Eurostrings, Cambridge, March 2015

2-D CFT and 3-D Gravity

A Supersymmetric Analysis

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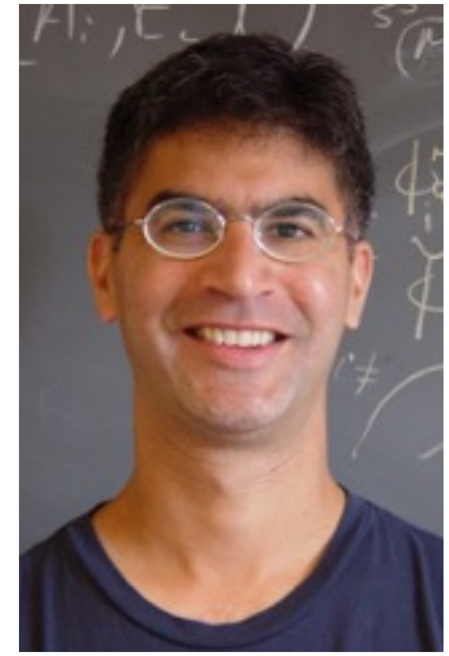
- Background & Basic Idea
- Some Details
- Examples
- Summary & Discussions

based on *arXiv:1503.14800* with

Nathan Benjamin



Shamit Kachru

Greg Moore

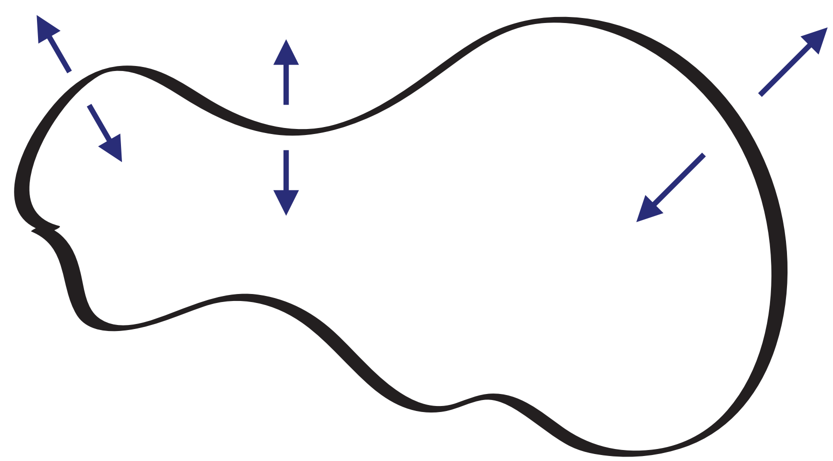
Natalie Paquette

as well as Hartman–Keller–Stoica 1405.5137

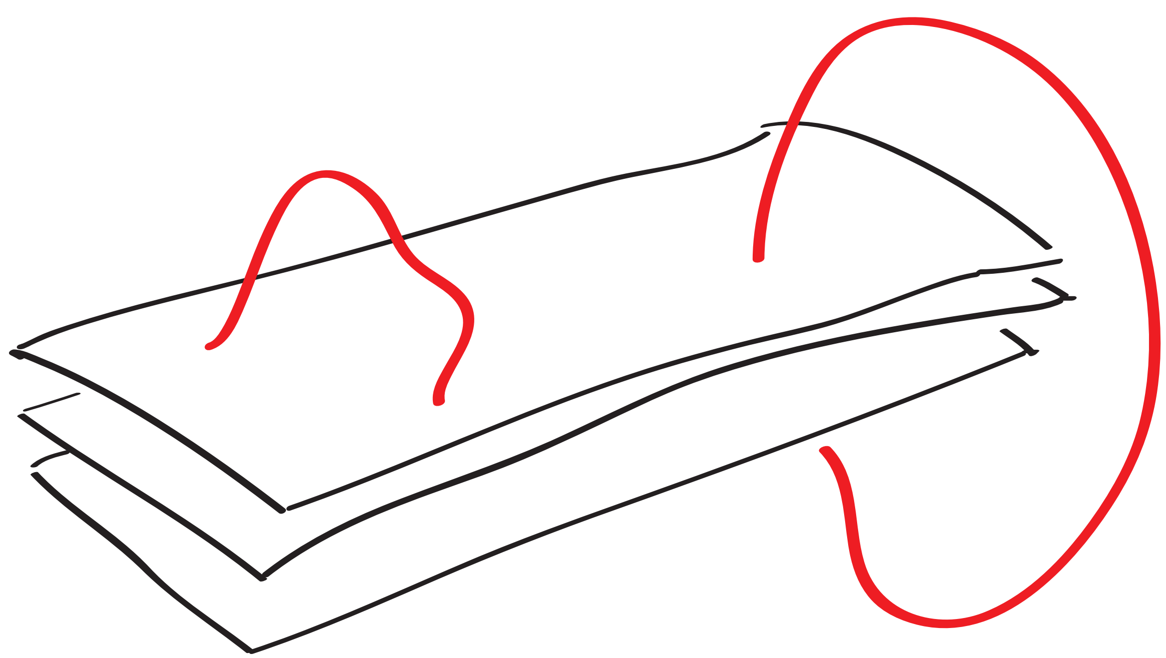
The Question:

What is Quantum Gravity?

Potential Answer:
String Theory



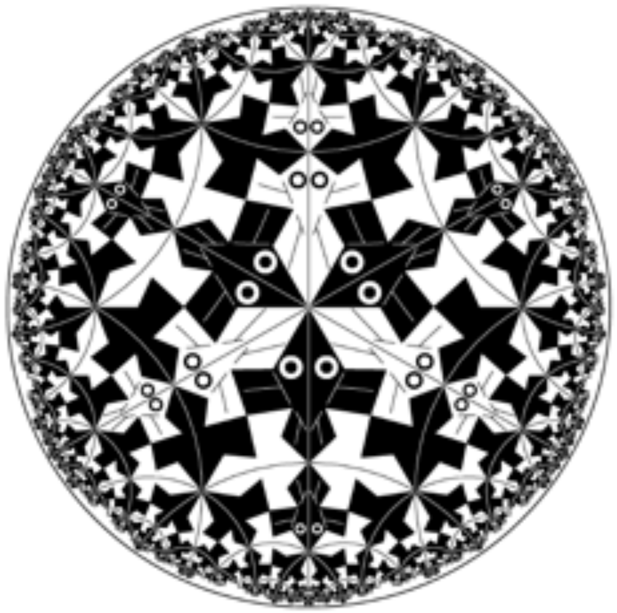
How about the non-perturbative aspects?

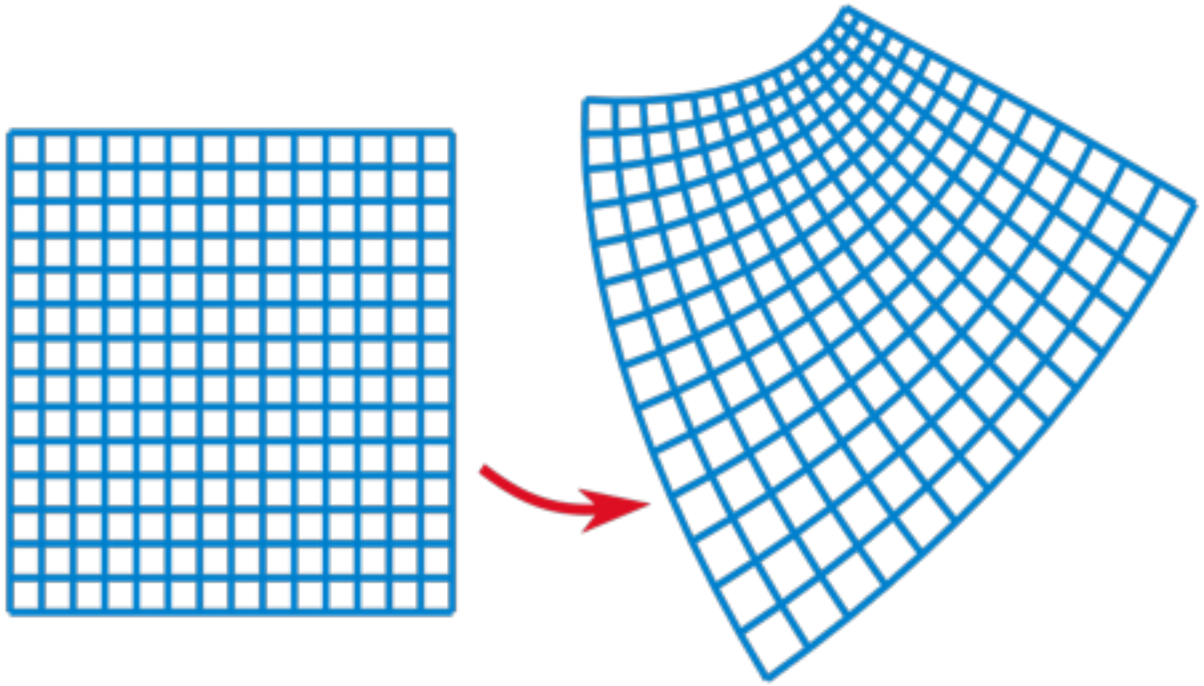


Potential Partial Answer:

AdS/CFT Correspondence

In some cases, a *CFT in d-dimensions* is equivalent to a *gravitational theory in the background of (d+1)-dimensional AdS space.*





When does AdS/CFT apply?

Q: What types of 2d CFTs admit a dual weakly coupled gravity description?

central charge ~ AdS radius in Planck units

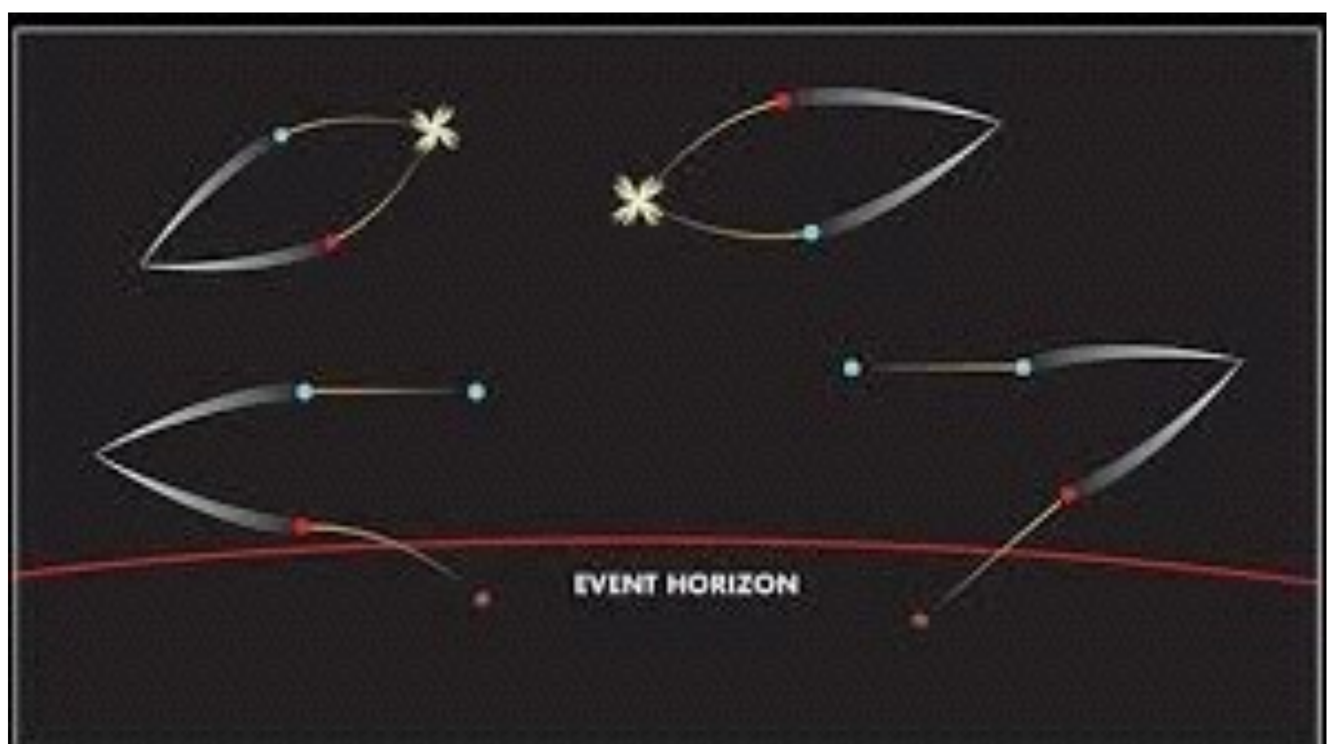
$$c = \frac{3\,\ell_{AdS}}{2\,G_N}$$

weakly coupled gravity ⇒ $c \gg 1$

Q: Consider families of 2d CFT with a $c \to \infty$ limit. What are the necessary conditions for such a family to admit a weakly coupled gravity description at $c \to \infty$?

Earmark of a *Weakly Coupled* Gravitational Theory

- Black Hole Solutions
- Hawking Radiation suggests that black holes are thermodynamical objects.



Weakly Coupled Gravitational Theory
⇒ Black Hole Thermodynamics

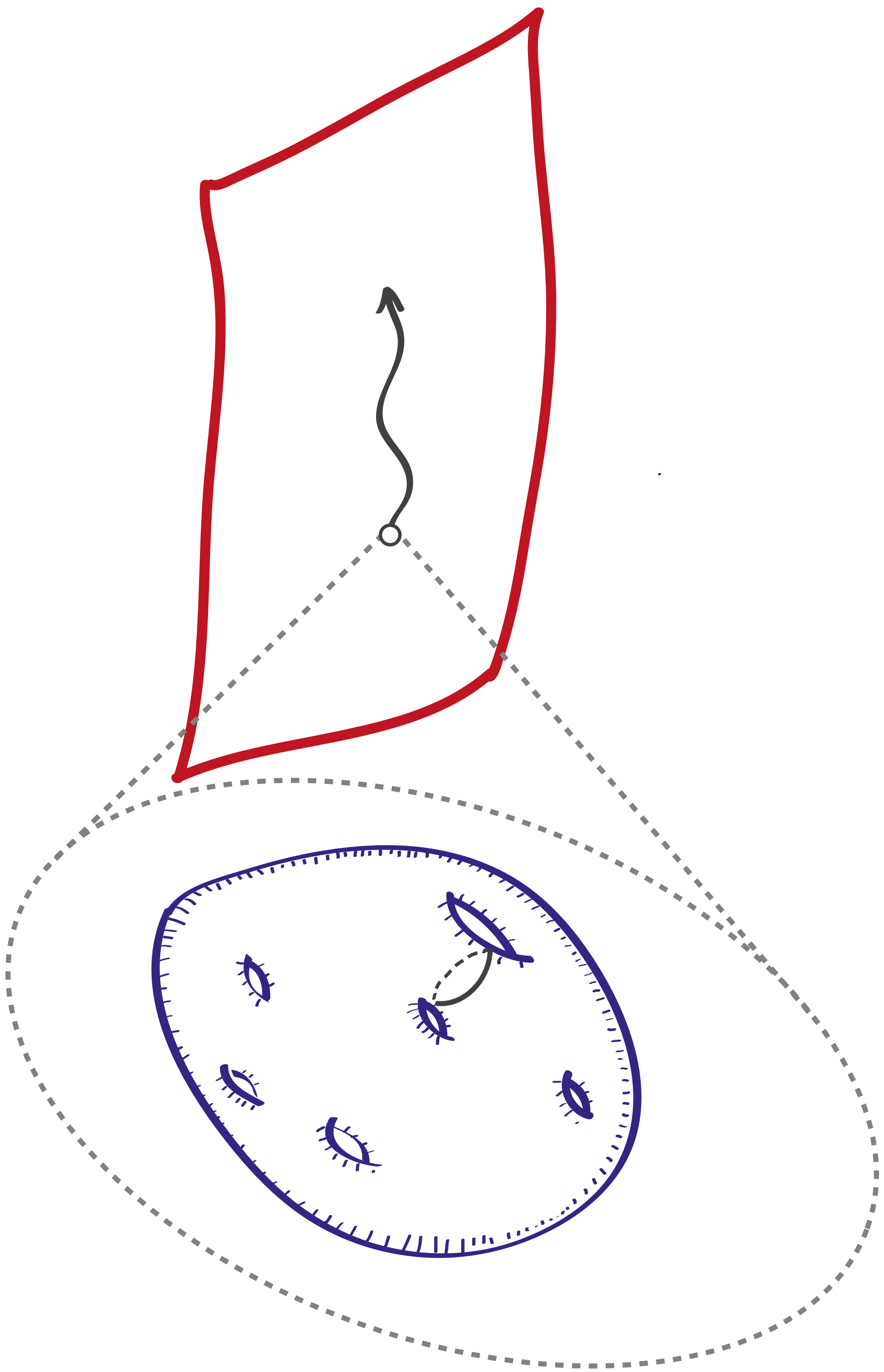
$$\boxed{S = \frac{\text{Area}}{4G_N}}$$

Bekenstein–Hawking Entropy

It is *universal* and does not depend on the details of the theory (matter content, UV completion, etc.).

This simple formula will be our main input.

String Theory and BH Entropy



String Theory successfully explain the BH entropy in various situations.

The D-branes making up the black holes are described by a *2d CFT* in the appropriate regime.

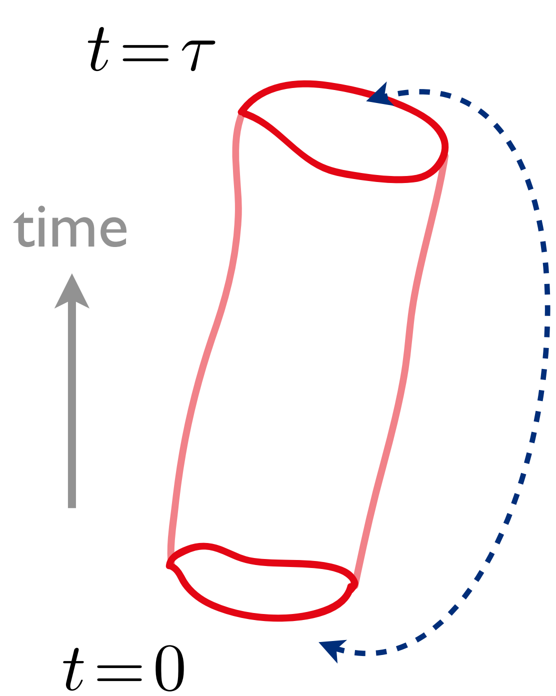
The black hole entropy can be then explained as the number of quantum states with a given mass and charges in the CFT.

[see for instance '96 Strominger–Vafa]

In fact, the derivation only relies on certain very *universal* feature of the 2d CFT.

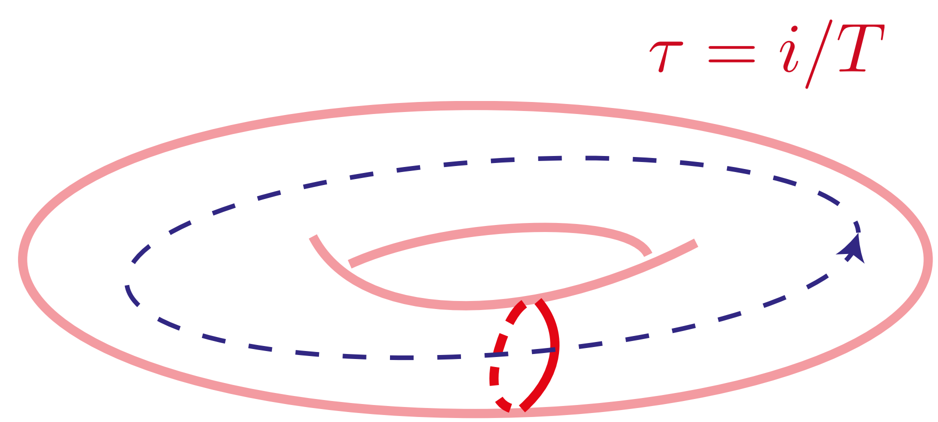
2d CFT Partition Functions are nice.

They are *Modular Forms*.



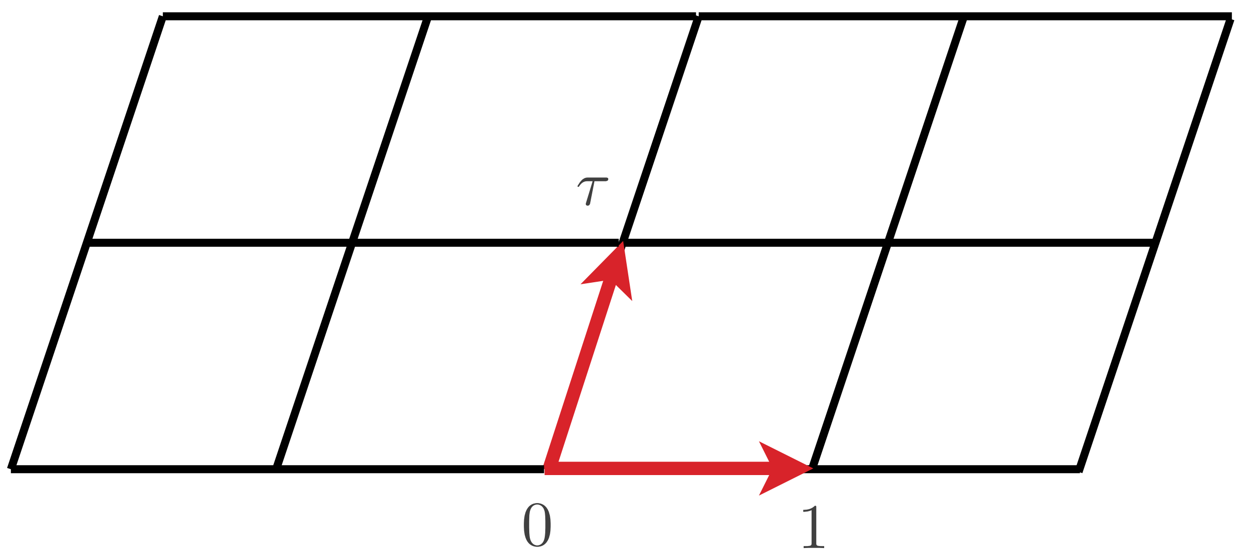
partition function

$$Z(\tau) = \mathrm{Tr}_{\mathcal{H}}\, e^{2\pi i \tau \hat{H}}$$



The partition functions are computed by identifying the initial and final time. This turns the cylinder into a torus. Moreover, conformal symmetries guarantee that we only care about the shape (complex structure) of the torus.

Torus and Modular Symmetries

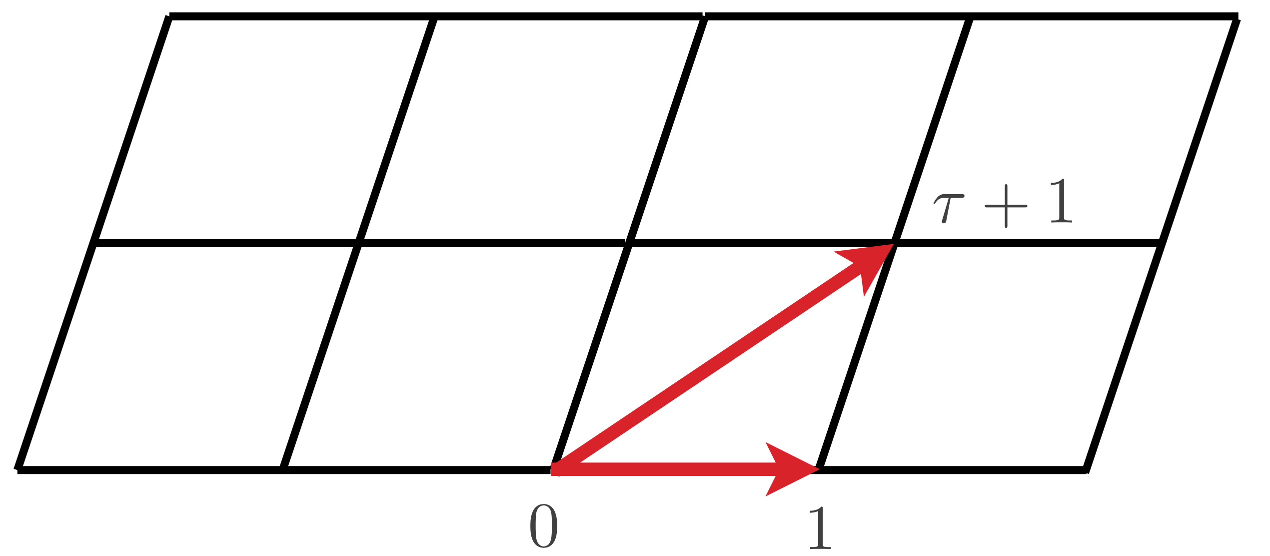


The $SL_2(Z)$ modular action

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

leaves the torus the same.

Torus and Modular Symmetries

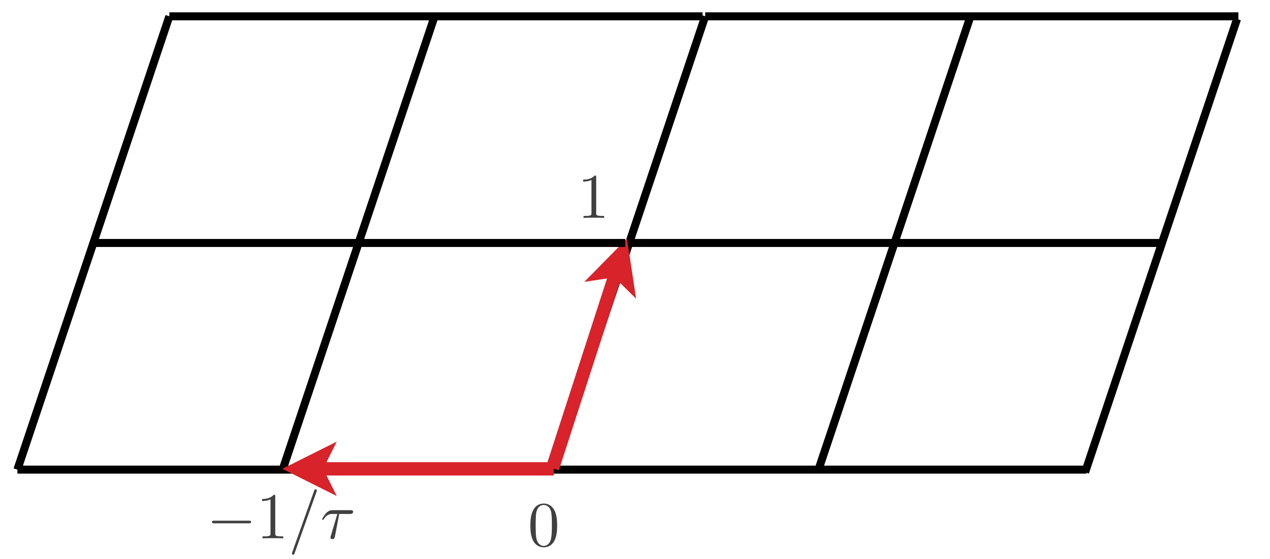


The $SL_2(Z)$ modular action generated by

$$\tau \mapsto \tau + 1 \; , \; \tau \mapsto -1/\tau$$

leaves the torus the same.

Torus and Modular Symmetries



The $SL_2(Z)$ modular action generated by

$$\tau \mapsto \tau + 1 \, , \; \tau \mapsto -1/\tau$$

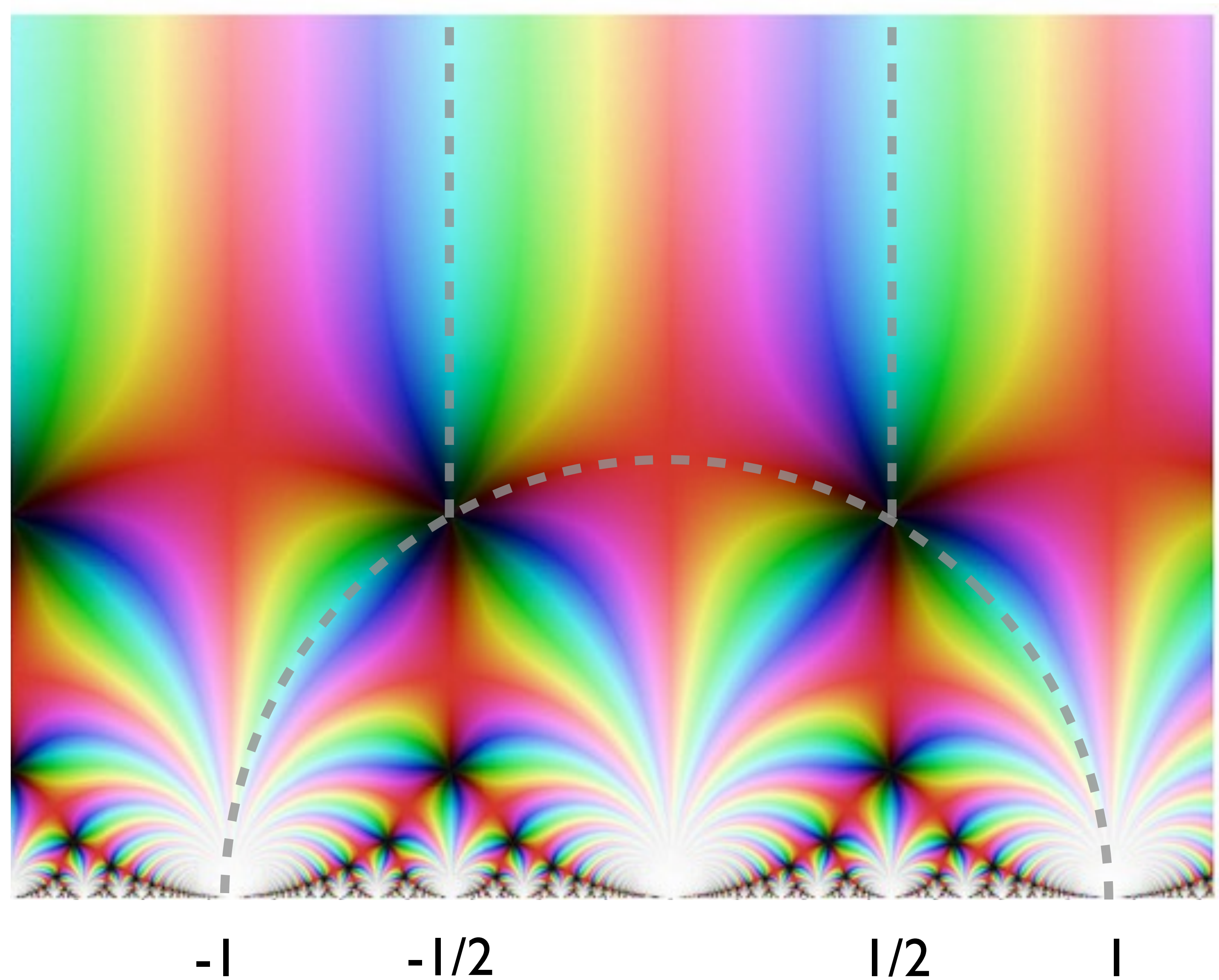
leaves the torus the same.

Modular Forms

are holomorphic functions transforming nicely under the modular group $SL_2(Z)$.

Example: the *J*-function

$$J(\tau) = J(\tau + 1) = J(-1/\tau)$$



Modularity ⇒ “Cardy Formula”

which dictates the growth of density of states with energy.

$$Z(\tau) = \sum_{E \geq -c/24} c_E \, e^{2\pi i \tau E} = Z(-1/\tau)$$

$$\Leftrightarrow c_E = \oint d\tau e^{-2\pi i \tau E} Z(\tau) = \oint d\tau e^{-2\pi i \tau E} Z(-1/\tau)$$

$$= \oint d\tau e^{-2\pi i \tau E} (e^{\frac{2\pi i}{\tau} \frac{c}{24}} + \dots)$$

ignore these

$$\tau_{saddle} = i\sqrt{\frac{c}{24E}} \Rightarrow \boxed{S(E) = log c_E \sim 2\pi\sqrt{\frac{cE}{6}}}$$

This is the BH entropy for extremal BTZ black holes with $r_+ = r_- = 2\sqrt{G_N E}$!

⇒ a microscopic explanation of BH entropy.

Black Hole vs. Cardy Regime

Q: Does the Cardy formula imply that *all* 2d CFT's have a spectrum compatible with BH entropy?

A: NO!

Range of Validity of Cardy Formula: fixed *c*, *E/c*≫ *1*

recall:

$$c_E = \oint d\tau e^{-2\pi i \tau E}\left(e^{\frac{2\pi i}{\tau}\frac{c}{24}} + \dots\right)$$

these were ignored

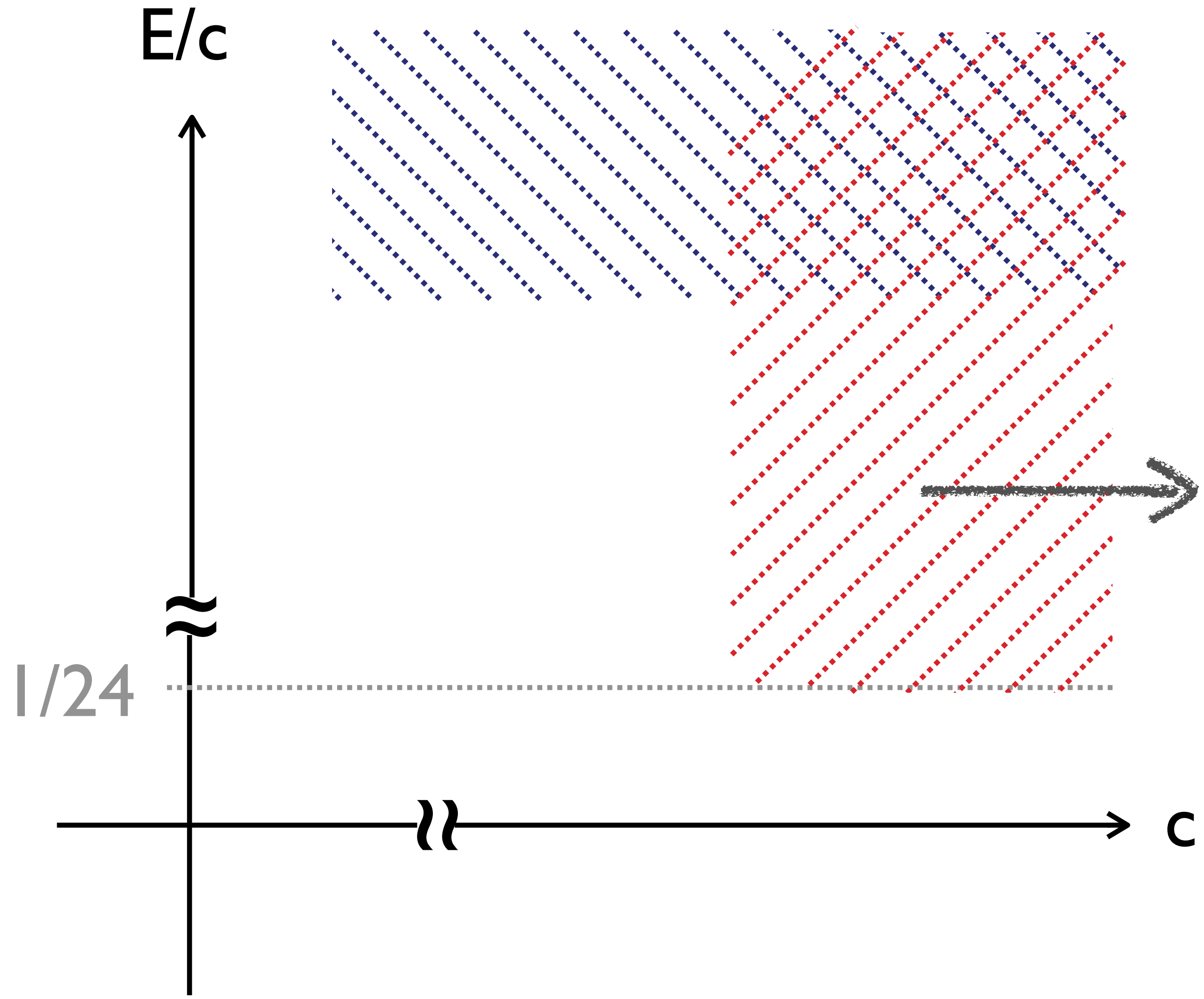
$$\tau_{saddle} = i\sqrt{\frac{c}{24E}} \Rightarrow \log c_E \sim 2\pi\sqrt{\frac{cE}{6}}$$

Black Hole vs. Cardy Regime

Range of Validity of Cardy Formula: fixed c, $E/c \gg 1$

BH entropy formula should be valid at: $c \gg 1$, $E > c/24$

when black holes start to be thermodynamically stable



A weakly-coupled gravity dual requires that the BH/Cardy formula to be valid outside the Cardy regime!

Main Idea

BH entropy, $E > \frac{c}{24}$

$$\Updownarrow$$

a universal "weak gravity polar bound" for the density of states with energy

$$-\frac{c}{24} < E < 0$$

$$\Updownarrow$$

A (Hawking-Page) first-order *phase transition*

AdS Space | BTZ black holes

$T = 1/2\pi$ ⟶

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Focussing on 2d SCFT

to obtain a universal bound that is:

- Applicable

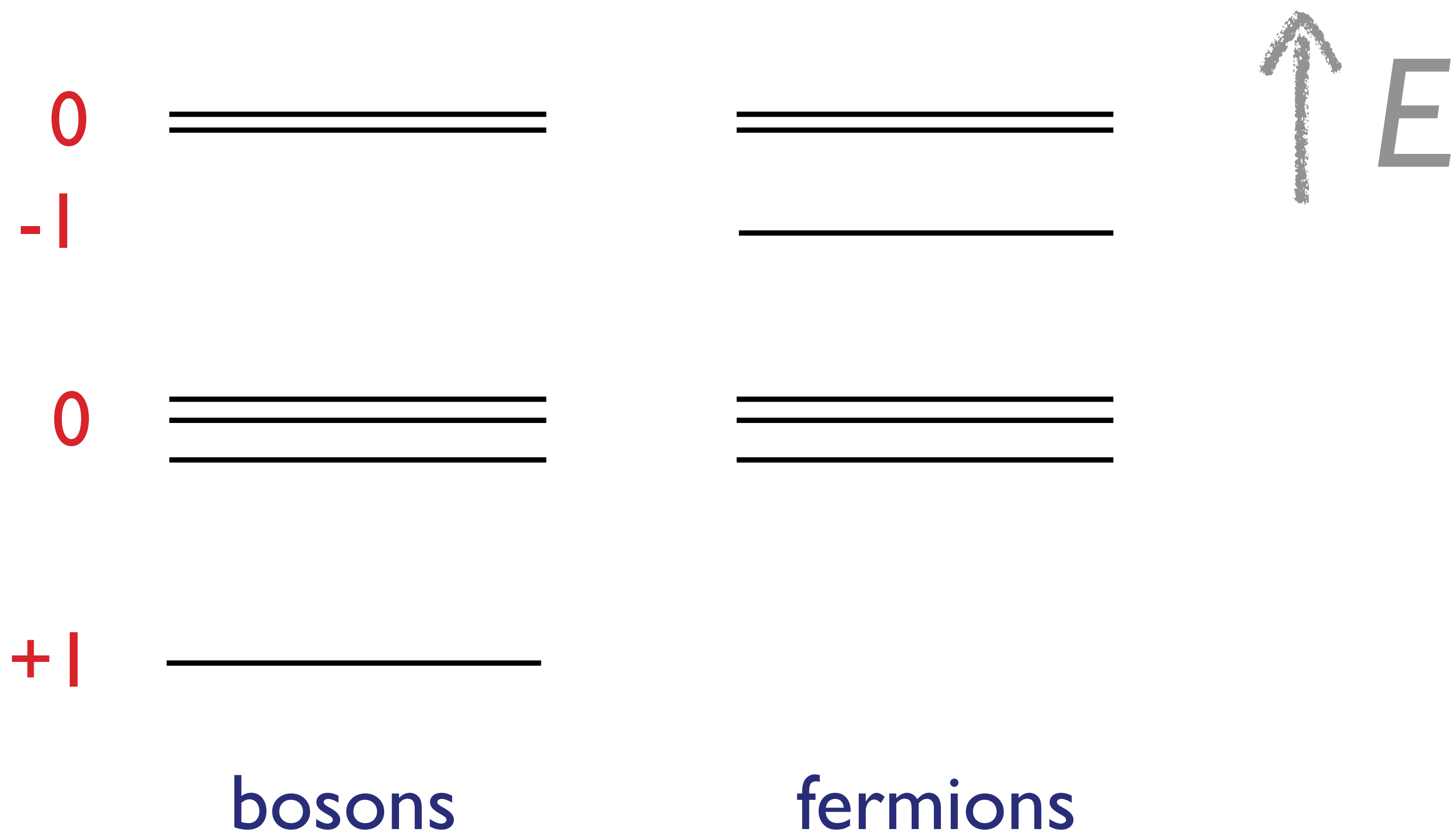
Theoretically, supersymmetry lends great computational control.
For instance, so far practically all string theoretic microscopic explanations for black hole entropy apply to supersymmetric theories.

- Powerful

Supersymmetric theories produce mathematical functions with nice properties (eg. holomorphicity).
Together with these properties, our *weak gravity bound* determines a great deal.

Supersymmetric Index

counting states with signs $(-1)^F$



Such an index is *rigid*:

It is invariant under deformation for compact theories
and hence more computable.

!!We assume a "non-cancellation hypothesis"!!

SCFT's with $N \geq (2,2)$ Susy

The symmetry algebra of the theory contains the → and ← super-Virasoro algebra, with bosonic generators L_n and J_n, $n \in Z$

These include the zero modes L_0 *and* J_0, satisfying $[L_0, J_0]=0$

Their eigenvalues are useful quantum numbers:

conformal weight~energy, U(1) charge

Elliptic Genus

counts susy states.

$$Z(\tau, z) = \mathrm{Tr}_{\mathcal{H}_{RR}} (-1)^F e^{2\pi i \tau (L_0 - c/24)} \, e^{2\pi i z J_0}$$
$$= \sum_{n \geq 0, \ell} \underline{c(n,\ell)} \, e^{2\pi i \tau n} \, e^{2\pi i z \ell}$$

['86 Schellekens–Warner, Witten]

↓

index counting number of states with a given conformal weight n and $U(1)$ charge ℓ

Note: EG can be defined more generally.

Symmetries of Elliptic Genus

$$Z(\tau, z) = \mathrm{Tr}_{\mathcal{H}_{RR}}(-1)^F e^{2\pi i\tau(L_0 - c/24)}\, e^{2\pi i z J_0} = \sum_{n\geq 0,\ell} c(n,\ell)\, e^{2\pi i\tau n}\, e^{2\pi i z\ell}$$

Modular Symmetry

$$Z(\tau, z) = e^{-2\pi i \frac{c}{6}\frac{z^2}{\tau}} Z(-\frac{1}{\tau}, \frac{z}{\tau})$$

Spectral Flow Symmetry

$$Z(\tau, z) = Z(\tau, z+1) = e^{2\pi i \frac{c}{6}(\tau + 2z)} Z(\tau, z+\tau)$$

$\Rightarrow$ $Z(\tau, z)$ is a *Jacobi form* of index c/6.

Spectral Flow Sym. of Elliptic Genus

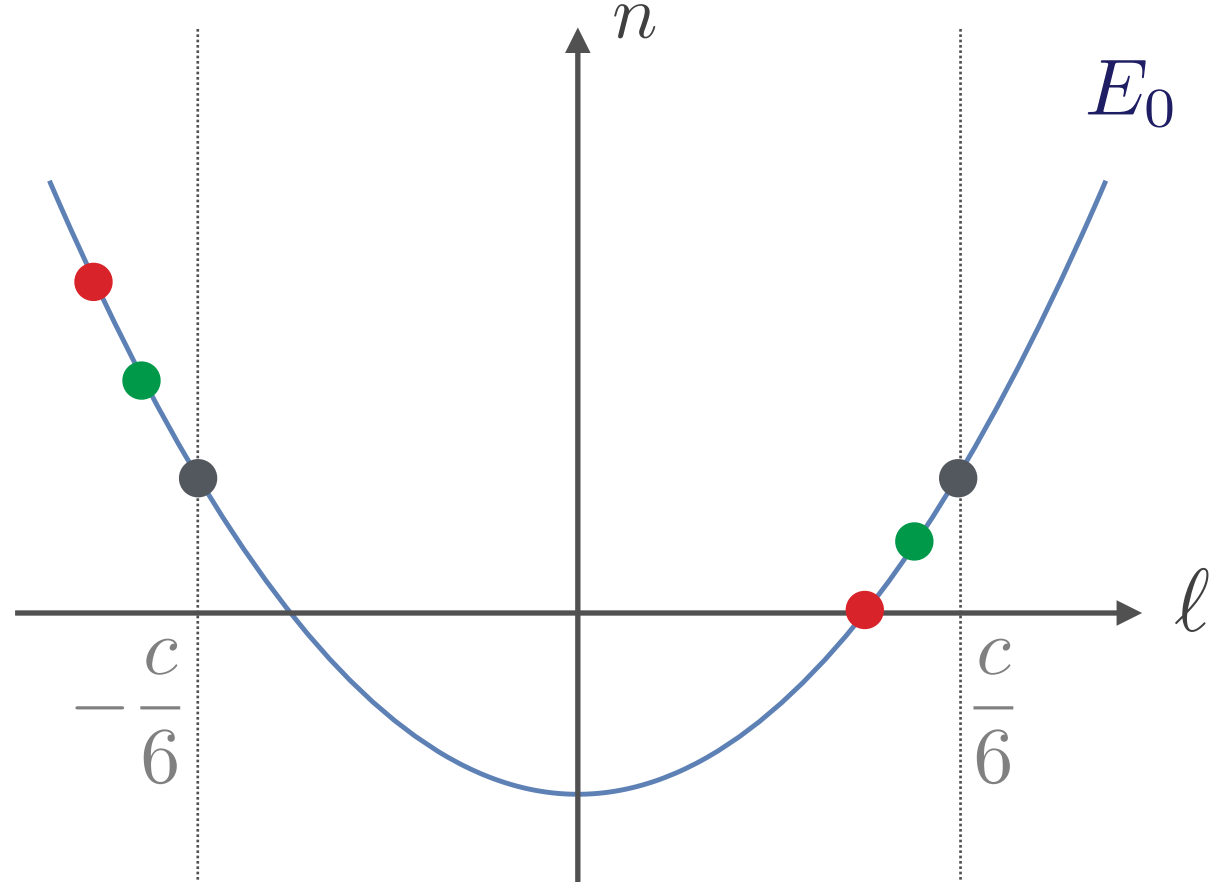
The spectrum is invariant when shifting L_0 *and* J_0 while keeping inv.

$$L_0^{red} = L_0 - \frac{c}{24} - \frac{3}{2c}J_0^2$$

More precisely, $c(n,\ell) = c(n',\ell')$ when

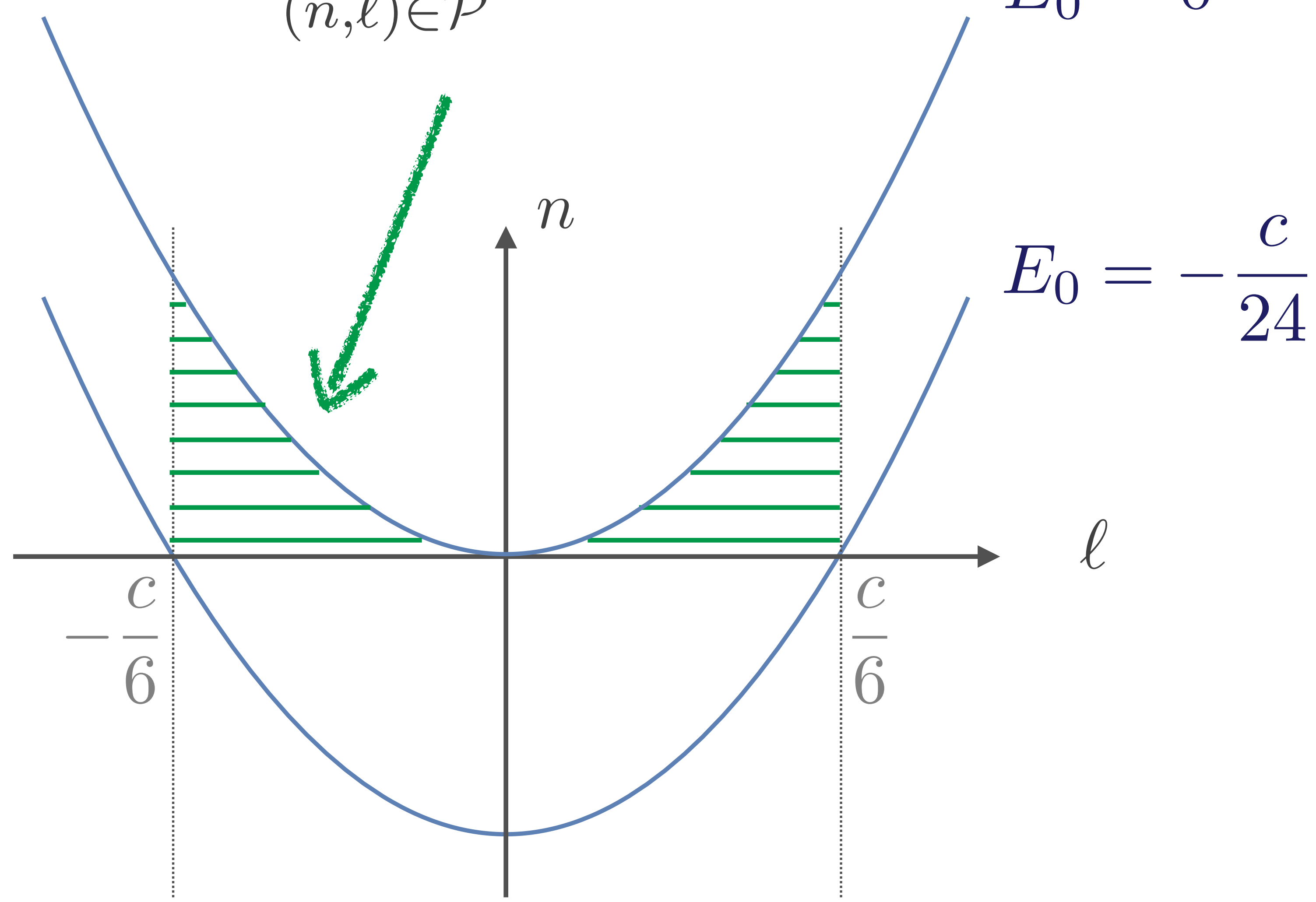
$$E_0 = n - \frac{3}{2c}\ell^2 = n' - \frac{3}{2c}\ell'^2, \quad \ell = \ell' \pmod{c/3}$$

It corresponds to a "large gauge transformation" on the gravity side.



Polar Part of Elliptic Genus

$$Z_P(\tau, z) = \sum_{\substack{n \geq 0, \ell \\ (n,\ell) \in \mathcal{P}}} c(n,\ell)\, e^{2\pi i \tau n}\, e^{2\pi i z \ell}$$ ~ c^3 terms



Math: The polar part Z_P determines the whole function Z!

Physics: These are the low energy states whose spectrum we will now constrain.

From BH Entropy to Free Energy

The Entropy

$$S(n,\ell) = 2\pi\sqrt{\frac{cE_0}{6}} = 2\pi\sqrt{\frac{c}{6}(n - \frac{3}{2c}\ell^2)}$$

['98 Cvetic–Larsen]

An Estimate

$$Z(\tau_\beta, z_{\beta,\mu}) = \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{2\pi i\tau_\beta(L_0 - c/24)} e^{2\pi i z_{\beta,\mu} J_0}$$

$\tau_\beta = i\frac{\beta}{2\pi},\ z_{\beta,\mu} = -i\frac{\beta\mu}{2\pi}$

$\beta, \mu \in \mathbb{R}$

$$= \sum_{\text{microstates}} e^{\beta(\mu Q - E)}$$

replacing sum with integral at large c

$$\sim \int dn \int d\ell\ e^{2\pi i\tau_\beta n}\, e^{2\pi i z_{\beta,\mu}\ell}\, e^{S(n,\ell)}$$

From BH Entropy to Free Energy

Input — allowed corrections — valid when

$$S(n,\ell) = 2\pi\sqrt{\frac{cE_0}{6}} = 2\pi\sqrt{\frac{c}{6}(n - \frac{3}{2c}\ell^2)} \qquad +O(\log c) \qquad \boxed{E_0 > c/24}$$

$$Z(\tau_\beta, z_{\beta,\mu}) \sim \int dn \int d\ell\; e^{2\pi i \tau_\beta n}\, e^{2\pi i z_{\beta,\mu}\ell}\, e^{S(n,\ell)}$$

with $\tau_\beta = i\frac{\beta}{2\pi}, \;\; z_{\beta,\mu} = -i\frac{\beta\mu}{2\pi}$

saddle point at: $n = \frac{c}{6}\left(\frac{\pi^2}{\beta^2} + \mu^2\right), \;\; \ell = \frac{c\mu}{3} \Rightarrow E_0 = \frac{c}{24}(\frac{2\pi}{\beta})^2$

Output

Free Energy: $-\frac{1}{\beta}\log Z(\tau_\beta, z_{\beta,\mu}) = -\frac{c}{6}\left(\frac{\pi^2}{\beta^2} + \mu^2\right) + O(\log c)$ $\qquad \boxed{\beta < 2\pi}$

From Free Energy to WG Polar Bound

$$Z(\tau_\beta, z_{\beta,\mu}) = \exp\left(\frac{c}{6}\left(\frac{\pi^2}{\beta}+\beta\mu^2\right)+O(\log c)\right) \ , \quad \mathrm{Im}\tau_\beta < 1$$

HIGH T

BH

modular property

LOW T

polar states, gas of particles

$$= e^{-2\pi i \frac{c}{6}\frac{z_{\beta,\mu}^2}{\tau_\beta}} Z(-\frac{1}{\tau_\beta}, \frac{z_{\beta,\mu}}{\tau_\beta}) = \exp\left(\frac{c}{6}\beta\mu^2\right)\sum_{n,\ell} c(n,\ell) e^{-\frac{2\pi i}{\tau_\beta}n} e^{2\pi i \ell \frac{z_{\beta,\mu}}{\tau_\beta}} \ , \ \mathrm{Im}(-\frac{1}{\tau_\beta}) > 1$$

Comparing the 2 equations, we obtain the *weak gravity polar bound*

$\Leftrightarrow$ ground state dominance

$$\boxed{|c(n,\ell)| \leq exp\left(2\pi(n+\frac{c}{12}-\frac{|\ell|}{2})+O(\log c)\right)} \ , \quad -\frac{c}{24} < E_0 = n - \frac{3}{2c}\ell^2 < 0$$

From BH Entropy to WG Polar Bound

$$Z(\tau_\beta, z_{\beta,\mu}) \sim \int dn \int d\ell \; e^{2\pi i \tau_\beta n} \, e^{2\pi i z_{\beta,\mu}\ell} \, e^{S(n,\ell)}$$

↓ BH entropy formula for $S(n,\ell)$

$$-\frac{1}{\beta}\log Z(\tau_\beta, z_{\beta,\mu}) = -\frac{c}{6}\left(\frac{\pi^2}{\beta^2} + \mu^2\right) \;, \quad \beta < 2\pi$$

↓ use modular property

$$Z(\tau_\beta, z_{\beta,\mu}) = \exp\left(\frac{c}{6}\left(\frac{\pi^2}{\beta} + \beta\mu^2\right) + O(\log c)\right) \;, \quad \mathrm{Im}(\tau_\beta) < 1$$

$$= e^{-2\pi i \frac{c}{6}\frac{z_{\beta,\mu}^2}{\tau_\beta}} Z\left(-\frac{1}{\tau_\beta}, \frac{z_{\beta,\mu}}{\tau_\beta}\right) = \exp\left(\frac{c}{6}\beta\mu^2\right) \sum_{n,\ell} c(n,\ell) e^{-\frac{2\pi i}{\tau_\beta}n} e^{2\pi i \ell \frac{z_{\beta,\mu}}{\tau_\beta}} \;, \; \mathrm{Im}\left(-\frac{1}{\tau_\beta}\right) > 1$$

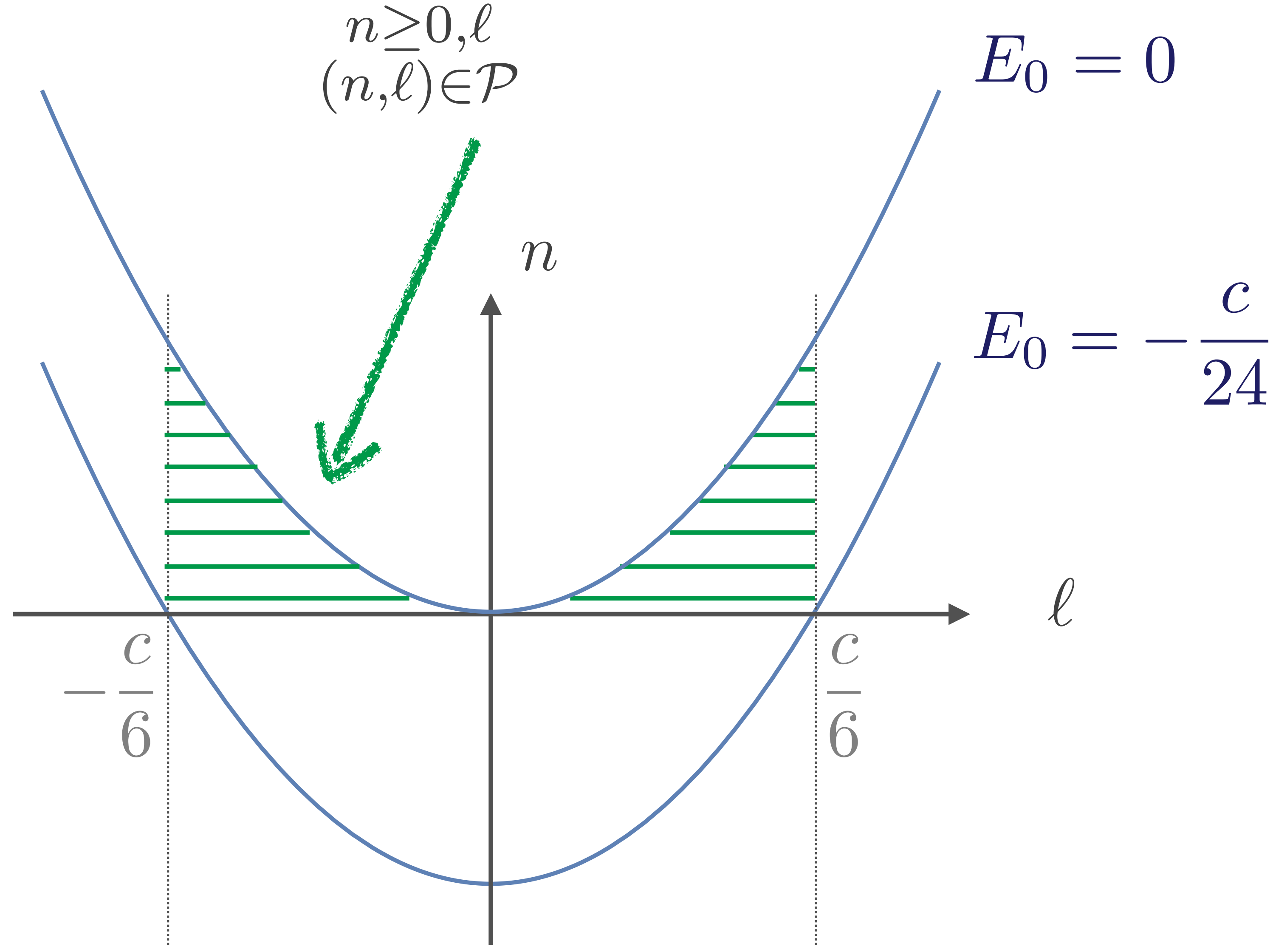
↓ compare the low T expansion with the free energy

$$\boxed{|c(n,\ell)| \leq exp\left(2\pi\left(n + \frac{c}{12} - \frac{|\ell|}{2}\right) + O(\log c)\right)} \;, \quad -\frac{c}{24} < E_0 = n - \frac{3}{2c}\ell^2 < 0$$

Weak Gravity Polar Bound

is a bound on the ~c^3 coefficients of

$$Z_P(\tau, z) = \sum_{\substack{n\geq 0,\ell \\ (n,\ell)\in\mathcal{P}}} c(n,\ell)\, e^{2\pi i \tau n}\, e^{2\pi i z \ell}$$



which determines the rest of the spectrum.

From WG Polar Bound to HP Phase Transition

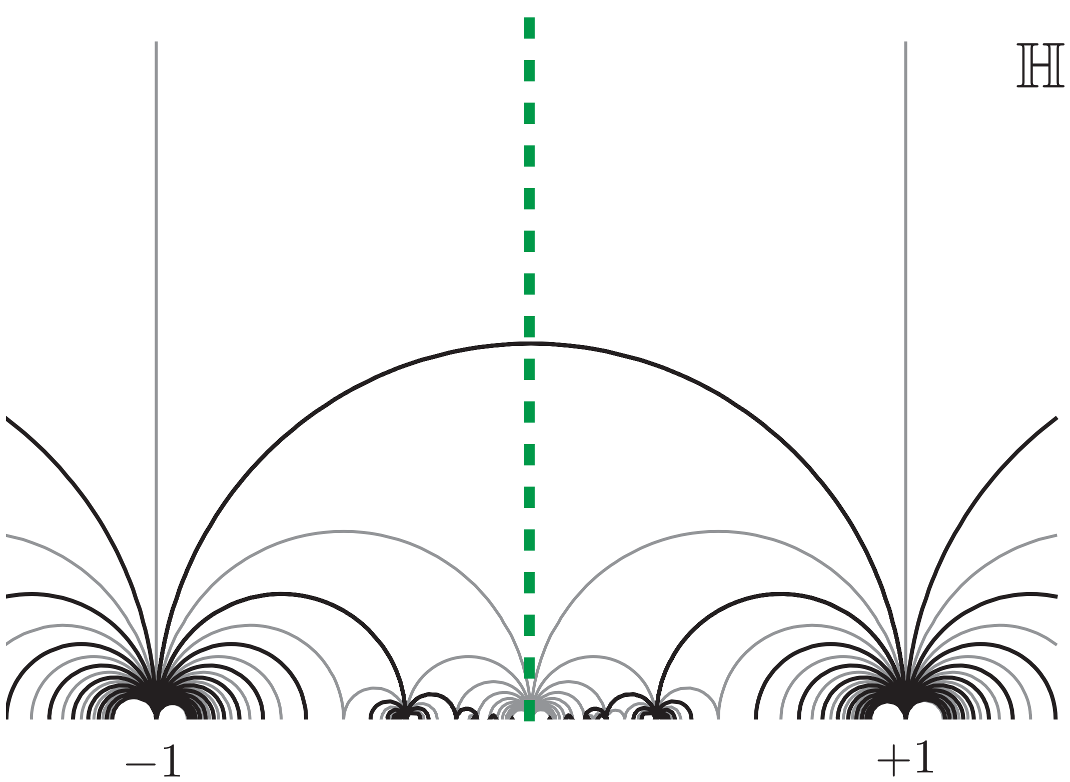
$$\log Z(\tau_\beta, 0) = \log Z(-\frac{1}{\tau_\beta}, 0) = \begin{cases} \frac{c\pi}{12}\frac{2\pi}{\beta}, & \beta < 2\pi \\ \frac{c\pi}{12}\frac{\beta}{2\pi}, & \beta > 2\pi \end{cases}$$

First order "Hawking–Page" phase transition

$$E_0 = -\partial_\beta \log Z = \begin{cases} \frac{c}{24} \\ -\frac{c}{24} \end{cases} \qquad \beta = 2\pi$$

black holes

empty AdS space



In fact, there is a more general (infinite) phase diagram, with each phase dominated by a distinct (Euclidean) black hole configuration.

[see also Dijkgraaf–Maldacena–Moore–Verlinde '07]

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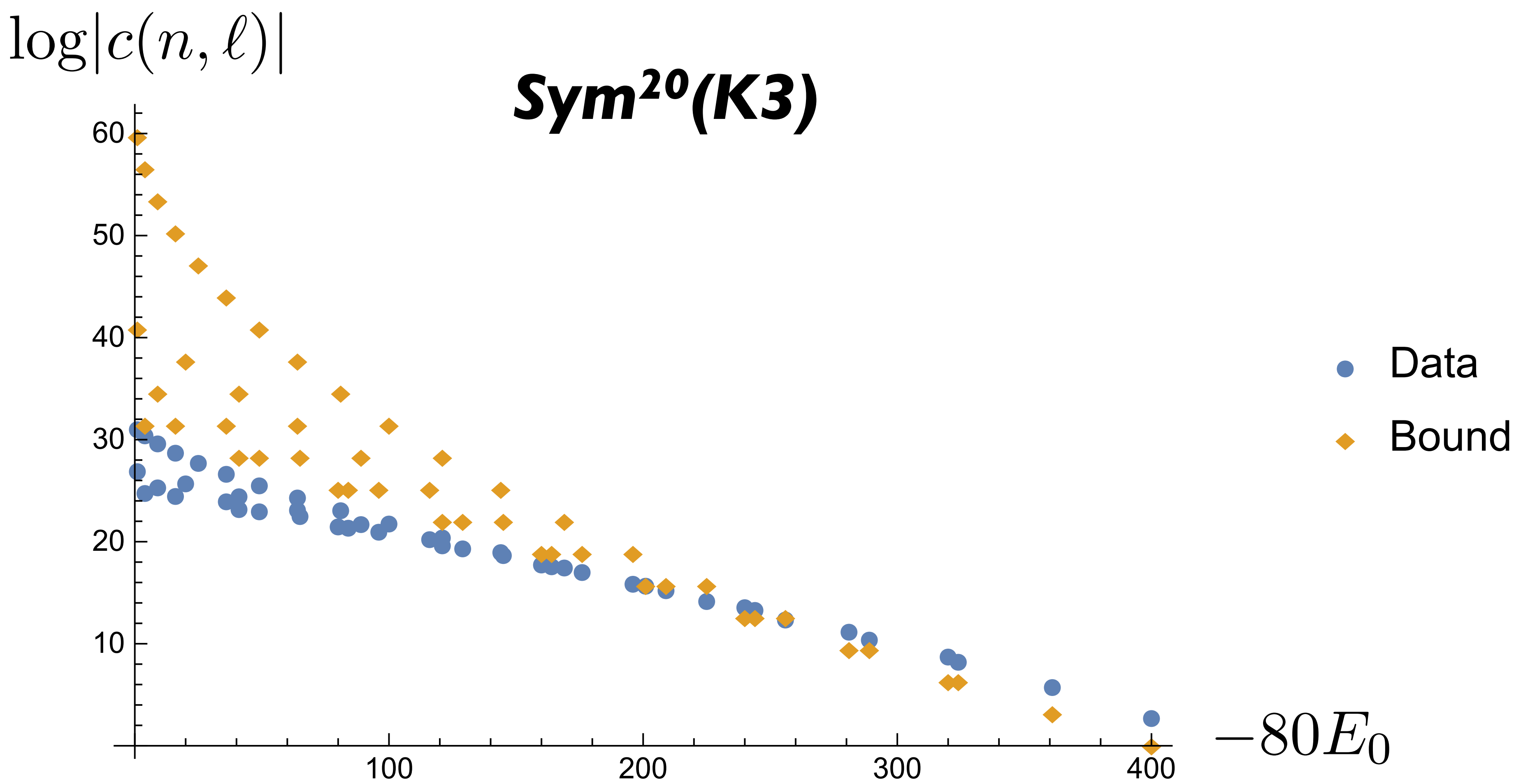
WG Polar Bound Test ✔

K3 is the simplest non-trivial Calabi-Yau manifold string theory can live on with supersymmetry. The SCFT with target space $Sym^N(K3) = K3^{\otimes N}/S_N$ has $c=6N$.

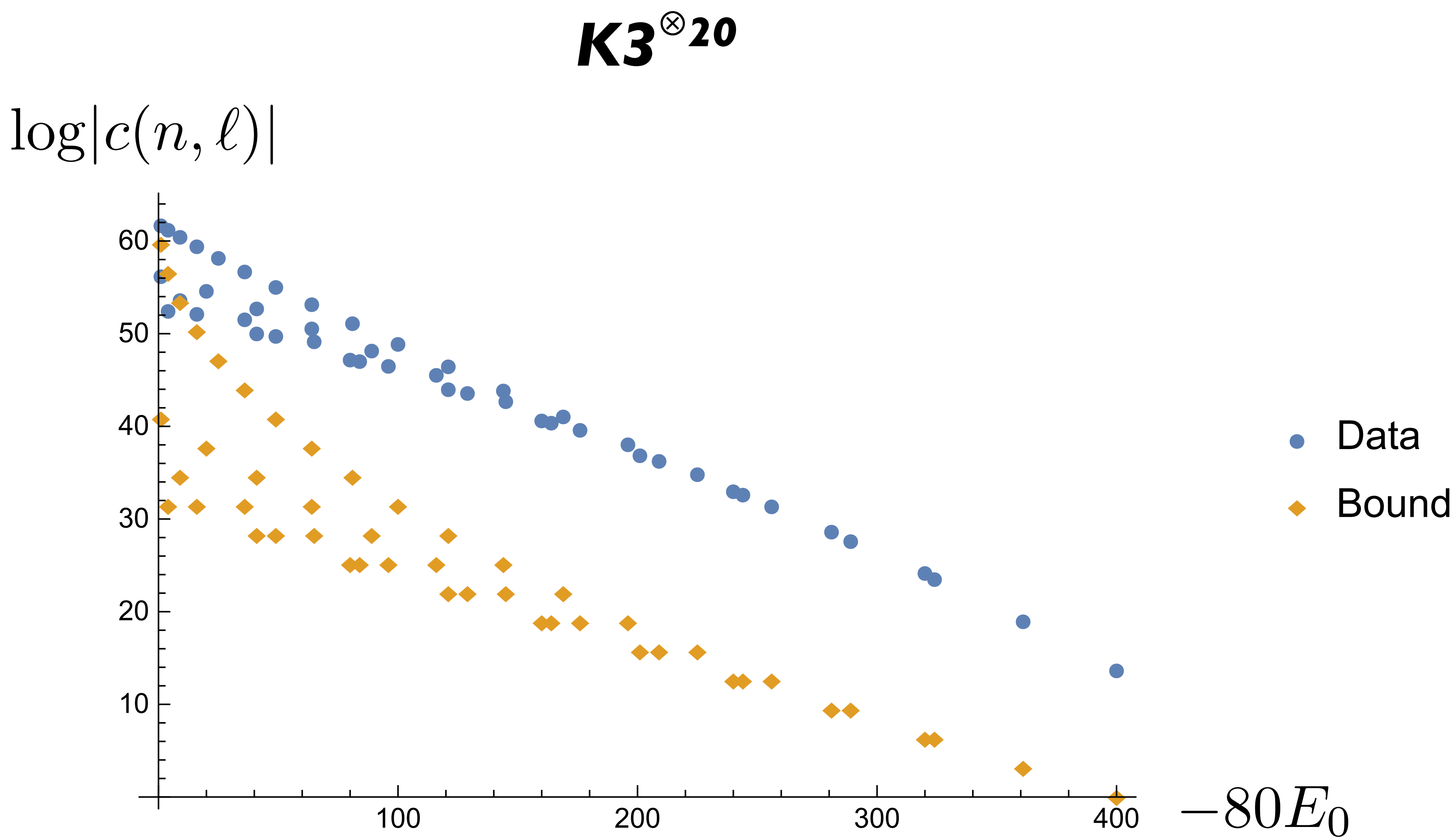


Indeed, it is widely believed that this theory has a supergravity dual description. See for instance [de Boer '98].
It can also be shown that all $Sym^N(X)$ satisfy the bound for reasonable X.

[Keller '11]

WG Polar Bound Test ✗

The SCFT with target space $K3^{\otimes N}$ clear violates the bound: an exponential growth of the ground states.



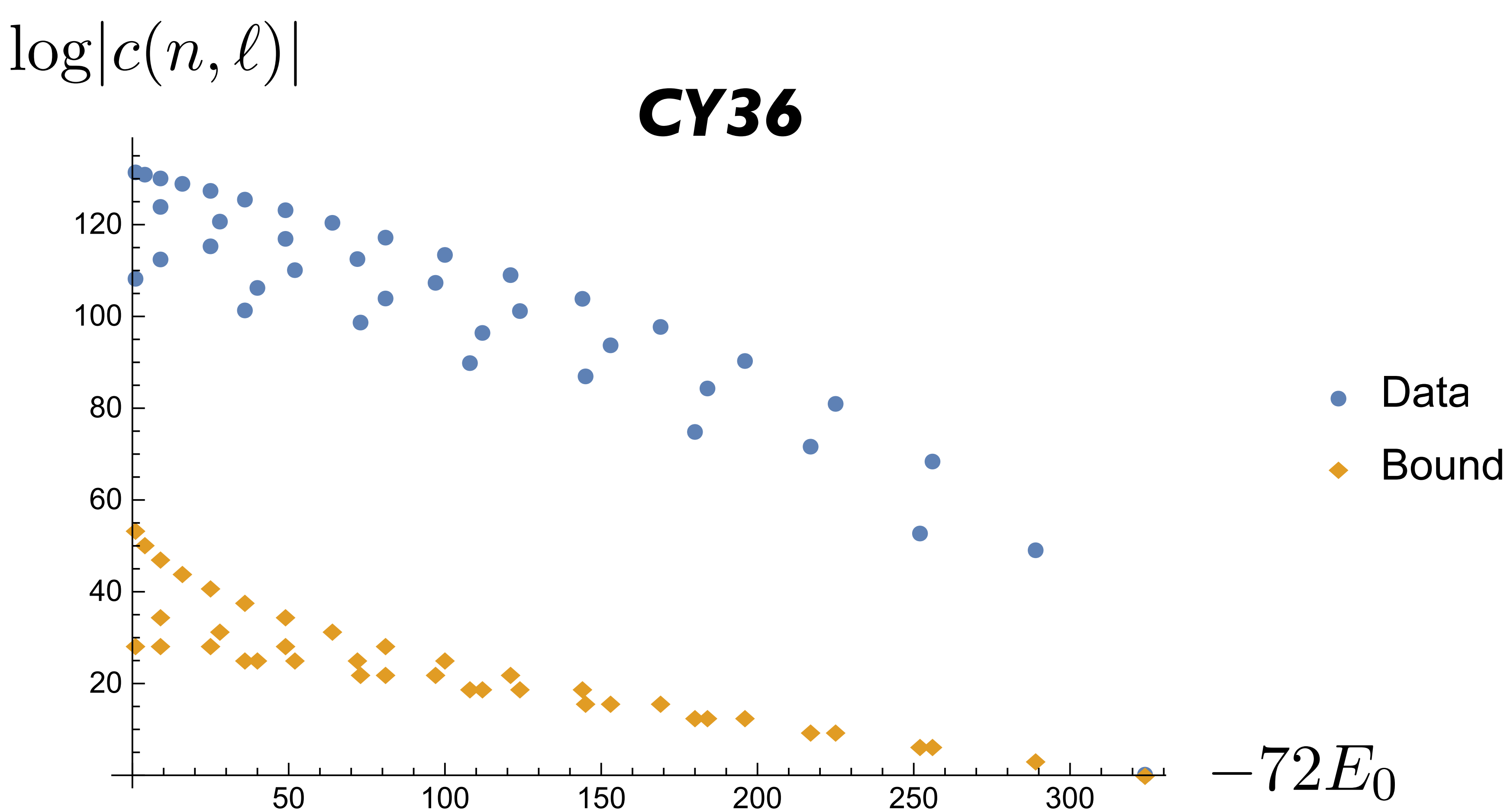
The same holds for $Sym^N(Sym^N(K3))$. ✗

WG Polar Bound Test

The SCFT with target space X_d: d-(complex)-dimensional Calabi-Yau given by the hypersurface of degree d+2 in CP^{d+1}

$$\sum_{i=0}^{d+1} z_i^{d+2} = 0$$

elliptic genus = $$\frac{1}{d+2}\sum_{k,\ell=0}^{d+1} y^{-\ell}\frac{\theta_1\left(\tau, -\frac{d+1}{d+2}z + \frac{\ell}{d+2}\tau + \frac{k}{d+2}\right)}{\theta_1\left(\tau, \frac{1}{d+2}z + \frac{\ell}{d+2}\tau + \frac{k}{d+2}\right)}$$



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A Question and an Answer

Question:

Consider families of 2d SCFTs with at least (2,2) susy with a $c \to \infty$ limit.

Is there a necessary condition such a family must satisfy for it to possibly admit a weakly coupled gravity description at $c \to \infty$ somewhere at its moduli space?

Recall, for compact SCFT, the elliptic genus

$$Z(\tau, z) = \mathrm{Tr}_{\mathcal{H}}(-1)^F q^{L_0 - c/24} y^{J_0}$$

$$= \sum_{n \geq 0, \ell} c(n, \ell)\, q^n y^\ell$$ is an invariant in its moduli space.

Answer: the Weak Gravity Polar Bound

$$\boxed{|c(n,\ell)| \leq exp\left(2\pi(n + \frac{c}{12} - \frac{|\ell|}{2}) + O(\log c)\right)}\ , \quad -\frac{c}{24} < E_0 = n - \frac{3}{2c}\ell^2 < 0$$

A Few Comments on the Answer

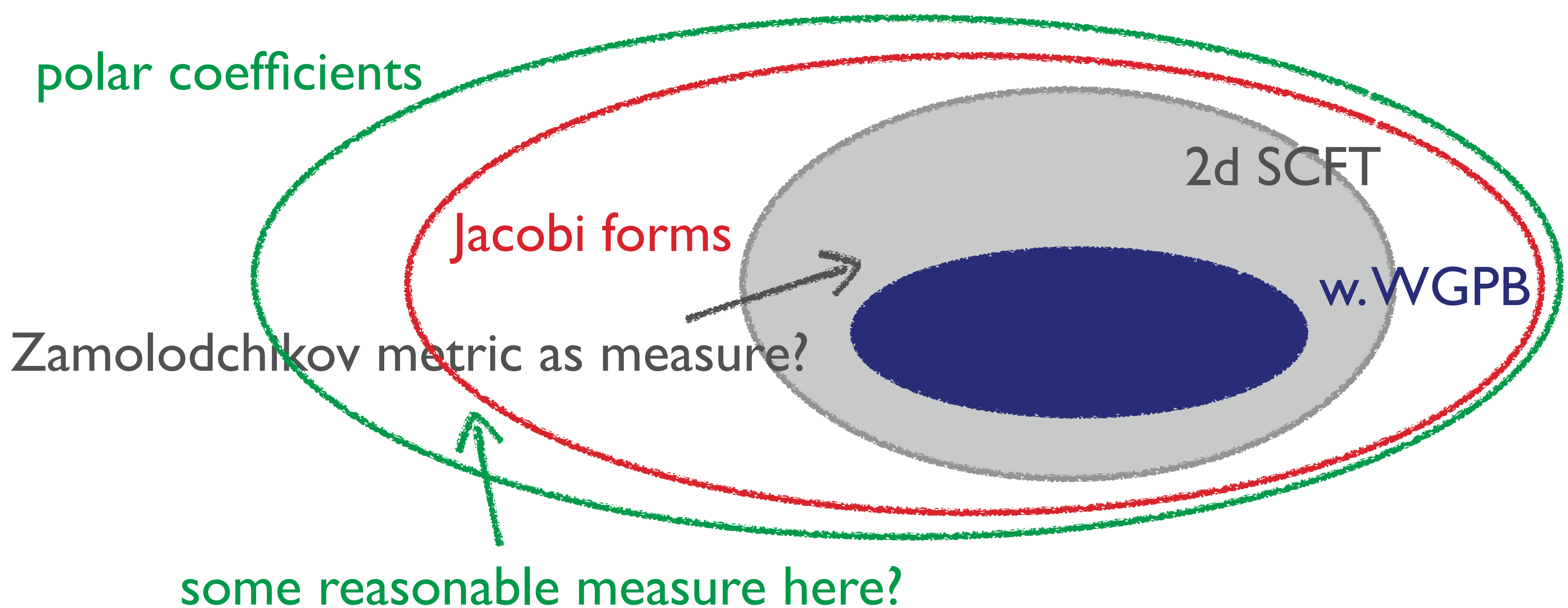
- The Weak Gravity Polar Bound is effective (not everyone passes) and useful (the polar terms determines the whole Z).

- It can be seen as a part of the effort to understand universal features of QFTs with gravity duals, inspired by the universal character of black hole thermodynamics. See also other works on HEE and works including [Haehl—Rangamani], [Bellin—Keller—Maloney], [Headrick—Maloney—Perlmutter—Zadeh], …….

A Few Further Questions

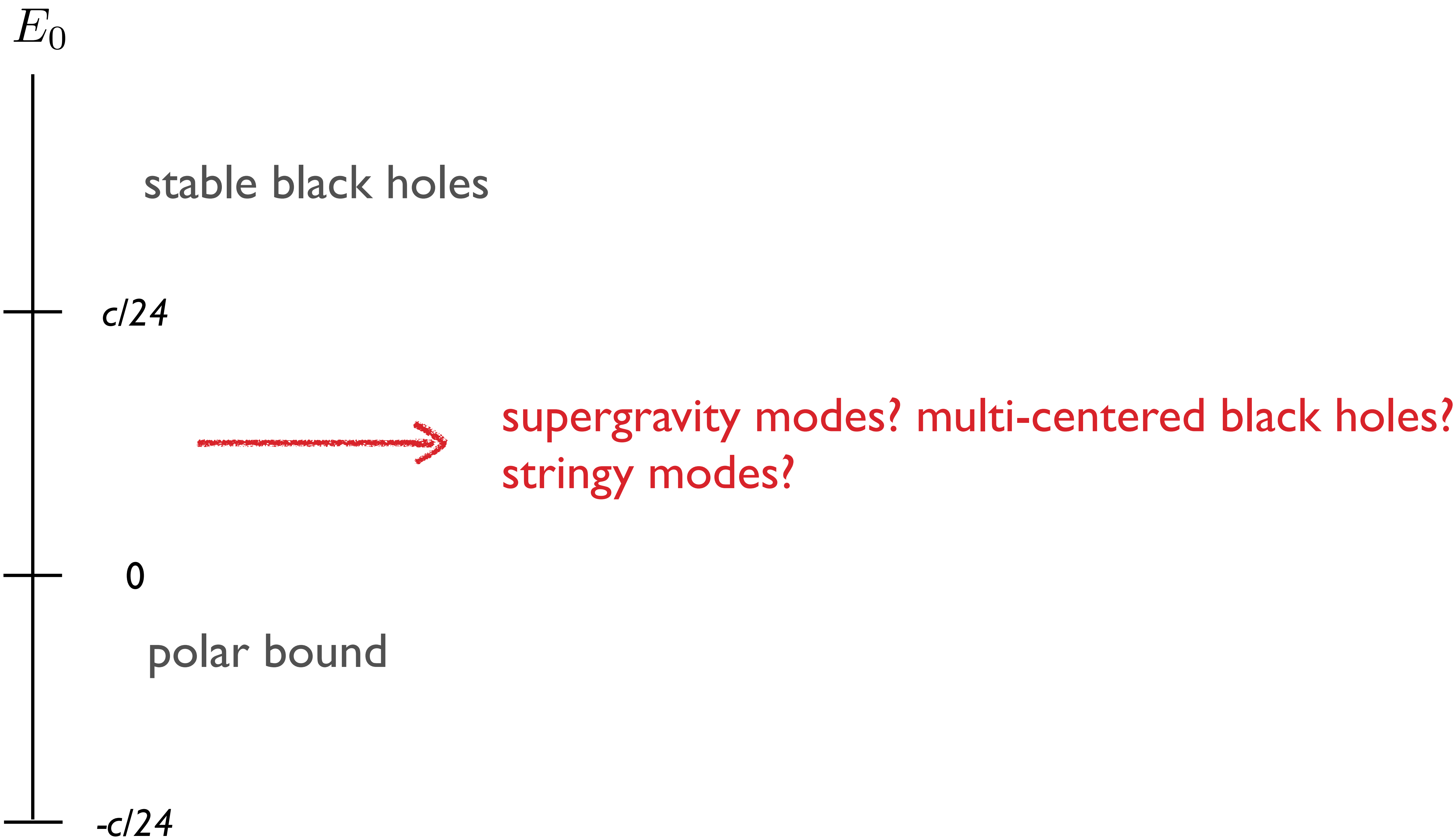
how general is WC gravity?

- What is the *fraction* of 2d SCFT that satisfy the WGPB?



A Few Further Questions

What happens in the enigmatic limbo $0 < E_0 < \frac{c}{24}$?



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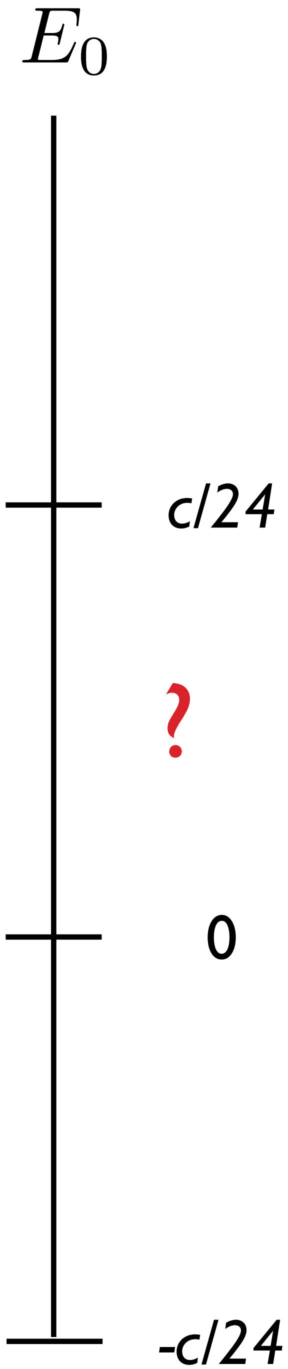


A related question is:

Assume that a family of SCFT satisfy our polar bound, then *what type of* weakly coupled gravitational dual does it have?
Einstein gravity? String theory? Higher spin theory?

A simple (and incomplete) diagnostic:

A $c(n,\ell) \sim e^{const \cdot E_0}$ growth is suggestive of a string theory like gravity dual. We have quite a few examples of CFT's with such a behaviour. For instance the symmetric product of the Monster CFT. See also other works we mentioned before.

Two Examples: Monstrous Stringy Growth

- The symmetric product $\mathrm{Sym}^N \mathcal{M}$ of the Monster CFT $\mathcal{M}$ exhibits Hagedorn growth.

[based on discussions with Xi Yin]

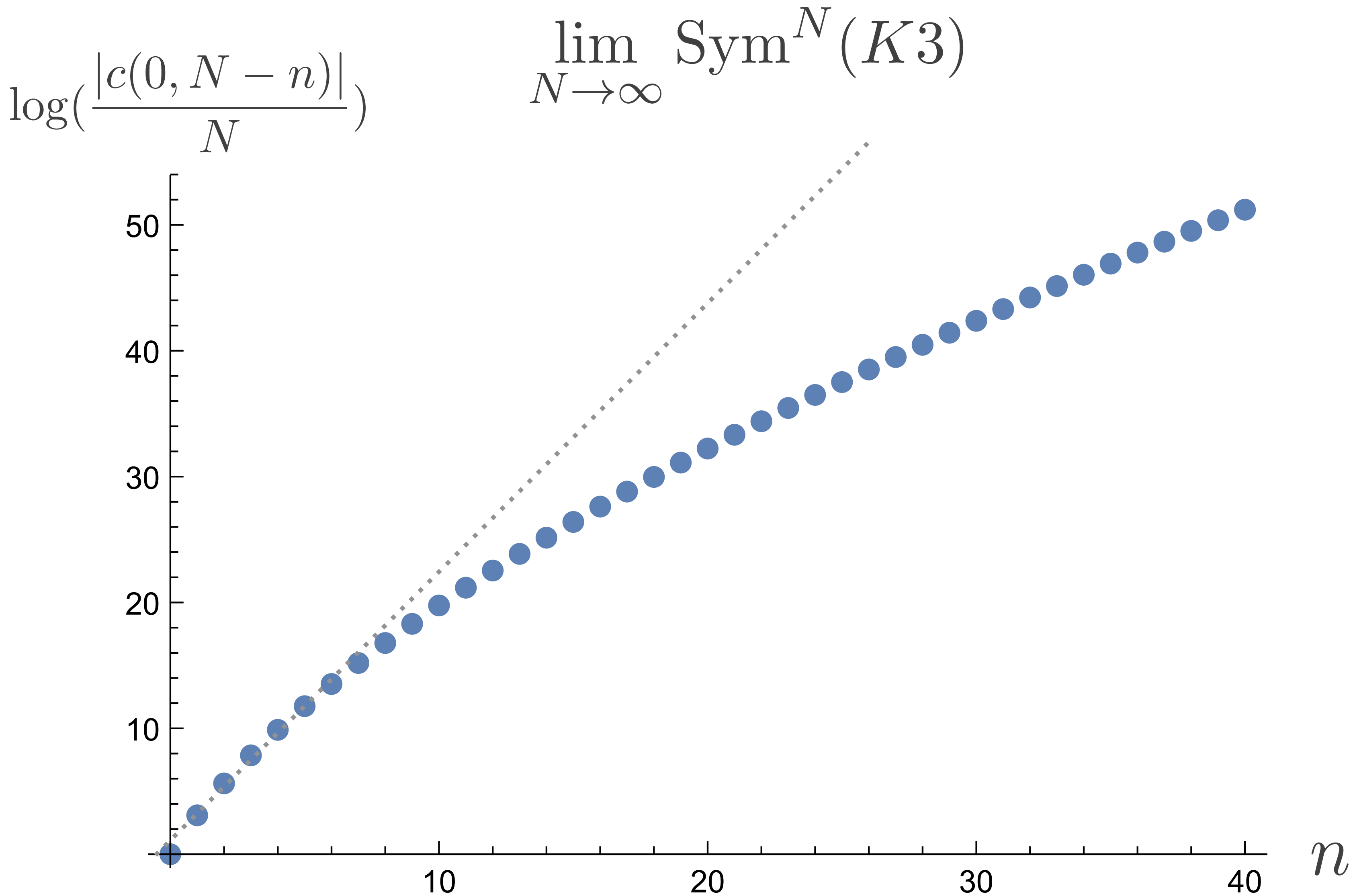
$$Z_{\mathcal{M}}(\tau) = J(\tau) = e^{-2\pi i\tau} + 196884 e^{2\pi i\tau} + \cdots = \sum_{n>-c/24} c_n e^{2\pi i n\tau} \quad , \quad c = 24$$

$$\sum_N p^N Z_{\mathrm{Sym}^N \mathcal{M}}(\tau) = \prod_{n>0, m\in\mathbb{Z}} \frac{1}{(1-p^n q^m)^{c_{nm}}}$$

$$\Rightarrow Z_{\mathrm{Sym}^N \mathcal{M}}(\tau) = q^{-N} + \sum_n q^{n-N} a_n \quad \text{where} \quad \boxed{a_n \sim e^{2\pi n}}$$

stringy Hagedorn growth

Two Examples: D1-D5 Supergravity



no exponential growth

A Few Further Questions

- What is the fraction of 2d SCFT that have Einstein-like gravity dual? Weakly coupled string theoretic dual?

- How about higher dimensions?

- Criteria beyond the spectrum?

Thank you for your attention!