a rather fruitful source of correspondences and lead Stephen Hawking and myself* to the result that the cosmological event horizon in de-Sitter spacetime also radiates thermally because the surface gravity

$$\boxed{\kappa = \sqrt{\frac{\Lambda}{3}}} \tag{2}$$

assuming quantum fields are in the De-Sitter invariant state which is well defined on S^4 †

*Cosmological Event Horizons, Thermodynamics, and Particle Creation. G.W. Gibbons, S.W. Hawking *Phys.Rev* **D15** :2738-2751,1977.

†now often called the Bunch-Davies vaccum state, Quantum Field Theory in De Sitter Space: Renormalization by Point-Splitting T. S. Bunch; P. C. W. Davies *Proceedings of the Royal Society of London* **A 360**(1978), 117-134

We also pointed out that the cosmological horizon should exhibit a no hair property similar to that of black holes: all perturbations would escape through the event horizon and the interior rapidly settle dowm to a de-Sitter invariant state.

John Barrow pointed out , some years later that this suggestion had to some extent been anticipated by Hermann Weyl * and also by Fred Hoyle and Jayant Narlikar †.

However the latter has little relation to the physics of event horizons, and was in the context of their own non-standard theory of gravity. The former is more interesting if somewhat abreviated.

*In the 1949 translation of *The Philosophy Mathematics and Natural Science*

†Mach's Principle and the Creation of Matter F. Hoyle; J. V. Narlikar *Proceedings of the Royal Society of London* **A 273**, (1963), 1-11

Incidentally the behavior of every world satisfying certain natural homogeneity conditions in the large (whether it is void or carries mass) follows this model asymptotically when, in the process of expansion, the world radius becomes essentially larger than a.

Of course there had been earlier and almost contemporary papers on quantum field theory in de-Sitter background two of which we cited *. The latter is the more interesting. However, being a two dimensional model, they were not able to relate their work to the properies of black holes or horizons.

*Consequences of field quantization in de Sitter type cosmological models. E.A. Tagirov . *Annals Phys* **76** (1973) 561-579 ; Interacting Relativistic Boson Fields in the de Sitter Universe with Two Space-Time Dimensions. R. Figari, R. Hoegh-Krohn, C.R. Nappi .. *Commun.Math.Phys* **44** 265-278,1975.

Other papers had been inspired by Phillips and Wigner's observation * (not known to me at the time) that the De-Sitter group $SO(41)$(unlike the Anti-de-Sitter group $SO(3,2)$) has no positive energy generators.

There is in fact quite a close conection between this fact, the existence of a Killing horizon and the thermal radiance of cosmological horizons †

*T. Philips and E. P. Wigner, Group Theory and its Applications, pp. 631-676. Academic Press, New York, 1968.

†A Group theoretical approach to causal structures and positive energy on space-times. Christophe Patricot [hep-th/0403040]

In the light of these, and future developments * it is ironic that de-Sitter introduced his four-dimensional No-boundary universe in response to Einstein's ill-fated three- dimensional no-boundary universe †.The fact that De-Sitter spacetime is the analytic continuation of S^4 lead him to believe that it had no boundary in the four-dimensional sense.

In fact one knows from the Gauss-Bonnet theorem that one cannot quotient de-Sitter by a discrete subgroup $\Gamma \subset SO(4,1)$ so that the quotient M/Γ is smooth, compact and without boundary ‡ .

*e.g. Hartle and Hawking's One-Boundary proposal

†

W de-Sitter, On the relativity of inertia. Remarks concerning Einstein's latest hypothesis *Koninklijke Academie van Wetenshappen te Amsterdam* **29** (1917)1217-1225

‡Calabi, E.; Markus, L. Relativistic space forms. *Ann. of Math* . **75** (1962) 63–76.

Although these theoretical results were of great conceptual interest they seemed at that time to have no immediate and direct cosmological application because any large scale cosmological constant had to be small, and if one extrapolated back in time it would the density of what is now called dark energy would have been tiny in comparison with the radiation density.

The possiblity of a first order phase transition of course invalidates that argument and at the previous Very Early Universe meeting I revisited the No-Hair property of De-Sitter spacetime with my then Research Student Wayne Boucher. For us, an important realisation was that while from a global point of view, perturbations are frozen in and do not decrease, from the local perspective of the interior of the cosmological horion, they decay exponentially fast.

Our work was greatly helped by conversations at the meeting with Alexei Starobinksy, who had worked on this problem earlier from a different point of view.

The problem remains of interest and there are interesting links to holography *

*see The Geometry of Large Causal Diamonds and the No Hair Property of Asymptotically de-Sitter Spacetimes. G.W. Gibbons, S.N. Solodukhin *Phys.Lett.***B** 652:103-110,2007. arXiv:0706.0603 [hep-th]

As well as the decay results, Boucher and I also conjectured some Uniqueness Results for De-Sitter spacetime analogous to Israel's Theorem. There are such results in four-spacetime dimensions but the uniqueness fails in higher dimensions, because Böhm's proof of the existence of infinitely many Riemannian Einstein metrics on spheres and products of spheres which admit hypersurface orthogonal Killing vectors, and thus analytically continue to give static spacetimes * .

*see Bohm and Einstein-Sasaki metrics, black holes and cosmological event horizons. G.W. Gibbons, Sean A. Hartnoll , C.N. Pope *Phys.Rev.***D67** :084024,2003. : hep-th/0208031

At the time of the Very Universe Meeting I was working on supergravity. I realised that (quite independently of the isssue of supersymmetry), simple supergravity theories do not admit de-Sitter type vacua or compactifications and eventually I produced a rather general proof *

Of course, as I suppose many people realised at the time, there is no problem if one couples N=1 supermatter to N=1 SUGRA, as is now rather fashionable.

*Aspects Of Supergravity Theories. G.W. Gibbons (Cambridge U.) . Print-85-0061 (CAMBRIDGE), Jun 1984. 53pp. Three lectures given at GIFT Seminar on Theoretical Physics, San Feliu de Guixols, Spain, Jun 4-11, 1984. Published in GIFT Seminar 1984:0123 (QCD161:G2:1984)

A major result, accepted by most workers by the end of the the Very Early Universe meeting, was that if inflation take place, it probably wasn't via a first order phase transition. Later, Belinksi and Khalatnikov on a visit to Cambridge raised the question, in the context of a mini-superspace model of slow-roll inflation with a massive scalar field:

How typical is inflation?

A formulation and answer to this question in precise terms, using the natural canonical measure was given by myself, Stephen Hawking and John Stewart *

*A Natural Measure On The Set Of All Universes. G.W. Gibbons, S.W. Hawking, J.M. Stewart . *Nucl.Phys* **B 281** :736,1987.

Leo Grishchuk posed an analogous question

What is a typical wave function for the universe

The answer is well defined and meaningful for a finite dimensional Hilbert space *.

It looks rather elusive if $\dim H_{\text{qm}} = \infty$.

*What Is A Typical Wave Function For The Universe? G.W. Gibbons, L.P. Grishchuk (Moscow State U.) *Nucl.Phys.* **B313** 736-748,(1989); Typical states and density matrices. G.W. Gibbons. *J.Geom.Phys* **8** :147-162,(1992) In an almost diametrically opposed approximation one may ask

what time-symmetric initial data for a closed universe

with radiation and a cosmological constant has largest entropy?

The answer is †

The Einstein Static Universe is a local maximum

There is no global maximum

†The Entropy And Stability Of The Universe. G.W. Gibbons *Nucl.Phys* **B292:**784,1987, Sobolev's Inequality, Jensen's Theorem And The Mass And Entropy Of The Universe. G.W. Gibbons *Nucl.Phys* **B 310** 636,1988.

This resolves a dispute between Stephen Hawking and Roger Penrose and which may be relevant for "emergent universe models". [‡] .

[‡] On the stability of the Einstein static universe. John D. Barrow, George F.R. Ellis, Roy Maartens, Christos G. Tsagas *Class.Quant.Grav.* **20** L155-L164,(2003) . e-Print: gr-qc/0302094

All past is prologue

Some of the previous considerations, were an attempt to make explicit and precise what was previouly and implicit and imprecise appeal to notions of probability, typicality, likelihod, fine-tuning etc. Currenty there appears to be little agreement about what these words actually mean and how they should be used. Very much the same holds for similar notions in elementary particle physics and String theory.

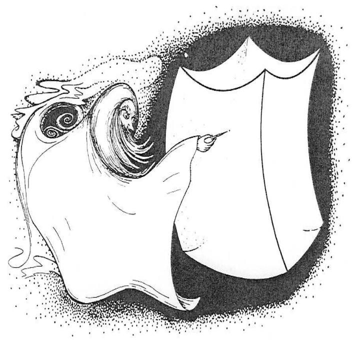

FIGURE 10. The Creator locating the tiny region of phase-space—one part in $10^{10^{123}}$—needed to produce a 10^{80}-baryon closed universe with a second law of thermodynamics in the form we know it.

An important consistency principle in this regard is due

to Humpty Dumpty

‘I don’t know what you mean by ”glory” ’, Alice said.

Humpty Dumpty smiled contemptuously. ‘Of course you don’t – till I tell you. I meant ”there’s a nice knock-down argument for you!”’

‘But ”glory” doesn’t mean ”a nice knock-down argument”,’Alice objected.

‘When I use a word,’Humpty Dumpty said, in rather a scornful tone, ‘it means just what I choose it to mean – neither more nor less.’

Take for example the words

Multiverse

and

Meta-Universe

They are frequently used in different ways sometimes by the same person on the same page of their book. An elision reminiscent of Beethoven at his finest.

Following Humpty Dumpty's lead, I will define **The Multiverse**

as the abstract and timeless set of all possible universes, i.e of all connected spacetimes satisfying the Einstein Equations.

This space, $M_{\text{Multiverse}}$ at least in mini-superspace examples, is even dimensional dim$M_{\text{Multiverse}} = 2n - 2$ and carries, by virtue of it being a reduced phase space, a natural symplectic structure, i.e a closed 2-form ω and hence measure

$$\frac{1}{(n-1)!}(-1)^{\frac{1}{2}(n-1)(n-2)}\omega^{n-1} \tag{3}$$

This is the measure originally advocated by myself , Stephen Hawking and John Stewart and revisited recently by Neil Turok and myself *

*The Measure Problem in Cosmology. G.W. Gibbons, Neil Turok *Phys Rev* in press. .e-Print: hep-th/0609095

The main difficulty is that the total measure of $M_{\mathsf{Multiverse}}$ diverges

$$\int_{M_{\mathsf{Multiverse}}} \frac{1}{(n-1)!}(-1)^{\frac{1}{2}(n-1)(n-2)}\omega^{n-1} = \infty \qquad (4)$$

and Probabilities cannot be normalised. Neil and I have made a suggestion for solving this problem. If one accepts our suggestion, then the set of classical histories which inflate is exponentionally small: $\propto e^{-3N}$.

I will turn later to a possible interpretation of this statement.

The Meta-Universe

As introduced by Alex Vilenkin *

> The world view suggested by quantum cosmology is that inflating universes with all possible values we are a "typical" civilization living in this metauniverse

this is a single connected 4-dimensional spacetime $M_{\text{Meta}-\text{Universe}}$ possibly containing many causally disjoint regions. Points of $M_{\text{Meta}-\text{Universe}}$ are called spacetime events and the probability of a set $U \subset M_{\text{Meta}-\text{Universe}}$

*Predictions from Quantum Cosmology Alexander Vilenkin *Phys Rev Lett* **74** (1995) 846 of such events, a pocket universe or a causal diamond is taken to be proportional to their spacetime volume †

$$\int_U \sqrt{g|} d^4 x \tag{5}$$

Again the main problem is one of normalizability,

$$\int_{M_{\mathrm{Meta-Universe}}} \sqrt{g|} d^4 x = \infty \,. \tag{6}$$

As Wittgenstein might have said ‡ Wovon man nicht rechnen kann, darüber muß man schweigen

†The Geometry of Large Causal Diamonds and the No Hair Property of Asymptotically de-Sitter Spacetimes. G.W. Gibbons, S.N. Solodukhin *Phys.Lett* **B 652** :103-110(2007). arXiv:0706.0603 [hep-th] ; The Geometry of small causal diamonds. G.W. Gibbons, S.N. Solodukhin *Phys.Lett***B** 649:317-324,2007

‡L Wittgentstein, *Tractatus Logico-Philosophicus*

Why should one want a probablity theory for cosmology in the first place?

One motivation is to apply Bayesian Reasoning to Cosmological Observations.

That is the viewpoint I will take from now on.

A priori probability distributions , "Priors" are integral part of the scientific method.

They are needed to to assess the probability that a hypothesis is true given a set of observations or measurements.

They are also required to assess the reliablity and information content of predictions since if a large measure of hypotheses give the same observations, then those observations don't tell us much.

Conversely if only small measure of hypotheses predict a set of well verified observations, one may have high confidence in those hypotheses.

However much confusion can result if different people use different priors, or more misleadingly, don't clearly articulate the that they priors they are using.

This problem is particularly prevalent in cosmology where there is at present no consensus on a suitable *a priori* measure on initial conditions.

In choosing *a priori* probabilities, it seems safest to assume as little as possible, consistent with making any progress. Indeed progress consists of using the results of observations and experiments to refine and update what are initially very flat *a priori* distributions so that they peak at some approximation for the real world.

In M/String theory all of physics is subsumed into history. Thus all coupling constants, and low energy laws of physics are determined by initial conditions. Issues of naturalness reduce to the naturalness of initial conditions.

Thus if M/String theory is to qualify as a science it requires convincing *a priori* measures.

One encouraging feature of String theory is that the total volume of the various spaces of the moduli , which determine coupling constants etc with respect to natural metrics and measures are are of finite volume.

Thus in principle, the concept of fine tuning is meaningful in String theory.

More formally if

$P(O|U)$ is the probability of making an observation O in universe U, (Likelihood)

$P(U|O)$ is the probability we are in universe U having made an observation O (*A posteriori probability*)

$P(O)$ is the probability of making an observation O in *any* universe

$P(U)$ is the probability that the universe is U (*A priori probability*)

$P(O \cap U)$ is the probability of making an observation O and the universe is actually U

Then ....

$$P(U|O)P(O) = P(O \cap U) = P(O|U)P(U) \tag{7}$$

whence the Cosmic Bayes's Theorem tells us that

$$P(U|O) = \frac{P(O|U)P(U)}{\int_M P(O|U)P(U)dU} \tag{8}$$

where

the integral is over the Multiverse and dU is a measure on the multiverse.

Thomas Bayes

quòdque ſolum, certa nitri ſigna præbere, ſed plura concurrere debere, ut de vero nitro producto dubium non relinquatur.

LII. *An Eſſay towards ſolving a Problem in the Doctrine of Chances. By the late Rev. Mr.* Bayes, *F. R. S. communicated by Mr.* Price, *in a Letter to* John Canton, *A. M. F. R. S.*

Dear Sir,

Read Dec. 23, 1763.

I Now ſend you an eſſay which I have found among the papers of our deceaſed friend Mr. Bayes, and which, in my opinion, has great merit, and well deſerves to be preſerved. Experimental philoſophy, you will find, is nearly intereſted in the ſubject of it; and on this account there ſeems to be particular reaſon for thinking that a communication of it to the Royal Society cannot be improper.

He had, you know, the honour of being a member of that illuſtrious Society, and was much eſteemed by many in it as a very able mathematician. In an introduction which he has writ to this Eſſay, he ſays, that his deſign at firſt in thinking on the ſubject of it was, to find out a method by which we might judge concerning the probability that an event has to happen, in given circumſtances, upon ſuppoſition that we know nothing concerning it but that, under the ſame circum-

Philosophical Transactions of the Royal Society **53** (1763) 269-271

If we adopt Laplaces's Principle of Indifference then $P(U)$ is independent of U.

Any other choice of $P(U)$ is a proposal for the state of the the universe

We could even define Kolmogorov's measure of the information content of a proposal $P(U)$ via

$$\int P(U) \ln P(U) dU \tag{9}$$

which should be least for Laplace's Proposal

But what is dU?

Within the limitations of mini-superspace models, and modulo the issue of normaliabilty we have a complete solution. We reduce the model to a Hamiltonian system constrained to have vanishing Hamiltonian. It then follows that the Poincaré invariant $d^{n-1}pd^{n-1}q$ restricted to the multiverse of classical histories dose indeeed depend only on the history and not how it is described.

It is important to realise that the measure on the multiverse so defined carries no information about the direction of time nor any preferred instant of time. Many discussions of the plausibility or otherwise of certain initial conditions make explicit, or more dangerously implicit, assumptions about either or both.

In fact the method works not only for gravity plus scalar fields but one may also consider the addition of fluids as well.

Extension to Perfect Fluids (wk.in progress with T Damour and N Turok)

$$X = -\nabla\psi^2. \tag{10}$$

Consider a Lagrangian$L = L(X)$. The energy momentum is that of a an irrotational perfect fluid

$$T_{\mu\nu} = (\rho + P)U^{\mu}U^{\nu} + P g_{\mu\nu} \tag{11}$$

$$U_{\mu} = \frac{\partial_{\mu}\psi}{|\partial\psi|}, \qquad P = L\,, \qquad \rho = 2XL_X - L\,. \tag{12}$$

e.g. radiation * $L = X^4 = (\nabla\psi)^4$

Constructing the Hamiltonian is now straightforward: the measure is related to the conserved entropy of the fluid!

*conformally invariant theory

In fact the entropy current is a Noether current for shift symmetry $\psi \to \psi + \text{constant}$

If one breaks the shift symmetry by coupling to the inflaton ϕ one obtains a Hamiltonian treatment of a dissipative fluid which provides a toy model describing the generation of entropy at the end of inflation.

Wigner Distribution Functions (wk.in progress with Hartle, Hertog and Turok) If inflation is no a priori probable we may need to alter our prior. This can be done by introducing a Wave function for the Universe $\Psi = \Psi(q^i) = \Psi(a = e^\lambda, \phi)$ satisfying (at least in some formal sense the Wheeler-De-Witt equation

$$H\Psi(a = e^\lambda, \phi) = 0 \,. \tag{13}$$

To obtain a probability distribution $P = P(U) = P(p, q)$ we need to calculate the Wigner distribution by

$$W(p_j, q^i) = \frac{1}{\pi^2} \int_{\mathbb{R}^n} d^2y \overline{\psi(q^i + y^i)} \psi(q^i - y^i) \exp\left(2iy^j p_j\right) . \tag{14}$$

This makes sense because e^λ, ϕ are naturally (flat) Cartesian coordinates on the configuration space (Wheeler's superspace).

Natural measures or, prors on coupling constants are essential for any rational discussion of fine tuning and anthropic arguments. In some cases this is unproblematic. For example in anthropic arguments based on the axion * one needs a measure on the phase $\equiv S^1 \equiv U(1)$. In other cases, such as Yang-Mill's couplings, mass matrices or CP violating parameters this is not so obvious. As mentioned above, String theory seems to improve things.

*Dimensionless constants, cosmology and other dark matters. Max Tegmark, Anthony Aguirre, Martin Rees, Frank Wilczek *Phys.Rev* **D73** 023505,2006. arXIv: [astro-ph/0511774]

Electric-Magnetic duality

In string theory one typically couples an axion and dilaton to a $U(1)$ field and maintain invariance of the equations of motion, but not the action, under $SL(2,R)$ classically, or a $SL(2,Z)$ quantum mechanically, if one takes the Lagrangian function

$$-\frac{1}{4}(\tau_2 F_{\mu\nu}F^{\mu\nu} - \tau_1 F_{\mu\nu} \star F^{\mu\nu})\,. \tag{15}$$

The electromagnetic field transform as

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} \to S \begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix}\,. \tag{16}$$

The relation to the conventional coupling constants g a theta angle θ is

$$\tau = \tau_1 + i\tau_2 = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}\,. \tag{17}$$

There is a natural metric on coupling constant space

$$ds^2 = \frac{1}{\tau_2}|d\tau|^2 \,, \tag{18}$$

S-duality acts as

$$\tau \to \frac{-c + a\tau}{d + -b\tau} \,. \tag{19}$$

To maintain the Dirac quantisation condition one needs $a, b, c, d \in Z$. Moreover two theories related by an $SL(2, Z)$ transformation are physically indistinguishable.

Normally these coupling constants span the entire upper half plane but in the present case they are restricted to the double coset, i.e. $D = SL(2,Z)\backslash SL(2,R)/SO(2)$, (also called the fundamental domain of the modular group), which is non-compact but nevertheless has finite area. A representative domain is $\tau_2 > 0$, $|\tau_1| < \frac{1}{2}$ and $|\tau| > 1$. There are two orbifold points, one at $\tau = i$, with deficit angle π and one at $\frac{1}{2} + i \pm \frac{\sqrt{3}}{2}$ with deficit $\frac{4\pi}{3}$.

The total area is finite

$$\int\int_D \frac{d\tau_1 d\tau_2}{\tau_2^2} = \frac{\pi}{3}. \qquad (20)$$

The calculation above may be understood geometrically and illustrates how compatifications in string theory gives rise to normalisable measures on almost all but one "modulus".

A unimodular metric on the 2-torus may be written as

$$ds^2 = \frac{1}{\tau_2}|dy^1 + \tau dy^2|^2 \,, \qquad (21)$$

with

$$\tau = \tau_1 + i\tau_2 \,. \qquad (22)$$

The De-Witte metric on the space of such metrics is

$$\mathrm{Tr} g^{-1} dg g^{-1} dg = 2 \frac{d\tau d\bar{\tau}}{\tau_2^2} \qquad (23)$$

Under

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \tag{24}$$

with $ab - cd = 1$, one finds that

$$\tau \to \frac{b + d\tau}{a + c\tau}. \tag{25}$$

Thus to maintain invariance one needs to compensate by the inverse

$$\tau \to \frac{-c + a\tau}{d + -b\tau}\,. \tag{26}$$

To maintain the torus periods one needs $a, b, c, d \in Z$.

The space of moduli is thus of finite total measure.

Deep theorems in geometry appear to show that the same is true for the space of Calabi-Yau's.

*

*Weil-Petersson volumes of the moduli spaces of CY manifolds. Andrey Todorov hep-th/0408033, Finiteness of volume of moduli spaces. Michael R. Douglas Zhiqin Lu hep-th/0509224 , On the Weil-Petersson volume and the first Chern class of the moduli space of Calabi-Yau manifolds. Zhiqin Lu, Xiaofeng Sun Commun.Math.Phys.261:297-322,2006. math/0510021 On the geometry of moduli space of polarized Calabi-Yau manifolds. Michael Douglas Zhiqin Lu . Submitted to Publ.Res.Inst.Ma math/0603414

The Kobayashi-Maskawa Matrix (wk with Ben Allanach and N Turok)

In the standard theory of CP violation the quark mass eigenstates are related by an $SU(3)$ matrix

$$U = G_1 C G_2 \,, \tag{27}$$

The $U(1)^2$ matrices G_1 and G_2 correspond the the freedom to phase the quarks , and are given by

$$G_1 = \begin{pmatrix} \exp \frac{2ip}{\sqrt{3}} & 0 & 0 \\ 0 & \exp i(-\frac{p}{\sqrt{3}} + q) & 0 \\ 0 & 0 & \exp -i(\frac{p}{\sqrt{3}} + q) \end{pmatrix} \tag{28}$$

$$G_2 = \begin{pmatrix} \exp i\frac{(r}{\sqrt{3}} + t) & 0 & 0 \\ 0 & \exp i(\frac{r}{\sqrt{3}} - t) & 0 \\ 0 & 0 & \exp -i\frac{2r}{\sqrt{3}} \end{pmatrix}, \tag{29}$$

The Kobayashi-Maskawa matrix C is an element of $SU(3)$, or strictly speaking the double coset $U(1)^2\backslash SU(3)/U(1)^2$.

Explicity

$$\begin{pmatrix} cycz & cysz & sy \exp -iw \\ -cxsz - sxsycz \exp iw & cxcz - sxsysz \exp iw & sxcy \\ sxsz - cxsycz \exp iw & -sxcz - cxsysz \exp iw & cxcy \end{pmatrix} \quad (30)$$

Thus p, q, r, t, x, y, z, w are coordinates on the group $SU(3)$ and the measurable physical coupling constants entering the Kobayashi-Maskawa matrix x, y, z, w may be regarded as coordinates on the double coset $U(1)^2 \backslash SU(3)/U(1)^2$. If $w \neq 0$, then the Lagrangian is CP violating.

There is a natural bi-invariant metric on $SU(3)$ given by

$$ds^2 = -\mathrm{Tr}\, U^{-1}dU\, U^{-1}dU \quad (31)$$

$$= \mathrm{Tr}\, dU\, dU^\dagger \quad (32)$$

$$= dU_{jk}\, d\bar{U}_{jk}\,. \quad (33)$$

The restriction to the Kobayashi-Maskawa matric is, according to Ozsvath and Schucking *

$$ds^2 = 2\{dx^2 + dy^2 + dz^2 + 2\sin y \cos w dx dz + \sin^2 y dw^2\}\,. \qquad (34)$$

Interestingly, this restricted metric is claimed to be in fact invariant under an action of $U(1)^3$. It is claimed by Ozsvath and Schucking that suitable coordinate change exists such that

$$ds^2 = 2\{du^2 + dx^2 + dz^2 + 2\cos u dx dz + \sin^2 u dv^2\} \qquad (35)$$

*I Ozsvath, *Working with Englebert* in *On Einstein's Path: Essays in Honour of Engelbert Schucking* ed A Harvey, Springer, New York (1996) 339-351

The induced normalizable but unnormalised measure coming from (34) is rather non-obvious:

$$\sqrt{g}\, dxdydzdw = \sin y(1 - \sin^2 y \cos^2 w)^{\frac{1}{2}} dxdydzdw\,. \qquad (36)$$

Translation into Particle Data Book Notation

The matrix C is usually called V with elements

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{37}$$

The angles are defined as

$$z = \theta_{12} \qquad x = \theta_{23} \qquad y = \theta_{13}\,, \qquad w = \delta\,. \tag{38}$$

In this notation the invariant introduced by Jarlskog is

$$J = c_{12} c_{23} {c_{13}}^2 s_{12} s_{23} s_{13} \sin\delta\,. \tag{39}$$

Given this probablity distribution, we can now talk about the statistical properties of J , in other words how "natural" is CP violation and presumably begin to answer the question

How likely is baryosynthesis and hence our existence.?

Mass matrices The natural (positive definite or Riemannian) metric on symmetric, or more generally Hermitian $n \times n$ matrices is

$$ds^2 = \mathrm{Tr}\, dM dM^\dagger . \tag{40}$$

and is invariant under conjugation by $U(n)$.

This is the standard metric used in Wigner's random matrix theory, an atempt to explain nuclear levels in tems of a randomly chosen Hamiltonian. Of course to normalise the induced Riemannian measure, one frequently multiplies it by a Gaussian factor, but that makes little difference to the considerations which follow.

If we demand that the mass matrices M are positive semi-definite they form a convex cone in $C^{\frac{1}{2}n(n+1}$. Fixing the trace or the trace of the square gives a compact space with a well defined measure One may write

$$M = S\Delta S^{\dagger} \tag{41}$$

where S is unitary and Δ is diagonal with entries λ_i. The case of 3×3 real symmetric matrices is described in detail by Giulini *. . The general Hermitian case is described by Zinn-Justin and Zuber †

*D. Giulini, A Euclidean Bianchi model based on S**3 / (D(8)*, *J. Geom. Phys.* **20** (1995) 149 [arXiv:gr-qc/9508040]

†P. Zinn-Justin and J. B. Zuber, On some integrals over the U(N) unitary group and their large N limit, *J. Phys.* **A 36** (2003) 3173 [arXiv:math-ph/0209019].

As far as masses are concerned we are not interested in the angles of the unitary group, but rather the eigen-values. Thus we simply average over the angles and obtain an induced metric on the eigenvalues. If these are non-negative we are confined to the positive orthant ${\mathbb{R}_+}^n$ of $\mathbb{R}^n$. Imposing a trace condition then limits one to a hyperplane plane $\sum_i \lambda_i = 1$, or a hypersphere $\sum_i \lambda_i^2 = 1$. The former case means that the measure is defined on an $(n-1)$-plex Σ_{n-1}. (i.e inside a (Dalitz style) triangle for the physically relevant 3×3 case).

The induced measure coming from the metric on ${R_+}^n$ is

$$\prod_{i<j} |\lambda_i - \lambda_j| d^n\lambda \tag{42}$$

Thus co-incident or near co-incincident masses are strongly disfavoured according to this measure. Given the quark masses and their error bars, it would be possible to calculate precisely the degree of fine tuning.

The Conclusions are that

- It remains a non-trvial problem to place priors on the space of initial conditions, coupling constants or of quantum states, except for cases with a finite number of classical degrees of freedom or Hilbert space dimensions.

- String theory, in which coupling constants are subsumed into initial conditions gives rise to some improvements but not as yet a solution.

- The absence of such priors challenges are our basic notions of science as a rational activity.