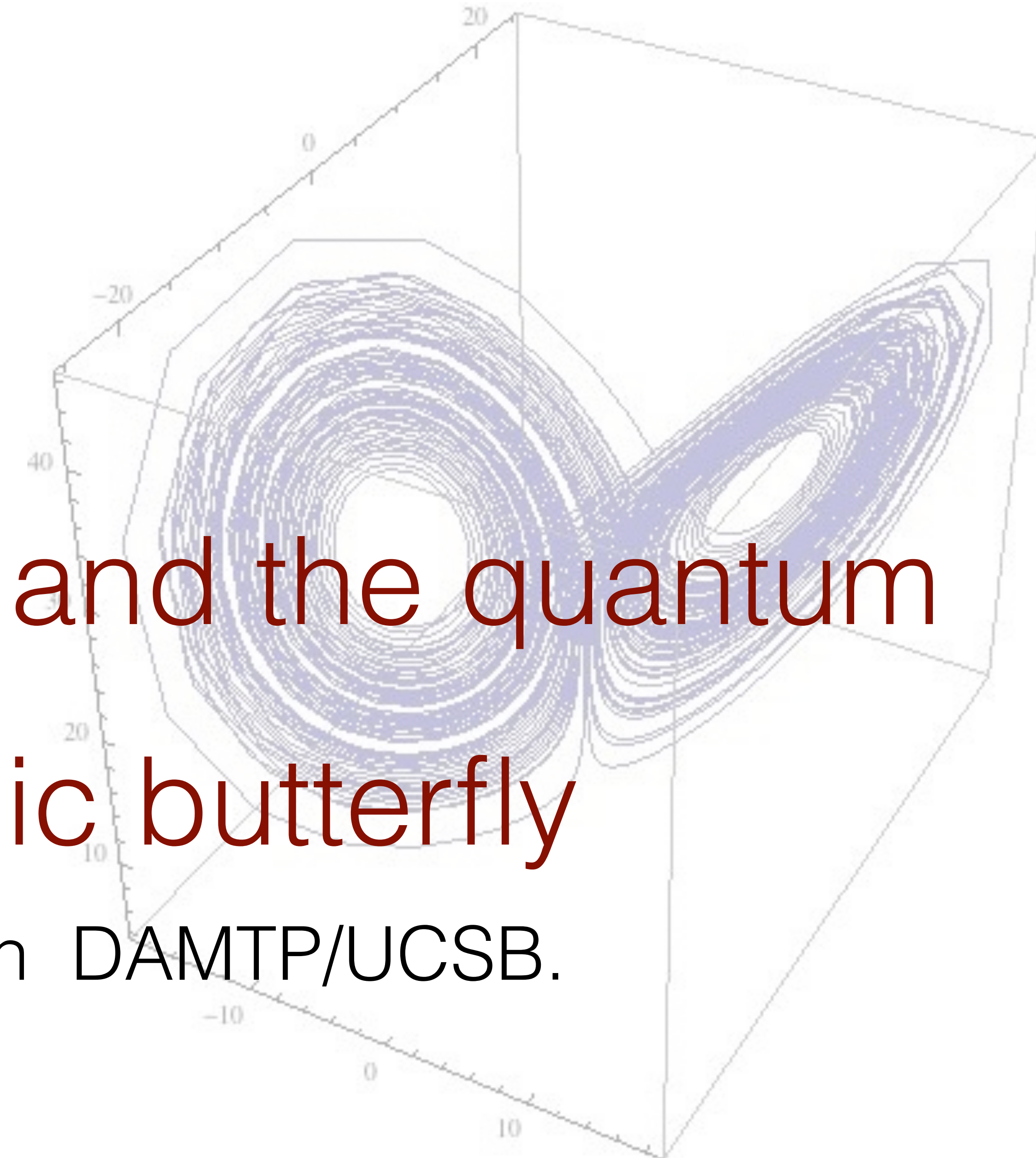
Entangling rates and the quantum holographic butterfly



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C. Asplund, D.B. arXiv:1503.04857
Work in progress with C. Asplund, A. Garcia Garcia

Questions

- If a black hole is in a pure state, where does its (horizon) entropy come from?
- How fast do objects scramble with the black hole when they fall into a black hole? -What does scrambling mean?
- How do we answer these questions in holographic dual?

Proposal: compute entanglement entropy for our choice of initial state for a (as yet to be determined) factorization of Hilbert space

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

Page's result: the typical state is essentially always maximally entangled.

Hilbert space is really big (infinite dimensional)

In this case there is no “canonical” splitting into factors of same size: all countable infinite dimensional Hilbert spaces are isomorphic.

We need more.

Need a preferred factorization determined by algebra of observables that we care about.

Observables are “simple” variables in a classical limit.

Splitting into sets of smooth variables with vanishing Poisson brackets between sets (e.g. product manifold).

One can study dependence on classical dimensionality of split.

Question

Is scrambling rate essentially universal?

Or is it extremely sensitive to choices?

How to formulate fast scrambling conjecture (Sekino-Susskind, also Hayden, Preskill) rigorously in dual field theory?

Strategy:

Try to understand classical scrambling in general finite dimensional Hamiltonian systems first, then understand how this relates to quantization.

Outline

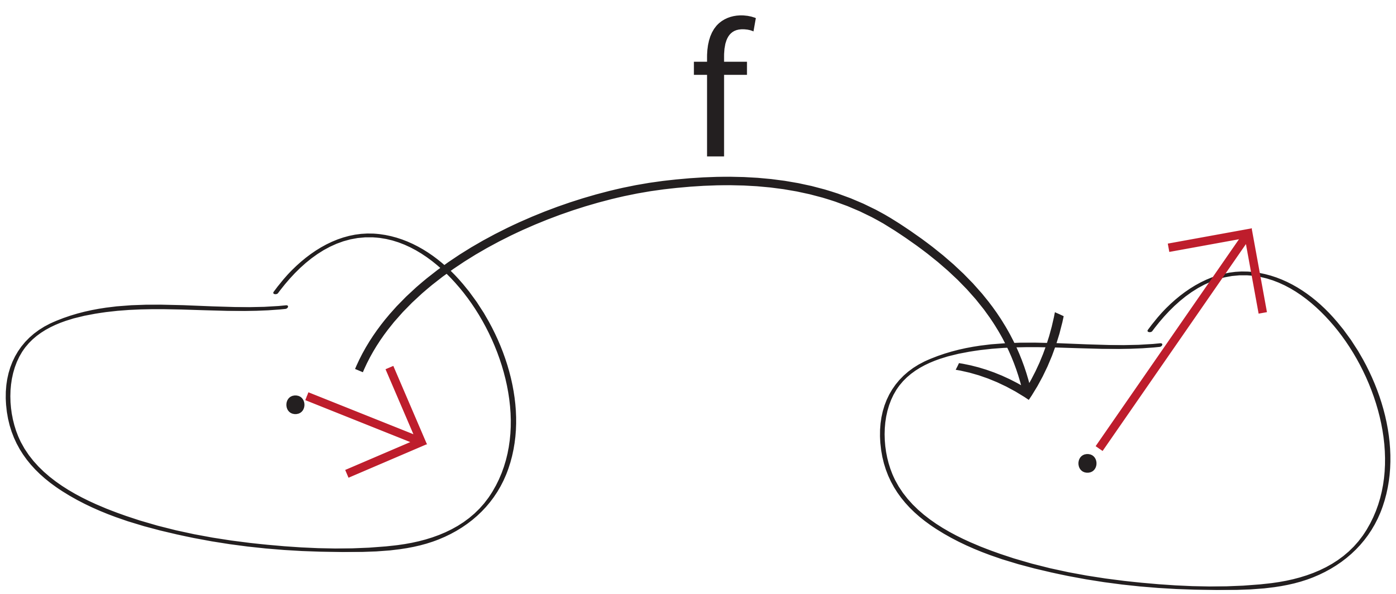
- Lyapunov exponents and (classical) topological entropy.

- Toy model of scrambling around periodic trajectories.

- Quantum butterfly effect.

- Speculations about black holes.

Lyapunov exponents and classical entropy



$$|f^{(n)}(p)_* v| \simeq \exp(n\lambda_{max})|v|$$

$\lambda_{max} = r_1$ Log. Rate of max stretching

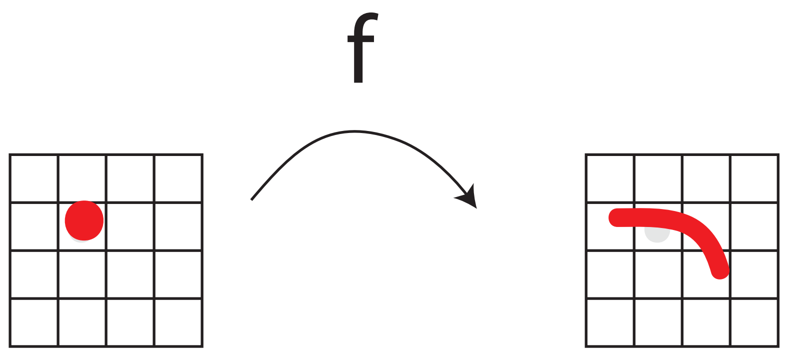
To define other Lyapunov exponents

$$r_1 + r_2 = \frac{1}{n} \log \left| \frac{f_*^{(n)} \delta A}{\delta A} \right|$$

$$r_1 + r_2 + r_3 = \frac{1}{n} \log \left| \frac{f_*^{(n)} \delta V^{(3)}}{\delta V^{(3)}} \right|$$

Etc.

Entropy



Take a small cell, and
count number of hits after f.

Refine cells and take limit of iteration.

$$KS \simeq \lim_{n\to\infty} \frac{1}{n} \lim_{\delta V^{(d)}\to 0} \log \left| \#\text{Hits of } f^{(n)}(\delta V^{(d)}) \right|$$

This definition of topological entropy is a rate.

Can be understood in terms of information theory:
how many new bits of information you need (on average) per iteration to determine on which cell one actually lands from a given initial condition .

$$KS = \sum_{r_i > 0} r_i$$

Pesin's formula.

Quantum scrambling?

Need dynamical system associated to a Symplectic manifold (maybe with a Hamiltonian flow, but one can measure stroboscopically).

Start with a pure state (close to classical), and evolve.

Integrate over half of d.o.f. or more.

Universal toy model

Typical chaotic systems have a dense set of periodic orbits.

Study initial “classical” conditions near one of these orbits.

Measure exactly on period of trajectories.

Classically end up nearby: linearized analysis is valid.

$\hbar$ is small

In a double scaled approx w.r.t. T (period)

Linear analysis can be done with
Bogolubov transformations.

Simplest model

2 Harmonic oscillator degrees of freedom mixing with a non trivial linear (symplectic) transformation.

C. Asplund, D.B. arXiv:1503.04857

Particle-antiparticle pairs

Preserve a U(1).

$$\begin{pmatrix} \tilde{a}^\dagger \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a^\dagger \\ b \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \exp(i\beta) \begin{pmatrix} e^{i\theta}\cosh\rho & e^{i\phi}\sinh\rho \\ e^{-i\phi}\sinh\rho & e^{-i\theta}\cosh\rho \end{pmatrix}$$

If applied to vacuum of a,b and compare to final Hilbert space, we get a squeezed state.

$$|0\rangle_0 \propto \exp(-CD^{-1}a^\dagger b^\dagger)|0\rangle_1$$

Trace over b, one gets a standard “thermal” density matrix for a-type particles.

In absence of phases upon iteration

$$C_n/D_n = \tanh n\rho$$

$$S_n \to \log(\exp(2n\hat{\rho})/4) + O(1) = 2n\hat{\rho} + O(1)\ .$$

Same result as classical entropy growth.

Lyapunov exponents

$$r_1 = r_2 = \max(\log|Eigenvalues|)$$

Same as eigenvalues of boost

Other two are negative and opposite (Liouville theorem)

If coefficients not real

Can be conjugated to similar form.

$$S_n = 2n\hat{\rho} + O(1)$$

All these choices go into the O(1) piece:
asymptotic entropy growth rate is universal,
but the offset depends on details.

$$S_n(\text{with phases}) > S_n(\text{without phases})$$

Can do general Bog. transformations without U(1) charge, or with mixing in choice of particle/antiparticle.

One ends up with a general gaussian density matrix

$$\rho_a(z,\bar{z}) = B_0 e^{B_1(z^2+\bar{z}^2)+B_2 z\bar{z}}$$

In this notation:

$$a^\dagger \simeq z$$

$$(a)_{\text{acting on right of }\rho} \simeq \bar{z}$$

Define:

$$x = \langle a^2 \rangle$$

$$y = \langle a^\dagger a \rangle$$

$$\chi^2 = (y - 1/2)^2 - |x|^2$$

One shows after some algebra that

$$S = (\chi + \tfrac{1}{2}) \log(\chi + \tfrac{1}{2}) - (\chi - \tfrac{1}{2}) \log(\chi - \tfrac{1}{2})$$

$$S \simeq \log \chi$$

We found this in

F. Benatti and R. Floreanini arXiv:hep- th/0010013

One can start with no mixing

$$|0\rangle_0 \propto \exp\left(\frac{1}{2}(\tanh n\rho_1)z^2 + \frac{1}{2}(\tanh n\rho_2)w^2\right) = \exp\left(\frac{s_1}{2}z^2 + \frac{s_2}{2}w^2\right)$$

And rotate basis

$$z = \cos(\theta)z_1 + i\sin(\theta)w_1$$
$$w = i\sin(\theta)z_1 + \cos(\theta)w_1$$

Do algebra

$$\chi^2 = \frac{1}{16}[3 + \cosh(2n(\rho_1 + \rho_2)) - 2\cos(4\theta)\sinh(n(\rho_1 + \rho_2)^2)]$$

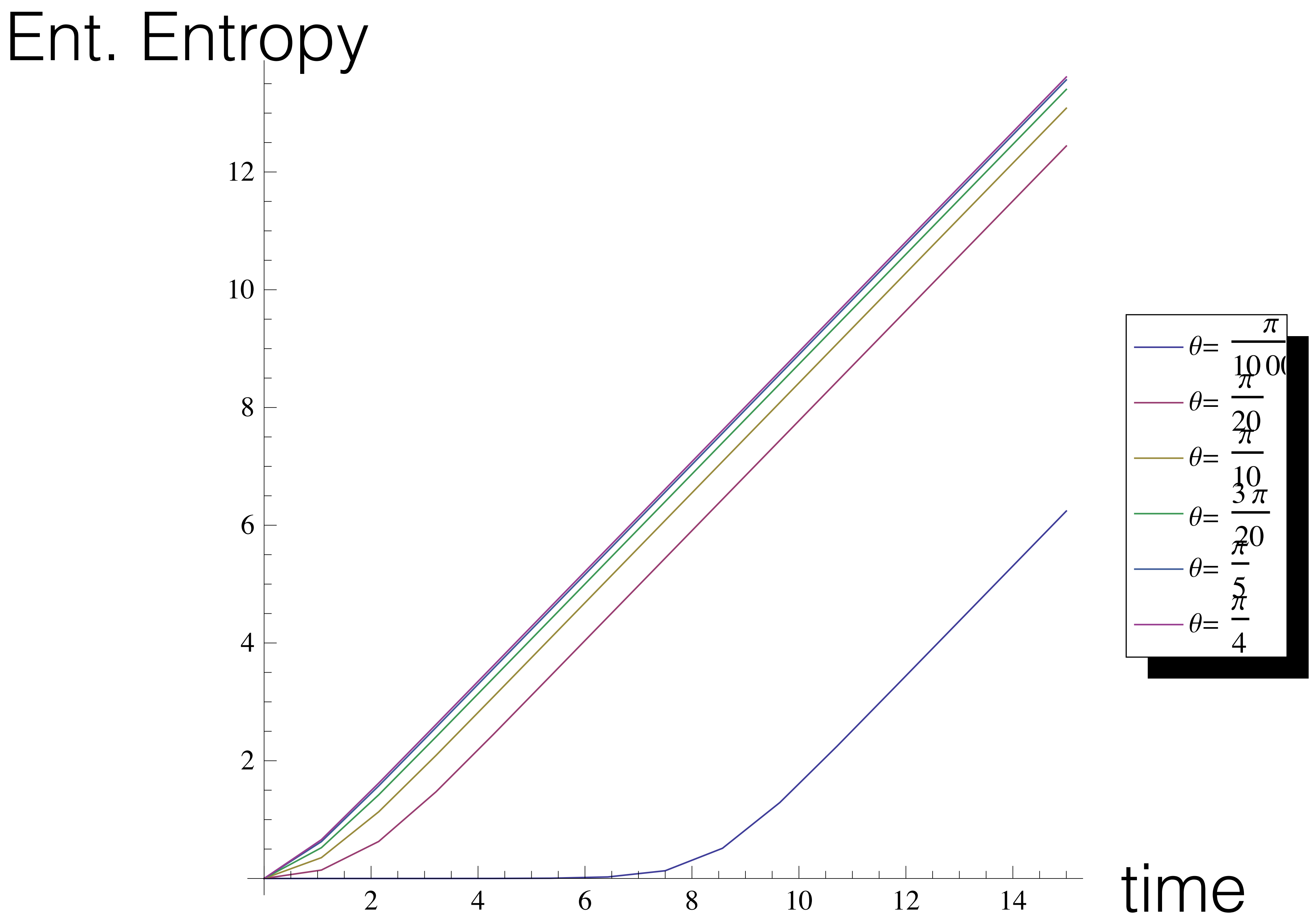
$$\log\chi \to n(\rho_1 + \rho_2)$$

Again, growth as classical entropy

Unless

$$\cos 4\theta = 1$$

Set of measure 0



Ent. Entropy
time
$\theta = \frac{\pi}{1000}$
$\theta = \frac{\pi}{20}$
$\theta = \frac{\pi}{10}$
$\theta = \frac{3\pi}{20}$
$\theta = \frac{\pi}{5}$
$\theta = \frac{\pi}{4}$

$$r_1 + r_2 = 1$$

If add more d.o.f and integrate all but 1, expect growth rate at sum of two largest Lyapunov exponents.

If add more d.o.f and integrate all but 2, expect growth rate at sum of four largest Lyapunov exponents.

Max rate is sum over half (positive) Lyapunov exponents

Quantum scrambling rate = classical scrambling rate.

Side comments

- This is usually phrased in quantum control in terms of two subsystems with weak interactions between them and the problem of decoherence (Zurek, Paz, etc) .
- Tweaking strength of interactions between subsystems is roughly the same as fine-tuning the angle theta that we saw.

corollary:

Choice of splitting into subsystems (sufficiently smoothly) does not really matter too much to define entropy growth asymptotically.

Back to black holes

- Scrambling rate universal at intermediate time (in collapse problem)
- Expect saturation at entropy of black hole.
- Details of which half of degrees of freedom we throw away does not seem to matter too much.
- How to choose “interior degrees of freedom” from entanglement is a red herring: roughly anything will do so long as we forget enough.

Quantum butterfly effect

- Shenker, Stanford;

Lyapunov exponents show elsewhere:

$$[f^{(n)}(X), Y] \simeq \hbar \exp(r_1 n) W$$

In growth of commutators.

$$\Delta X_n \Delta Y \geq \hbar \exp(r_1 n)$$

We loose predictivity eventually:
uncertainty becomes large.

Ehrenfest time

$$\Delta X_n \simeq \text{Size of system}$$

$$t \simeq 1/r \log(\hbar)$$

Parametrically larger than scale of classical dissipation (autocorrelation).

New claim

Maldacena, Shenker, Stanford

$$r < T$$

Can we understand this?

Lets start with commutators

Simple toy model

Arnold cat maps (well known chaotic system)
on fuzzy torus

$$U, V$$

$$UV = \omega VU$$

$$U^n = V^n = 1$$

$$\omega^n = 1$$

$$\hbar \simeq 1/n$$

Two standard automorphisms

$T:$

$$U \to UV\gamma$$
$$V \to V$$

$S:$

$$U \to V$$
$$V \to U^{-1}$$

Generate an SL(2,Z) mapping class of torus

$$SL(2, Z_n)$$

Up to phases

$$U \to U^a V^b \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$V \to U^c V^d$$

When computing

$$[U^{(m)}, U] \simeq (1 - \exp(2\pi i b_m/n))U^{(n)}U \simeq b_m/n \simeq \lambda^m/n$$

Grows like powers of eigenvalues: also Lyap. exponent.

Estimate in some holographic examples

IN BFSS

At high T it is classical physics

Virial theorem:

$$X \simeq T^{1/4}$$

$$P \simeq T^{1/2}$$

$$r \simeq P/X \simeq T^{1/4}$$

No problem saturating inequality.

Exactly near T of order 1 we expect crossover to classical black hole.

Argument of Wiseman 2013

$$P \simeq XT$$

With right factors of N, and assuming configuration approximately diagonal,
v^4/r^7 potential one can estimate free energy of black hole and get things right.

Conclusion

- Scrambling rates can be fairly independent of details of “factorization” of Hilbert space, esp. in a classical limit.
- Suggests scrambling in black hole duals is sufficiently Universal.

- Can one prove quantum entropy rate is classical entropy rate not just about periodic trajectories?
- Lyapunov exponents also show in commutators: there is a butterfly effect in quantum systems.
- Intriguing maximal rate for Lyapunov exponents in very quantum systems.