Ultraviolet Surprises in Gravity

EuroStrings 2015

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Zvi Bern, UCLA

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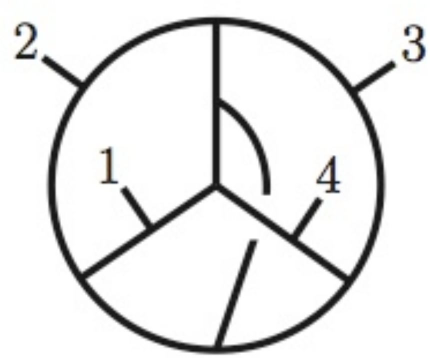
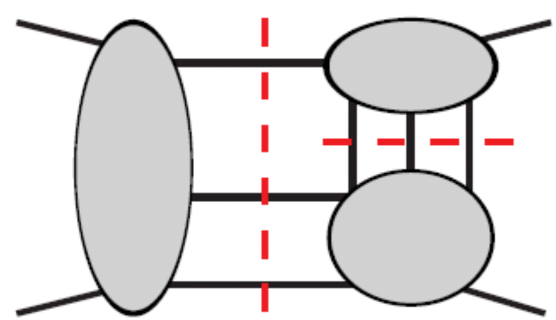




Outline

1) **Basic tools and ideas.**

2) **Review of ultraviolet properties of supergravity and standard arguments.**

3) **"Enhanced" UV cancellations. UV cancellation beyond the ones understood from standard symmetry or counterterm arguments.**

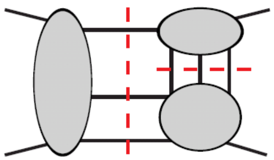
4) **Explicit calculations demonstrating enhanced UV cancellations in $N = 4, 5$ supergravity at 3, 4 loops.**

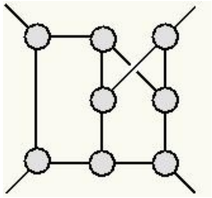
5) **Revisiting pure Einstein gravity at 2 loops. More interesting than usual arguments suggest.**

Our Basic Tools

We have powerful tools for computing amplitudes and for discovering new structures:

See Trnka's talk.





- **Unitarity Method.**

  ZB, Dixon, Dunbar, Kosower
  ZB, Carrasco, Johansson , Kosower

- **Duality between color and kinematics.**

  ZB, Carrasco and Johansson

- **Advanced loop integration technology.**

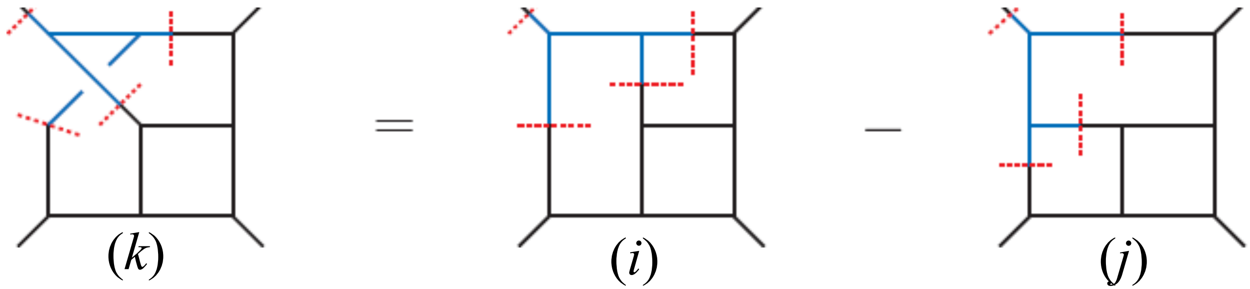
  Chetyrkin, Kataev and Tkachov; A.V. Smirnov; V. A. Smirnov, Vladimirov; Marcus, Sagnotti; Cazkon; etc

Many other tools and advances that I won't discuss here.

Duality Between Color and Kinematics

ZB, Carrasco, Johansson (2010)

Conjecture: kinematic numerators exist with same algebraic properties as color factors



color factor

$$c_k = c_i - c_j$$

$$n_k = n_i - n_j$$

kinematic numerator

If you have a set of duality satisfying kinematic numerators.

$$n_i \sim k_1 \cdot l_1 \, k_3 \cdot l_2 \, \varepsilon_1 \cdot l_3 \, \varepsilon_2 \cdot k_3 \, \varepsilon_3 \cdot l_2 \, \varepsilon_4 \cdot k_3 + \ldots$$

gauge theory ⟶ gravity theory

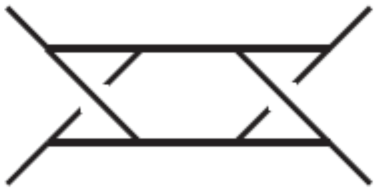
simply take

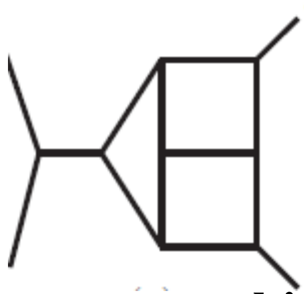
color factor ⟶ kinematic numerator

Gravity loop integrands are trivial to obtain once we have gauge theory in a form where duality holds.

Conjecture: Duality Between Color and Kinematics

BCJ





$$c_i + c_j + c_k = 0$$
$$n_i + n_j + n_k = 0$$

sum is over diagrams

kinematic numerator

color factor

$$\frac{(-i)^L}{g^{n-2+2L}} \mathcal{A}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

gauge theory

propagators

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}} \mathcal{M}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

gravity

$$n_i \sim k_1 \cdot l_1 \, k_3 \cdot l_2 \, \varepsilon_1 \cdot l_3 \, \varepsilon_2 \cdot k_3 \, \varepsilon_3 \cdot l_2 \, \varepsilon_4 \cdot k_3 + \ldots$$

symmetry factor

- **Crucial part is to find numerators that satisfy duality.**
- **Unitarity method used to prove that whenever we have such numerators gravity amplitudes correct.**

Gravity From Gauge Theory

BCJ

$$-i\left(\frac{2}{\kappa}\right)^{(n-2)} \mathcal{M}_n^{\text{tree}}(1,2,\ldots,n) = \sum_i \frac{n_i\,\tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

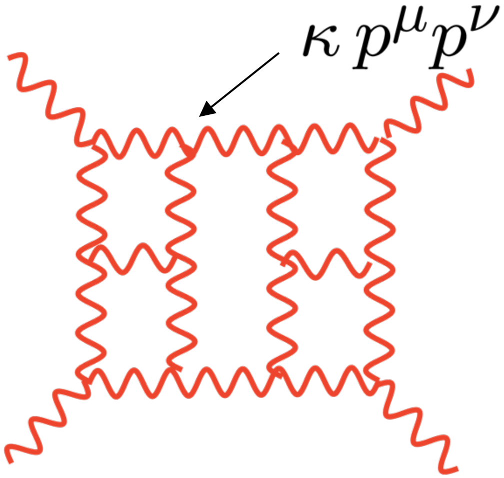
	n	$\tilde{n}$
$N = 8$ sugra:	**($N = 4$ sYM)$\times$**	**($N = 4$ sYM)**
$N = 5$ sugra:	**($N = 4$ sYM)$\times$**	**($N = 1$ sYM)**
$N = 4$ sugra:	**($N = 4$ sYM)$\times$**	**($N = 0$ sYM)**

Spectrum controlled by simple tensor product of YM theories.
Recent papers show more sophisticated lower-susy cases.

Anastasiou, Bornsten, Duff; Duff, Hughs, Nagy; Johansson and Ochirov; Carrasco, Chiodaroli, Günaydin and Roiban; ZB, Davies, Dennen, Huang and Nohle; Nohle; Chiodaroli, Günaydin, Johansson, Roiban.

Is a UV finite field theory of gravity possible?

$\kappa = \sqrt{32\pi G_N}$ ⟵ Dimensionful coupling



Gravity: $$\int \prod_{i=1}^{L} \frac{dp_i^D}{(2\pi)^D} \frac{(\kappa\, p_j^\mu p_j^\nu)\cdots}{\text{propagators}}$$

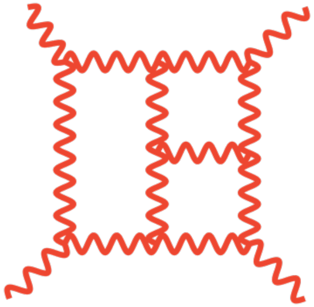
Gauge theory: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(g\, p_j^\nu)\cdots}{\text{propagators}}$$

- **Extra powers of loop momenta in numerator means integrals are badly behaved in the UV.**
- **Much more sophisticated power counting in supersymmetric theories but this is basic idea.**

- **With more susy expect better UV properties.**
- **High symmetry implies simplicity.**

See also comments in Vanhove's talk.

UV Finiteness of Supergravity?



Consensus opinion from the late 1970's and early 1980's:
***All* supergravity theories should diverge by three loops and therefore are not viable as fundamental theories.**

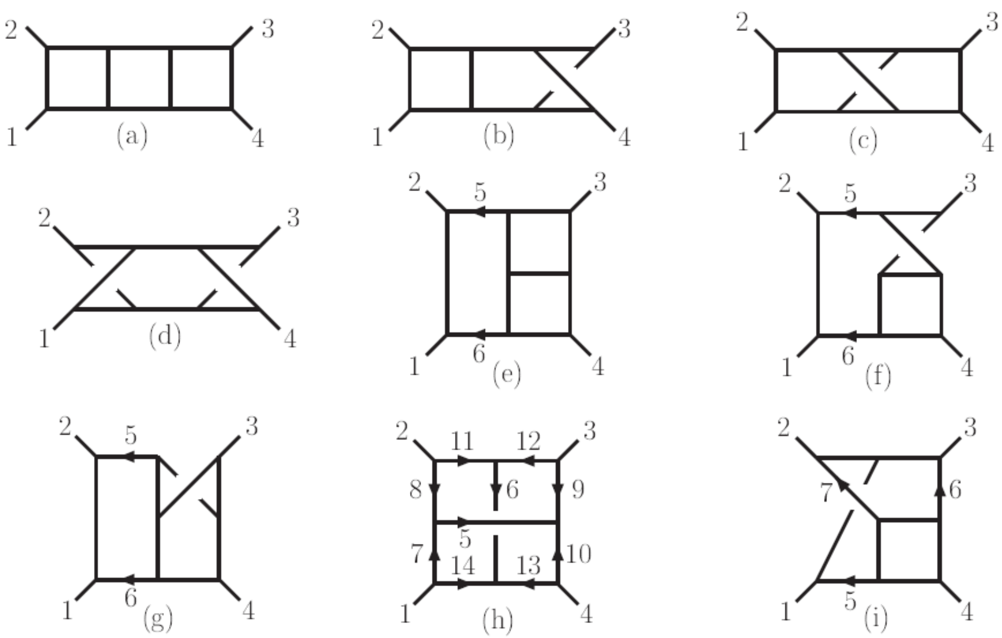
Green, Schwarz, Brink; Howe and Stelle; Marcus and Sagnotti; etc.

Is this true?

$N = 8$ Supergravity at Three Loops

Analysis of unitarity cuts shows highly nontrivial all-loop cancellations. ZB, Dixon and Roiban (2006); ZB, Carrasco, Forde, Ita, Johansson (2007)

To test completeness of cancellations, we decided to directly calculate potential three-loop divergence.



ZB, Carrasco, Dixon, Johansson, Kosower, Roiban (2007)

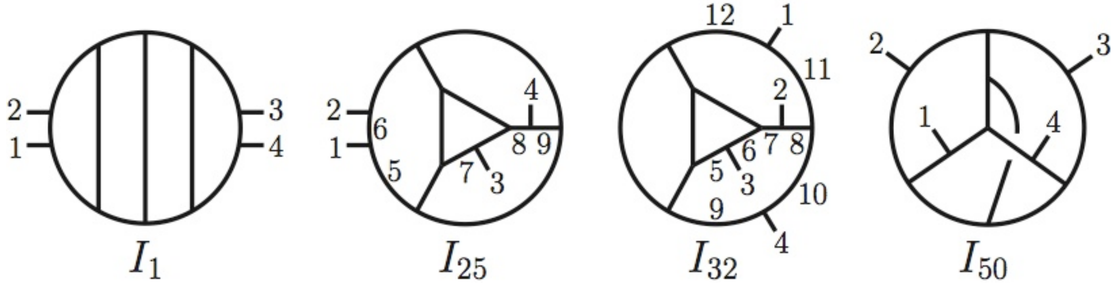
Three loops is not only ultraviolet finite, it is very finite— finite for $D < 6$.

Obtained via on-shell unitarity method.

Four-Loop $N = 8$ Supergravity Amplitude Construction

ZB, Carrasco, Dixon, Johansson, Roiban (2009)

Get 85 distinct diagrams or integrals.



$$M_4^{\text{4-loop}} = \left(\frac{\kappa}{2}\right)^{10} stu M_4^{\text{tree}} \sum_{S_4} \sum_{i=1}^{50} c_i I_i$$

Integral

symmetry factor

leg perms

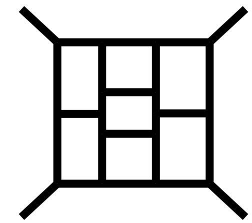
UV finite for *D* < 11/2
It is very finite!
Same behavior as *N* = 4 sYM

Duality between color and kinematic discovered by doing this calculation.

Current Status of $N = 8$ Divergences

Consensus that in $N = 8$ supergravity trouble starts at 5 loops and by 7 loops we have valid UV counterterm in $D = 4$ under all known symmetries (suggesting divergences).
susy + $E_{7(7)}$ duality symmetry.

Bossard, Howe, Stelle; Elvang, Freedman, Kiermaier; Green, Russo, Vanhove ; Green and Björnsson ; Bossard , Hillmann and Nicolai; Ramond and Kallosh; Broedel and Dixon; Elvang and Kiermaier; Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger



For $N = 8$ sugra in $D = 4$:

- **All counterterms ruled out until 7 loops.**
- **D^8R^4 counterterm available at 7 loops under all known symmetries.**
- **Oddly, it is not a full superspace integral.**

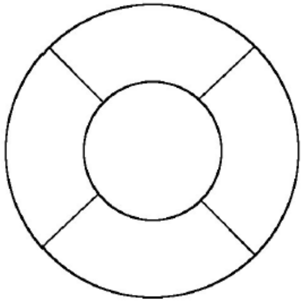
Bossard, Howe, Stelle and Vanhove

Based on this, a reasonable person would conclude that $N = 8$ supergravity almost certainly diverges at 7 loops in $D = 4$.

Predictions of Ultraviolet Cancellations

Björnsson and Green developed a first quantized formulation of Berkovits' pure-spinor formalism.

Key point: *all* supersymmetry cancellations are exposed.



They identify contributions that are poorly behaved.

Poor UV behavior, unless new types of cancellations between diagrams exist that are "not consequences of supersymmetry in any conventional sense":

Bjornsson and Green

- **$N = 8$ sugra should diverge at 7 loops in $D = 4$.**
- **$N = 8$ sugra should diverge at 5 loops in $D = 24/5$.**

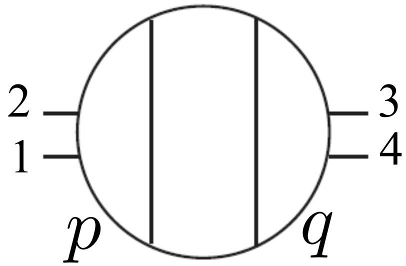
Maximal Cut Power Counting

ZB, Davies, Dennen

Maximal cuts of diagrams poorly behaved:

already log divergent

$N=4$ sugra



$N = 4$ sugra: pure YM x $N = 4$ sYM

$$n_i \sim s^3 t A_4^{\text{tree}} \, (p\cdot q)^2 \, \varepsilon_1 \cdot p \, \varepsilon_2 \cdot p \, \varepsilon_3 \cdot q \, \varepsilon_4 \cdot q + \ldots$$

This diagram is log divergent

$N = 8$ sugra should diverge at 7 loops in $D = 4$. **Bet with David Gross**

$N = 8$ sugra should diverge at 5 loops in $D = 24/5$ **Bet with Kelly Stelle**

$N = 4$ sugra should diverge at 3 loops in $D = 4$

$N = 5$ sugra should diverge at 4 loops in $D = 4$

Unfortunately no bets

This result equivalent to Björnsson and Green's approach: Identify poorly behaved terms and count.

All other groups that looked at the question of symmetries agree. Looked like a safe bet that these divergences are present.

Enhanced UV Cancellations

Suppose diagrams in any possible covariant diagrammatic representation are UV divergent.

$\mathcal{N} = 4$ sugra

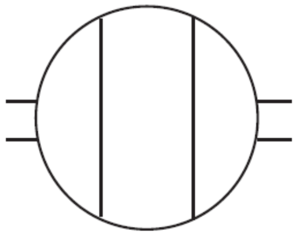


Diagram necessarily is log divergent

If sum over diagrams is UV finite by definition this is an "enhanced cancellation"

- The Björnsson and Green power counting does *not* include enhanced cancellations.
- Through four loops in $\mathcal{N} = 8$ sugra, UV cancellations are *not* enhanced and can be understood from standard symmetries.
- Standard UV cancellations in susy gauge theory *not* enhanced.

Examples of Enhanced Cancellations

Four explicitly known examples in gravity:

1) $N = 4$ supergravity in $D = 4$ at 3 loops.

2) $N = 5$ supergravity in D = 4 at 4 loops.

3) Half-maximal supergravity in $D = 5$ at 2 loops.

4) Pure Einstein gravity at 1 loop simplest example.

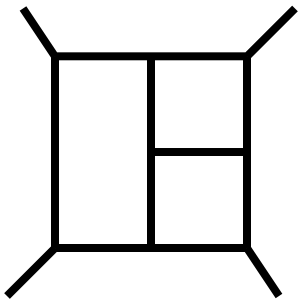
- First 3 not explained by standard symmetry arguments.
- Last one explained by Gauss-Bonnet theorem.

Three-Loop $\mathcal{N} = 4$ Supergravity Construction

ZB, Davies, Dennen, Huang

$\mathcal{N} = 4$ sugra : ($\mathcal{N} = 4$ sYM) x ($\mathcal{N} = 0$ YM)

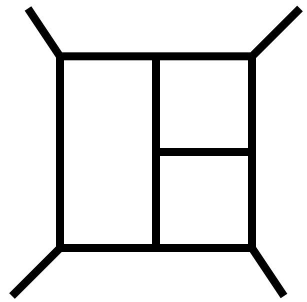
$\mathcal{N} = 4$ sYM



$$l \cdot k \, s^2 t A_4^{\rm tree}$$

BCJ representation

pure YM



$$(\varepsilon \cdot l)^4 \, l^4$$

Feynman representation

$c_i \rightarrow n_i$

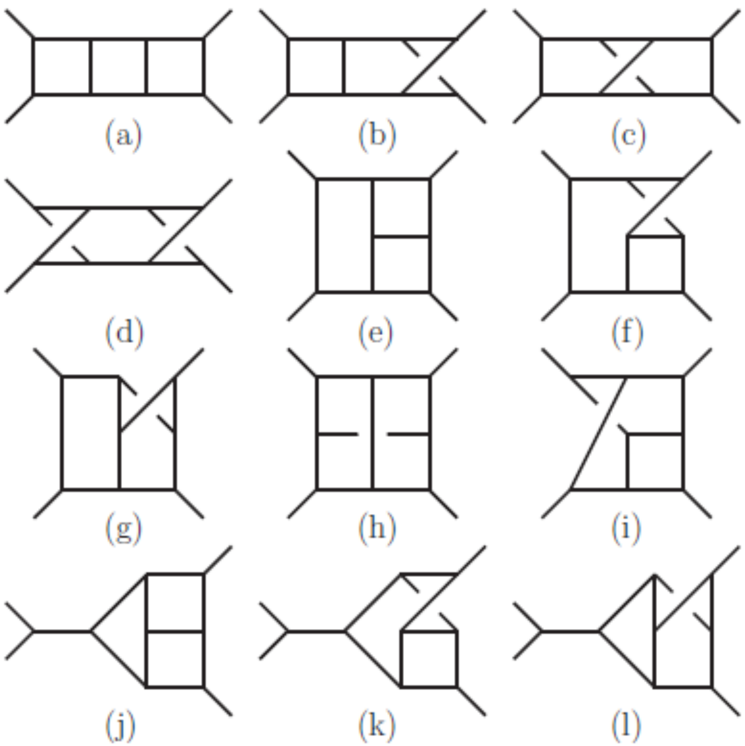
$\mathcal{N} = 4$ sugra diagrams linearly divergent

$$\int (d^D l)^3 \, \frac{k^7 l^9}{l^{20}}$$

- **Ultraviolet divergences are obtained by series expanding small external momentum (or large loop momentum).**
- **Introduce mass regulator for IR divergences.**
- **In general, subdivergences must be subtracted.**

Vladimirov; Marcus and Sagnotti

$\mathcal{N} = 4$ Supergravity UV Cancellation



$D = 4 - 2\epsilon$ ZB, Davies, Dennen, Huang

Graph	$(\text{divergence})/(\langle 12\rangle^2[34]^2 st A^{\text{tree}}(\frac{\kappa}{2})^8)$
(a)-(d)	0
(e)	$\frac{263}{768}\frac{1}{\epsilon^3} + \frac{205}{27648}\frac{1}{\epsilon^2} + \left(-\frac{5551}{768}\zeta_3 + \frac{326317}{110592}\right)\frac{1}{\epsilon}$
(f)	$-\frac{175}{2304}\frac{1}{\epsilon^3} - \frac{1}{4}\frac{1}{\epsilon^2} + \left(\frac{593}{288}\zeta_3 - \frac{217571}{165888}\right)\frac{1}{\epsilon}$
(g)	$-\frac{11}{36}\frac{1}{\epsilon^3} + \frac{2057}{6912}\frac{1}{\epsilon^2} + \left(\frac{10769}{2304}\zeta_3 - \frac{226201}{165888}\right)\frac{1}{\epsilon}$
(h)	$-\frac{3}{32}\frac{1}{\epsilon^3} - \frac{41}{1536}\frac{1}{\epsilon^2} + \left(\frac{3227}{2304}\zeta_3 - \frac{3329}{18432}\right)\frac{1}{\epsilon}$
(i)	$\frac{17}{128}\frac{1}{\epsilon^3} - \frac{29}{1024}\frac{1}{\epsilon^2} + \left(-\frac{2087}{2304}\zeta_3 - \frac{10495}{110592}\right)\frac{1}{\epsilon}$
(j)	$-\frac{15}{32}\frac{1}{\epsilon^3} + \frac{9}{64}\frac{1}{\epsilon^2} + \left(\frac{101}{12}\zeta_3 - \frac{3227}{1152}\right)\frac{1}{\epsilon}$
(k)	$\frac{5}{64}\frac{1}{\epsilon^3} + \frac{89}{1152}\frac{1}{\epsilon^2} + \left(-\frac{377}{144}\zeta_3 + \frac{287}{432}\right)\frac{1}{\epsilon}$
(l)	$\frac{25}{64}\frac{1}{\epsilon^3} - \frac{251}{1152}\frac{1}{\epsilon^2} + \left(-\frac{835}{144}\zeta_3 + \frac{7385}{3456}\right)\frac{1}{\epsilon}$

Spinor helicity used to clean up table, but calculation for all states

***All* three-loop divergences and subdivergences cancel completely!**

3-loop 4-point $\mathcal{N} = 4$ sugra UV finite contrary to predictions

A pity we did not bet on this theory

Tourkine and Vanhove understood this result by extrapolating from two-loop heterotic string amplitudes.

Explanations?

Key Question:

Is there an ordinary symmetry explanation for this?
Or is something extraordinary happening?

Bossard, Howe and Stelle (2013) showed that 3 loop finiteness of N =4 sugra can be explained by ordinary superspace + duality symmetries, *assuming* a 16 supercharge off-shell superspace exists.

If true, there is a perfectly good "ordinary" symmetry explanation.

Does this superspace exist in D = 5 or D = 4?

Prediction of superspace: If you add N = 4 vector multiplets, amplitude should develop no *new* 2, 3 loop divergences.

Bossard, Howe and Stelle (2013)

Subsequent explicit calculation proves new divergences at 2, 3 loops.
Conclusion: currently no viable standard-symmetry understanding.

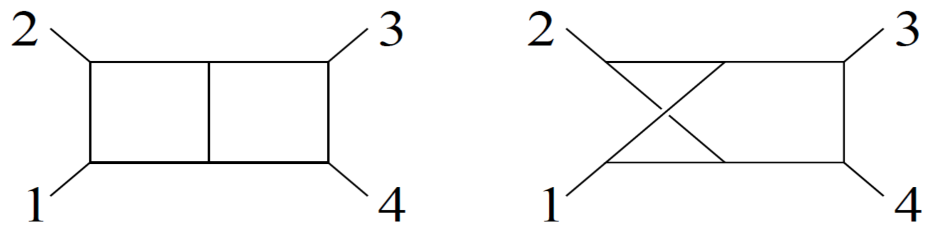
ZB, Davies, Dennen (2013)

What is new magic?

ZB, Davies, Dennen, Huang

To analyze we need a simpler example: Half-maximal supergravity in *D* = 5 at 2 loop.

Similar to *N* = 4, *D* = 4 sugra at 3 loops, except much simpler.



Quick summary:

— Finiteness in *D* = 5 tied to double-copy structure.

— Sugra UV cancellation explained in terms of corresponding cancellation of divergences in certain forbidden color structures of pure non-susy YM theory.

Theory has more structure than just susy and duality symmetry

Not easy to prove UV link to YM beyond two loops.

Four-loop $N = 4$ Supergravity Divergences

ZB, Davies, Dennen, Smirnov, Smirnov

To make a deeper probe we calculated four-loop divergence in $N = 4$ supergravity.

Industrial strength software needed: FIRE5 and C++

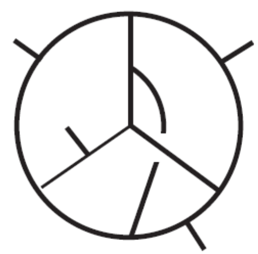
$N = 4$ sugra: ($N = 4$ sYM) x ($N = 0$ YM)

$N = 4$ sYM

$$(l \cdot k)^2 s^2 t A_4^{\text{tree}}$$

BCJ representation

pure YM



$$(\varepsilon \cdot l)^4 l^6$$

Feynman representation

$N = 4$ sugra diagrams quadratically divergent

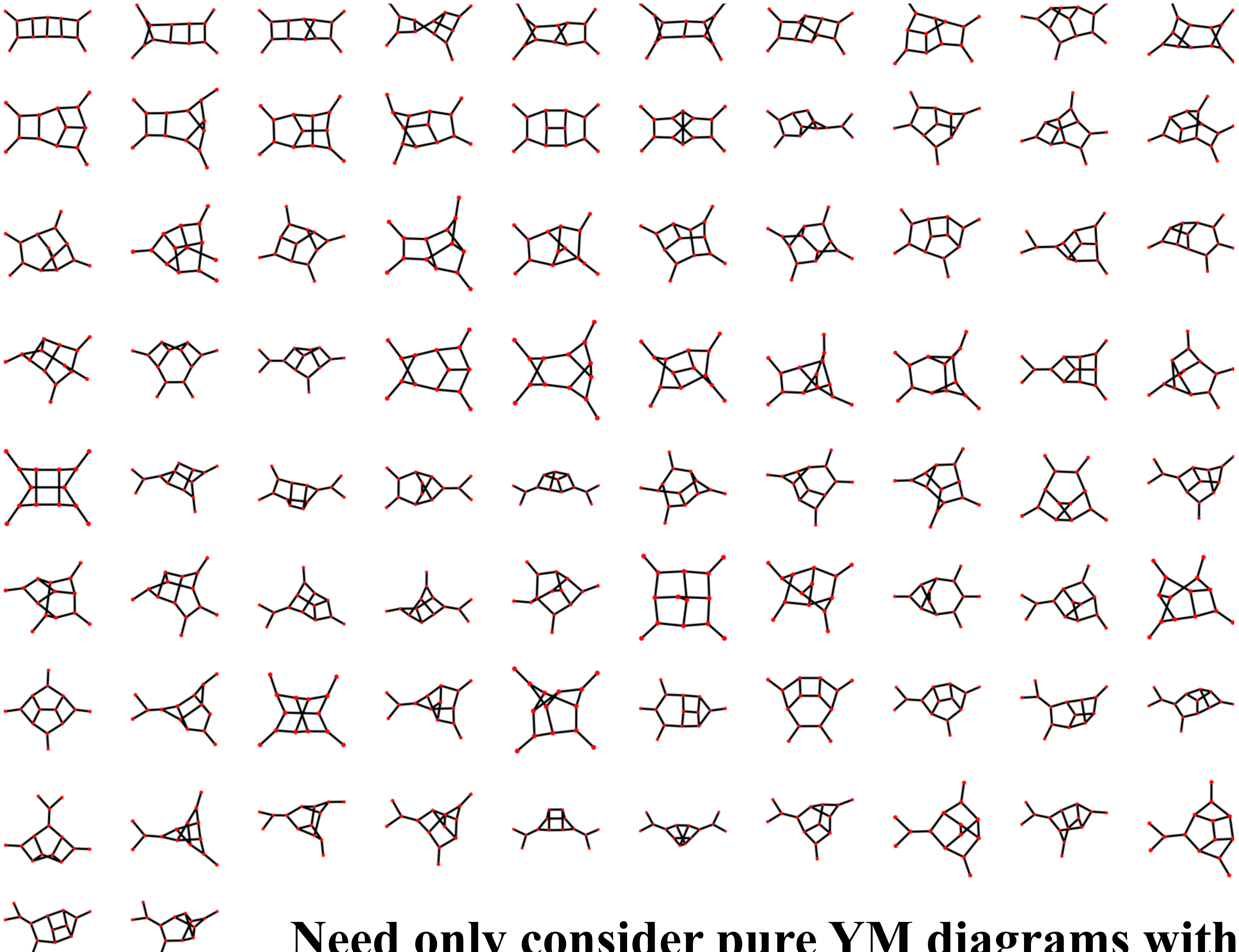
$$\int (d^D l)^4 \frac{k^8 l^{12}}{(l^2)^{13}}$$

$D^2 R^4$ counterterm

82 nonvanishing diagram types using $N = 4$ sYM BCJ form.

82 nonvanishing numerators in BCJ representation

ZB, Carrasco, Dixon, Johansson, Roiban ($\mathcal{N} = 4$ sYM)



Need only consider pure YM diagrams with color factors that match these.

The 4 loop Divergence of $N = 4$ Supergravity

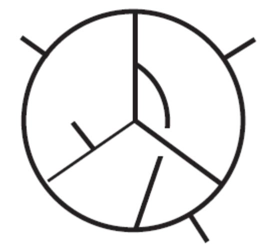
ZB, Davies, Dennen, Smirnov, Smirnov

Pure $N = 4$ supergravity is divergent at 4 loops with divergence

Result is for Siegel dimensional reduction.

dim. reg. UV pole

$$\boxed{\mathcal{M}^{\text{4-loop}}\Big|_{\text{div.}} = \frac{1}{(4\pi)^8}\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^{10}\frac{1}{144}(1-264\zeta_3)\,\mathcal{T}}$$



$D = 4 - 2\epsilon$

$$\mathcal{T} = st A^{\text{tree}}_{\mathcal{N}=4}\left(\mathcal{O}_1 - 28\mathcal{O}_2 - 6\mathcal{O}_3\right)$$

$$s = (k_1+k_2)^2$$

$$t = (k_2+k_3)^2$$

$$\mathcal{O}_1 = \sum_{\mathcal{S}_4}(D_\alpha F_{1\mu\nu})(D^\alpha F_2^{\mu\nu})\,F_{3\rho\sigma}F_4^{\rho\sigma}$$

$$\mathcal{O}_2 = \sum_{\mathcal{S}_4}(D_\alpha F_{1\mu\nu})(D^\alpha F_2^{\nu\sigma})\,F_{3\sigma\rho}F_4^{\rho\mu}$$

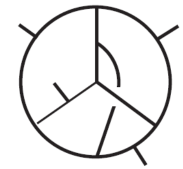
$$\mathcal{O}_3 = \sum_{\mathcal{S}_4}(D_\alpha F_{1\mu\nu})(D_\beta F_2^{\mu\nu})\,F_{3\sigma}{}^{\alpha}F_4^{\sigma\beta}$$

$$F_j^{\mu\nu} \equiv i(k_j^\mu \varepsilon_j^\nu - k_j^\nu \varepsilon_j^\mu)\,,$$

$$D^\alpha F_j^{\mu\nu} \equiv -k_j^\alpha(k_j^\mu \varepsilon_j^\nu - k_j^\nu \varepsilon_j^\mu)$$

Valid for all nonvanishing 4-point amplitudes of pure $N = 4$ sugra

Some Peculiar Properties



Linear combinations to expose $D = 4$ helicity structure

Refers to helicities of pure YM component

$$\mathcal{O}^{--++} = \mathcal{O}_1 - 4\mathcal{O}_2$$
$$\mathcal{O}^{-+++} = \mathcal{O}_1 - 4\mathcal{O}_3$$
$$\mathcal{O}^{++++} = \mathcal{O}_2$$

$$\mathcal{O}^{--++} = 4s^2 t \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\,,$$

$$\mathcal{O}^{-+++} = -12 s^2 t^2 \frac{[2\,4]^2}{[1\,2]\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[4\,1]}\,,$$

$$\mathcal{O}^{++++} = 3st(s+t)\frac{[1\,2]\,[3\,4]}{\langle 1\,2\rangle\,\langle 3\,4\rangle}\,,$$

The latter two configurations would vanish if the *U*(1) symmetry were not anomalous.

See Carrasco, Kallosh, Tseytlin and Roiban

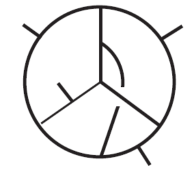
All three independent configurations have similar divergence!

Very peculiar because the nonanomalous sector should have a very different analytic structure. Not related by any supersymmetry Ward identities.

For anomaly:

- ***D* = 4 generalized cuts decomposing into tree amplitudes vanish.**
- **Anomaly is ϵ/ϵ (UV divergence suppressed by ϵ).**

Relation to $U(1)$ Anomaly



unitarity cut

Anomalous sector feeds poor UV behavior into non-anomalous sector

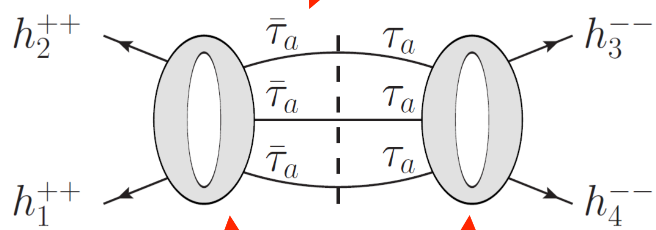


Figure from arXiv:1303.6219
Carrasco, Kallosh, Tseytlin and Roiban

Anomalous 1-loop amplitudes

- **As pointed out by Carrasco, Kallosh Roiban,Tseytlin the anomalous amplitudes are poorly behaved and contribute to a 4-loop UV divergence (unless somehow canceled).**
- **Via anomaly it is easy to understand why all three sectors can have similar divergence structure.**
- **The dependence of the divergence on vector multiplets matches anomaly.**

$$\mathcal{M}^{4\text{-loop}}_{n_V}\Big|_{\text{div.}} = \frac{1}{(4\pi)^8}\left(\frac{\kappa}{2}\right)^{10} \boxed{\frac{n_V+2}{2304}} \Big[\frac{6(n_V+2)n_V}{\epsilon^2} + \frac{(n_V+2)(3n_V+4) - 96(22-n_V)\zeta_3}{\epsilon}\Big]\mathcal{T}$$

anomaly has exactly this factor

n_V is number vector multiplets

Bottom line: The divergence looks specific to $N = 4$ sugra and likely due to an anomaly. Won't be present in $N \geq 5$ sugra.

If anything, this suggests $N = 8$ sugra UV finite at 8 loops.

$\mathcal{N} = 5$ Supergravity at Four Loops

ZB, Davies and Dennen

No anomaly in $\mathcal{N} = 5$ sugra so expect no divergences

$\mathcal{N} = 5$ sugra: ($\mathcal{N} = 4$ sYM) x ($\mathcal{N} = 1$ sYM)

Again crucial help from FIRE5 and (Smirnov)^2

$\mathcal{N} = 4$ sYM

$\mathcal{N} = 1$ sYM

Had we made susy cancellation manifest we would have expected log divergence

Straightforward but nontrivial following what we did in $\mathcal{N} = 4$ sugra.

$\mathcal{N} = 5$ supergravity has no D^2R^4 divergence at four loops.

Another example of an *enhanced cancellation* analogous to 7 loops in $\mathcal{N} = 8$ sugra.

A pity we did not bet on this theory as well!

$\mathcal{N}=5$ supergravity at Four Loops

ZB, Davies and Dennen (2014)

graphs	(divergence) $\times\, u/(-i/(4\pi)^8\langle 12\rangle^2[34]^2 st A^{\text{tree}}(\frac{\kappa}{2})^{10})$
1–30	$\frac{1}{\epsilon^4}\left[\frac{7358585}{7962624}s^2+\frac{2561447}{2654208}st-\frac{872683}{1990656}t^2\right]+\frac{1}{\epsilon^3}\left[\frac{75972559}{35389440}s^2+\frac{240984061}{26542080}st+\frac{1302037}{1310720}t^2\right] +\frac{1}{\epsilon^2}\Big[\zeta_3\left(-\frac{369234283}{11059200}s^2-\frac{257792411}{4915200}st-\frac{101847769}{14745600}t^2\right)+\zeta_2\left(\frac{7358585}{3981312}s^2+\frac{2561447}{1327104}st-\frac{872683}{995328}t^2\right) -\mathrm{S2}\left(\frac{1223621}{49152}s^2+\frac{46816475}{442368}st+\frac{2639903}{221184}t^2\right)+\frac{206093335871}{11466178560}s^2+\frac{320983191023}{3822059520}st+\frac{53309416589}{2866544640}t^2\Big] +\frac{1}{\epsilon}\Big[\zeta_5\left(-\frac{84777347}{368640}s^2+\frac{382194721}{1474560}st+\frac{417476581}{1474560}t^2\right)-\zeta_4\left(\frac{3062401}{2457600}s^2+\frac{3881051}{3276800}st-\frac{112081813}{29491200}t^2\right) +\zeta_3\left(\frac{28162691399797}{53747712000}s^2+\frac{19354492750651}{35831808000}st-\frac{22092683352811}{107495424000}t^2\right)-\zeta_2\left(\frac{70861961}{17694720}s^2+\frac{227180689}{13271040}st +\frac{105727243}{53084160}t^2\right)+\mathrm{T1ep}\left(-\frac{1223621}{663552}s^2-\frac{46816475}{5971968}st-\frac{2639903}{2985984}t^2\right)-\mathrm{S2}\left(\frac{11916028151}{5898240}s^2 +\frac{72637733971}{13271040}st+\frac{17223563447}{53084160}t^2\right)+\mathrm{D6}\left(-\frac{9001177}{552960}s^2-\frac{264491}{10240}st-\frac{2610157}{552960}t^2\right) +\frac{110945914744727}{1146617856000}s^2+\frac{16989492195991}{127401984000}st-\frac{21362122998269}{573308928000}t^2\Big]$
31–60	$\frac{1}{\epsilon^4}\left[-\frac{5502451}{2654208}s^2-\frac{3675877}{884736}st+\frac{11269}{497664}t^2\right]+\frac{1}{\epsilon^3}\left[\frac{38102993}{26542080}s^2-\frac{291607201}{106168320}st-\frac{565798829}{318504960}t^2\right] +\frac{1}{\epsilon^2}\Big[\zeta_3\left(\frac{108955183}{2211840}s^2+\frac{653019571}{8847360}st+\frac{9453043}{1769472}t^2\right)+\zeta_2\left(-\frac{5502451}{1327104}s^2-\frac{3675877}{442368}st+\frac{11269}{248832}t^2\right) +\mathrm{S2}\left(\frac{16797481}{1327104}s^2+\frac{1172969}{16384}st+\frac{978427}{82944}t^2\right)-\frac{304243754383}{19110297600}s^2-\frac{2032063711381}{19110297600}st-\frac{257798086613}{7166361600}t^2\Big] +\frac{1}{\epsilon}\Big[\zeta_5\left(\frac{33327659}{122880}s^2+\frac{13276219}{24576}st+\frac{22251887}{184320}t^2\right)+\zeta_4\left(\frac{12299887}{1474560}s^2+\frac{258056147}{5898240}st+\frac{46913759}{5898240}t^2\right) +\zeta_3\left(-\frac{26846001990157}{42998169600}s^2-\frac{337106527201}{265420800}st-\frac{5298324906787}{42998169600}t^2\right)+\zeta_2\left(\frac{282283789}{39813120}s^2+\frac{975199319}{53084160}st +\frac{60394451}{159252480}t^2\right)+\mathrm{T1ep}\left(\frac{16797481}{17915904}s^2+\frac{1172969}{221184}st+\frac{978427}{1119744}t^2\right)+\mathrm{S2}\left(\frac{10516980893}{4976640}s^2 +\frac{389045625329}{53084160}st+\frac{216032337589}{159252480}t^2\right)+\mathrm{D6}\left(\frac{503413}{23040}s^2+\frac{12342607}{552960}st+\frac{3661}{184320}t^2\right) -\frac{166777358259461}{1146617856000}s^2-\frac{565137511429117}{1146617856000}st-\frac{21629055712141}{191102976000}t^2\Big]$
61–82	$\frac{1}{\epsilon^4}\left[\frac{285899}{248832}s^2+\frac{1058273}{331776}st+\frac{275869}{663552}t^2\right]+\frac{1}{\epsilon^3}\left[-\frac{380329649}{106168320}s^2-\frac{74703227}{11796480}st+\frac{124701919}{159252480}t^2\right] +\frac{1}{\epsilon^2}\Big[\zeta_3\left(-\frac{1371419}{86400}s^2-\frac{236241539}{11059200}st+\frac{4326077}{2764800}t^2\right)+\zeta_2\left(\frac{285899}{124416}s^2+\frac{1058273}{165888}st+\frac{275869}{331776}t^2\right) +\mathrm{S2}\left(\frac{8120143}{663552}s^2+\frac{1893289}{55296}st+\frac{92293}{663552}t^2\right)-\frac{58867708103}{28665446400}s^2+\frac{71191292711}{3185049600}st+\frac{83016363427}{4777574400}t^2\Big] +\frac{1}{\epsilon}\Big[\zeta_5\left(-\frac{1520563}{36864}s^2-\frac{1178767861}{1474560}st-\frac{595491677}{1474560}t^2\right)-\zeta_4\left(\frac{6539029}{921600}s^2+\frac{313837819}{7372800}st+\frac{21665663}{1843200}t^2\right) +\zeta_3\left(\frac{20790944575597}{214990848000}s^2+\frac{6505876281371}{8957952000}st+\frac{70676991239557}{214990848000}t^2\right)+\zeta_2\left(-\frac{491377507}{159252480}s^2-\frac{66476563}{53084160}st +\frac{128393639}{79626240}t^2\right)+\mathrm{T1ep}\left(\frac{8120143}{8957952}s^2+\frac{1893289}{746496}st+\frac{92293}{8957952}t^2\right)+\mathrm{S2}\left(-\frac{14810628499}{159252480}s^2 -\frac{19698937889}{10616832}st-\frac{10272602953}{9953280}t^2\right)+\mathrm{D6}\left(-\frac{616147}{110592}s^2+\frac{1939907}{552960}st+\frac{1299587}{276480}t^2\right) +\frac{9307894793789}{191102976000}s^2+\frac{206124003456599}{573308928000}st+\frac{21562322533673}{143327232000}t^2\Big]$

graphs	(divergence) $\times\, u/(-i/(4\pi)^8\langle 12\rangle^2[34]^2 st A^{\text{tree}}(\frac{\kappa}{2})^{10})$
1–30	$\frac{1}{\epsilon^4}\left[\frac{1052159}{995328}s^2+\frac{509789}{331776}st-\frac{121001}{497664}t^2\right]+\frac{1}{\epsilon^3}\left[\frac{9042569}{1474560}s^2+\frac{34360945}{1327104}st+\frac{73518401}{13271040}t^2\right] +\frac{1}{\epsilon^2}\Big[\zeta_3\left(-\frac{11443919}{2764800}s^2+\frac{32520079}{552960}st+\frac{5836531}{230400}t^2\right)+\zeta_2\left(\frac{1052159}{497664}s^2+\frac{509789}{165888}st-\frac{121001}{248832}t^2\right) -\mathrm{S2}\left(\frac{637991}{6144}s^2+\frac{10978729}{27648}st+\frac{5080825}{55296}t^2\right)+\left(\frac{270806866183}{7166361600}s^2+\frac{89848068067}{597196800}st+\frac{218093645149}{7166361600}t^2\right)\Big] +\frac{1}{\epsilon}\Big[\zeta_5\left(\frac{100843}{360}s^2+\frac{17118043}{30720}st-\frac{30266471}{92160}t^2\right)+\zeta_4\left(\frac{11435323}{614400}s^2+\frac{232002227}{1843200}st+\frac{22211783}{460800}t^2\right) +\zeta_3\left(\frac{223300432349}{3359232000}s^2-\frac{178732984847}{716636160}st+\frac{951659436383}{53747712000}t^2\right) -\zeta_2\left(\frac{5492357}{245760}s^2+\frac{53468887}{663552}st+\frac{129714599}{6635520}t^2\right)+\mathrm{T1ep}\left(-\frac{637991}{82944}s^2-\frac{10978729}{373248}st-\frac{5080825}{746496}t^2\right) +\mathrm{S2}\left(-\frac{5700088747}{3686400}s^2-\frac{69470348491}{16588800}st-\frac{713512871}{6635520}t^2\right)+\mathrm{D6}\left(-\frac{357421}{43200}s^2-\frac{2891743}{230400}st-\frac{470219}{138240}t^2\right) -\frac{3571506237341}{28665446400}s^2-\frac{1611591325291}{5971968000}st+\frac{2301084608777}{143327232000}t^2\Big]$
31–60	$\frac{1}{\epsilon^4}\left[-\frac{150715}{82944}s^2-\frac{668333}{221184}st-\frac{7213}{1990656}t^2\right]+\frac{1}{\epsilon^3}\left[-\frac{68021833}{13271040}s^2-\frac{36852103}{1327104}st-\frac{298377299}{39813120}t^2\right] +\frac{1}{\epsilon^2}\Big[\zeta_3\left(-\frac{36448033}{2764800}s^2-\frac{455889533}{2764800}st-\frac{82059281}{1382400}t^2\right)+\zeta_2\left(-\frac{150715}{41472}s^2-\frac{668333}{110592}st-\frac{7213}{995328}t^2\right) +\mathrm{S2}\left(\frac{13910839}{165888}s^2+\frac{1340033}{4096}st+\frac{26303855}{331776}t^2\right)-\frac{68286245653}{2388787200}s^2-\frac{20649690431}{119439360}st-\frac{351701043553}{7166361600}t^2\Big] +\frac{1}{\epsilon}\Big[\zeta_5\left(-\frac{2362679}{9216}s^2-\frac{178668311}{92160}st-\frac{1268313}{10240}t^2\right)+\zeta_4\left(-\frac{124344121}{1843200}s^2-\frac{491722333}{1843200}st-\frac{68141309}{921600}t^2\right) -\zeta_3\left(\frac{630084012997}{53747712000}s^2-\frac{1250670277213}{663552000}st-\frac{6913218302303}{13436928000}t^2\right) +\zeta_2\left(\frac{352368061}{19906560}s^2+\frac{35509679}{663552}st+\frac{227699801}{19906560}t^2\right)+\mathrm{T1ep}\left(\frac{13910839}{2239488}s^2+\frac{1340033}{55296}st+\frac{26303855}{4478976}t^2\right) +\mathrm{S2}\left(\frac{188312318729}{99532800}s^2+\frac{110749829741}{16588800}st+\frac{5056299197}{3981312}t^2\right)+\mathrm{D6}\left(\frac{1220779}{76800}s^2+\frac{44791}{6912}st-\frac{1159831}{230400}t^2\right) +\frac{2755666297013}{28665446400}s^2+\frac{5622513975899}{35831808000}st-\frac{196197363193}{1769472000}t^2\Big]$
61–82	$\frac{1}{\epsilon^4}\left[\frac{756421}{995328}s^2+\frac{985421}{663552}st+\frac{163739}{663552}t^2\right]+\frac{1}{\epsilon^3}\left[-\frac{1670161}{1658880}s^2+\frac{415193}{221184}st+\frac{4863881}{2488320}t^2\right] +\frac{1}{\epsilon^2}\Big[\zeta_3\left(\frac{110861}{6400}s^2+\frac{16293841}{153600}st+\frac{9408019}{276480}t^2\right)+\zeta_2\left(\frac{756421}{497664}s^2+\frac{985421}{331776}st+\frac{163739}{331776}t^2\right) +\mathrm{S2}\left(\frac{1657459}{82944}s^2+\frac{7734025}{110592}st+\frac{4181095}{331776}t^2\right)-\frac{8243516153}{895795200}s^2+\frac{558349337}{24883200}st+\frac{11133949867}{597196800}t^2\Big] +\frac{1}{\epsilon}\Big[\zeta_5\left(-\frac{1094509}{46080}s^2+\frac{63657091}{46080}st+\frac{5210161}{11520}t^2\right)+\zeta_4\left(\frac{11254769}{230400}s^2+\frac{129860053}{921600}st+\frac{23717743}{921600}t^2\right) -\zeta_3\left(\frac{2745647960587}{53747712000}s^2+\frac{3654260151947}{2239488000}st+\frac{5720906529119}{10749542400}t^2\right) +\zeta_2\left(\frac{11564107}{2488320}s^2+\frac{2244901}{82944}st+\frac{40360999}{4976640}t^2\right)+\mathrm{T1ep}\left(\frac{1657459}{1119744}s^2+\frac{7734025}{1492992}st+\frac{4181095}{4478976}t^2\right) +\mathrm{S2}\left(-\frac{420043}{1215}s^2-\frac{825589625}{331776}st-\frac{5785239343}{4976640}t^2\right)+\mathrm{D6}\left(-\frac{210731}{27648}s^2+\frac{4196129}{691200}st+\frac{1457647}{172800}t^2\right) +\frac{33976742047}{1194393600}s^2+\frac{4046536311847}{35831808000}st+\frac{212357840779}{2239488000}t^2\Big]$

Adds up to zero: no divergence. Enhanced cancellations!

Enhanced Cancellations

Many of you are saying: "There has to be a better way"

- **Yes, take it as a challenge.**
- **Enhanced cancellations, so standard arguments will *not* work.**

Two approaches:

- **Study the cancellations.**
- **Study divergences to understand why they occur.**

$N = 4$ sugra complicated because of high loop order.
Need simpler theory to analyze.

That simpler theory is Einstein gravity:

2 loop divergence has surprising similarities to 4 loop divergence of $N = 4$ sugra.

One Loop Pure Gravity

Standard finiteness argument for 1 loop finiteness or pure gravity:

't Hooft and Veltman (1974)

$\cancel{R^2}$ $\cancel{R_{\mu\nu}^2}$

$\cancel{R_{\mu\nu\rho\sigma}^2}$

Counterterms vanish by equation of motion and can be eliminated by field redefinition.

In *D* = 4 flat space Gauss-Bonnet theorem eliminates Riemann-square term.

Pure gravity divergence with nontrivial topology:

$$\mathcal{L}^{\rm GB} = \frac{1}{(4\pi)^2}\frac{53}{90\epsilon}(R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\rho\sigma}^2)$$

$$\int d^4x\,\sqrt{-g}(R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\rho\sigma}^2) = 32\pi^2\chi$$

Capper and Duff (1974)
Tsao (1977); Critchley (1978)
Gibbons, Hawking, Perry (1978)
Goroff and Sagnotti (1986)
Bornsen and van de Ven (2009)

This divergence is behind conformal or trace anomaly.

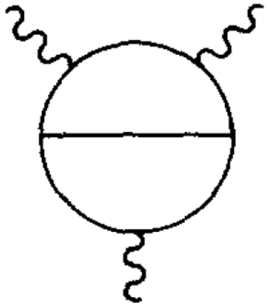
Euler characteristic vanishes in flat space. 't Hooft and Veltman (1974)

Dimensional regularization makes it subtle. Capper and Kimber (1980)

Two-Loop Pure Gravity

Goroff and Sagnotti (1986); Van de Ven (1992)

Using standard MS-bar prescriptions Goroff and Sagnotti showed Einstein gravity diverges at 2 loops.



$$\mathcal{L}^{R^3} = \frac{209}{2880}\frac{1}{(4\pi)^4}\frac{1}{2\epsilon}\sqrt{-g}\,R^{\alpha\beta}{}_{\gamma\delta}R^{\gamma\delta}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta}$$

$D = 4 - 2\epsilon$

Unclear in their calculation what role Gauss-Bonnet operator plays in this. Subtractions done integral by integral.

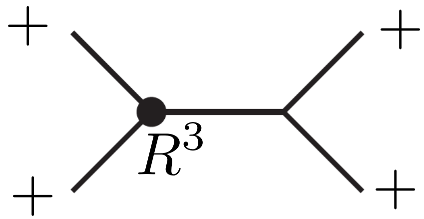
30 years later we have more potent methods:

- **Get complete amplitudes not just divergent part.**
- **Track the role of Gauss-Bonnet divergence.**
- **Demonstrate that topological Gauss-Bonnet term does contribute in flat-space perturbation theory by two loops. Evanescent operator.**

Two Loop Identical Helicity Amplitude

ZB, Cheung, Chi, Davies, Dixon and Nohle (to appear)

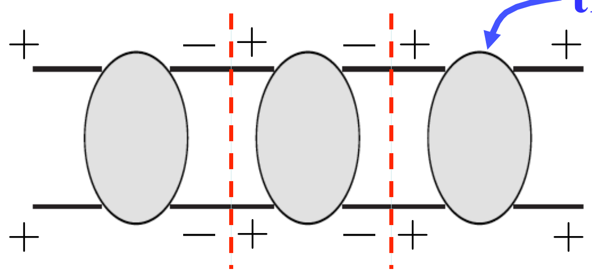
Standard counterterm arguments might have you believe that R^3 divergence is generic with no special properties. This is false.



Pure gravity identical helicity amplitude sensitive to Goroff and Sagnotti divergence.

Curious feature:

tree amplitude vanishes



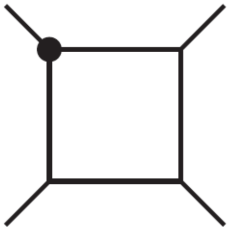
- **Unitarity cut vanishes for four-dimensional loop momenta.**
- **Nonvanishing because of ϵ-dimensional loop momenta.**

A surprise:

- **Divergence is *not* generic but is tied to anomalous behavior.**
- **By anomalous behavior we mean that divergence would vanish, if not for 0/0 behavior in dim. reg. Just like chiral anomaly.**

Gauss-Bonnet

Gauss-Bonnet insertion:



$$\mathcal{L}^{\rm GB} = \alpha(R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\rho\sigma}^2)$$

- **total derivative in $D = 4$**
- **evanescent operator**

The 2-loop 4-point identical helicity amplitude shifts by:

$$\Delta M_4^{(2)} = -\alpha\left(\frac{\kappa}{2}\right)^6 \frac{i}{(4\pi)^2}\, stu\, \mathcal{T}^2\, \frac{26}{15} \qquad \mathcal{T} \equiv \frac{[12][34]}{\langle 12\rangle\langle 34\rangle}$$

For one-loop Einstein gravity counterterm:

$$\alpha = \frac{1}{4\pi^2}\frac{53}{90\epsilon}$$

UV divergence

$$\mathcal{M}_4^{(1)\rm GB} = \left(\frac{\kappa}{2}\right)^6 \frac{i}{(4\pi)^4}\mathcal{T}^2\Big[stu\left(\frac{689}{675\epsilon} + \frac{2968}{4500}\right) - \frac{689}{2025}\left(s^3\ln(-s) + \text{perms}\right)\Big]$$

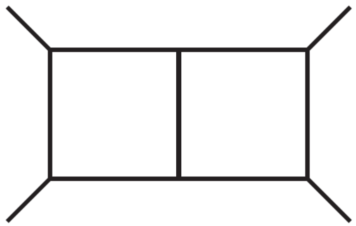
Need evanescent Gauss-Bonnet counterterm to reproduce G&S divergence

- ***D* = 4 topological term contributes at two loops even in flat space!**
- **It plays an important role in UV structure.**

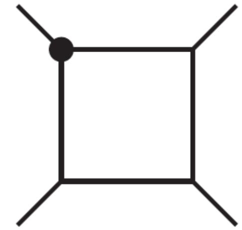
Two-Loop Identical-Helicity Amplitude

ZB, Cheung, Chi, Davies, Dixon and Nohle (to appear)

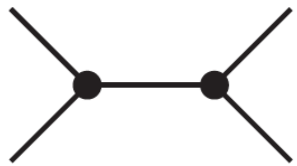
bare:



single GB counterterm:



double GB counterterm:



Subtract known IR singularity

Weinberg; Naculich, Nastase, Schnitzer; Akhoury, Saotome, Sterman

Full two-loop identical-helicity amplitude:

Goroff and Sagnotti divergence

$$\mathcal{M}_4^{(2)} = \left(\frac{\kappa}{2}\right)^6 \frac{i}{(4\pi)^4} \mathcal{T}^2 \Bigg[\left(\frac{209}{24\epsilon} + \frac{117617}{21600} \right) stu + \left(\frac{1}{120}(s^2+t^2+u^2)s\ln^2(-s) - \frac{1}{60}s(s^2-6tu)\ln(-s) + \text{perms} \right) \Bigg]$$

$$\mathcal{T} \equiv \frac{[12][34]}{\langle 12\rangle\langle 34\rangle}$$

This is pure Einstein gravity. No Gauss-Bonnet modification.

Key Points

UV divergence of pure Einstein gravity surprising structure:

— **One loop finiteness part of a pattern that we see in $N = 4$, 5 sugra: "*enhanced cancellations*".**

— **2 loop divergence arises from 0/0 effect. Reminiscent of anomalies and structure of $L = 4$, $N = 4$ sugra divergence.**

— **Topological Gauss Bonnet has nontrivial contribution by two loops even in flat space. Counterterm.**

Surprise: **Both known divergences in pure (super)gravity are linked to 0/0 anomaly-like behavior.**

Hope to have much more to say about these points in near future.

ZB, Cheung, Chi, Davies, Dixon and Nohle (to appear)

Question: **Can we avoid divergences? $N \geq 5$ supergravities should not have this anomaly-like behavior.**

Summary

- **At sufficiently high loop orders in *any* (super)gravity theory covariant diagrams will be UV divergent:**
  - **— Björnsson and Green world-line formalism.**
  - **— maximal cut power counting.**
- **Phenomenon of "enhanced cancellations": Divergences cancel between diagrams. Demonstrated in $N = 4, 5$ sugra at 3, 4 loops.**
- **$N = 4$ sugra diverges at 4 loops, but curious structure tied to duality symmetry anomaly.**
- **Pure gravity:**
  - **— Well known 1 loop finiteness is enhanced cancellation.**
  - **— By two loops topological Gauss-Bonnet term becomes nontrivial.**
- **Both explicitly known pure (super) gravity divergences appear as a 0/0 effect. Very much like an anomaly. Don't expect $N \geq 5$ sugra to have this.**

We can expect many more surprises as we probe gravity theories using modern perturbative tools.