Resurgent Methods in Exact QFTs

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-Cambridge-

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DESY

“Divergent series are the invention of the *Devil*, and it is shameful to base on them any demonstration whatsoever ...”

Niels Henrik Abel (1802 – 1829)

“Divergent series are the invention of the *Devil*, and it is shameful to base on them any demonstration whatsoever ...”

“That most of these things [summation of divergent series] are correct, in spite of that, is extraordinarily surprising. I am trying to find a reason for this; it is an exceedingly interesting question.”

Some physical motivations:

Perturbation Theory in

- Asymptotic Nature of QM, QFT, String Theory

- IR Renormalon Puzzle in asymptotically free QFTs

- Non-perturbative phys. /wo Instantons

Role of non-BPS saddles?

The Bigger scheme:

- Non-pert. definition of asymptotically free QFTs

- Analytic continuation of path integrals

Lefschetz thimbles

“New” Idea ⟶ Resurgence

Ecalle (1980), Stokes (1850)

- Maths: ODE,dynamical systems, finite difference

  [Sauzin,Garoufalidis,Costin]

- Phys:QM, Topological Strings, QFTs susy (and non-)

  [Zinn-Justin,Jentschura,Voros]

  [Mariño,Schiappa,Aniceto,Vonk]

  [Argyress,Cherman,Basar,Dunne,Unsal,DD]

“New” Idea $\longrightarrow$ Resurgence

Ecalle (1980), Stokes (1850)

Unification of Perturbative and
Non-Perturbative Physics

<u>“Philosophical” shift</u>:

Semi-classical expansion

$\downarrow$

Exact answer
BUT
encoded in
a Transseries Form

“New” Idea $\longrightarrow$ Resurgence

Ecalle (1980), Stokes (1850)

Unification of Perturbative and Non-Perturbative Physics

<u>“Philosophical” shift</u>:

Semi-classical expansion

$$\Big\downarrow$$

$$f(g^2) = \sum_n \sum_k \sum_p c_{n,k,p}\, g^{2n} \left[\exp\left(-\frac{S}{g^2}\right)\right]^k \left(\log\frac{1}{g^2}\right)^p$$

most general ansatz soln to a non-linear problem

"New" Idea $\longrightarrow$ Resurgence

Ecalle (1980), Stokes (1850)

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Perturbative Fluctuations

“New” Idea $\longrightarrow$ Resurgence

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Multi-Instantons

“New” Idea $\longrightarrow$ Resurgence

Ecalle (1980), Stokes (1850)

Unification of Perturbative and
Non-Perturbative Physics

<u>“Philosophical” shift</u>:

Semi-classical expansion

$$\Big\downarrow$$

(Resonances in Painleve)

$$f(g^2) = \sum_n \sum_k \sum_p c_{n,k,p}\, g^{2n} \left[\exp\left(-\frac{S}{g^2}\right)\right]^k \left(\log\frac{1}{g^2}\right)^p$$

quasi-zero-modes

What is Resurgence?

"..resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the "origin". Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities.."
-Ecalle ('80)

What is Resurgence?

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Take Away Message: reconstruct NP physics from the perturbative series

Back to the Basics:

How do we compute physical quantities?

$\longrightarrow$ Unless Magic Happens (i.e. localization, integrability,..) : Perturbation Theory

Just by diagram counting (Dyson, Lipatov)

$$f(g) = \sum_{n=0}^{\infty} c_n g^n \longrightarrow c_n \sim n!$$

Idea: Insert factor $1 = \frac{1}{n!}\int_0^\infty dt\, t^n\, e^{-t}$

Commute Sum w/ Integral

Borel Transform:

Take $f(g) = \sum_{n=0}^{\infty} c_n g^n$

Consider $B[f](t) = \sum_{n=1}^{\infty} \frac{c_n}{(n-1)!} t^{n-1}$

Germ of analytic functions
at the origin

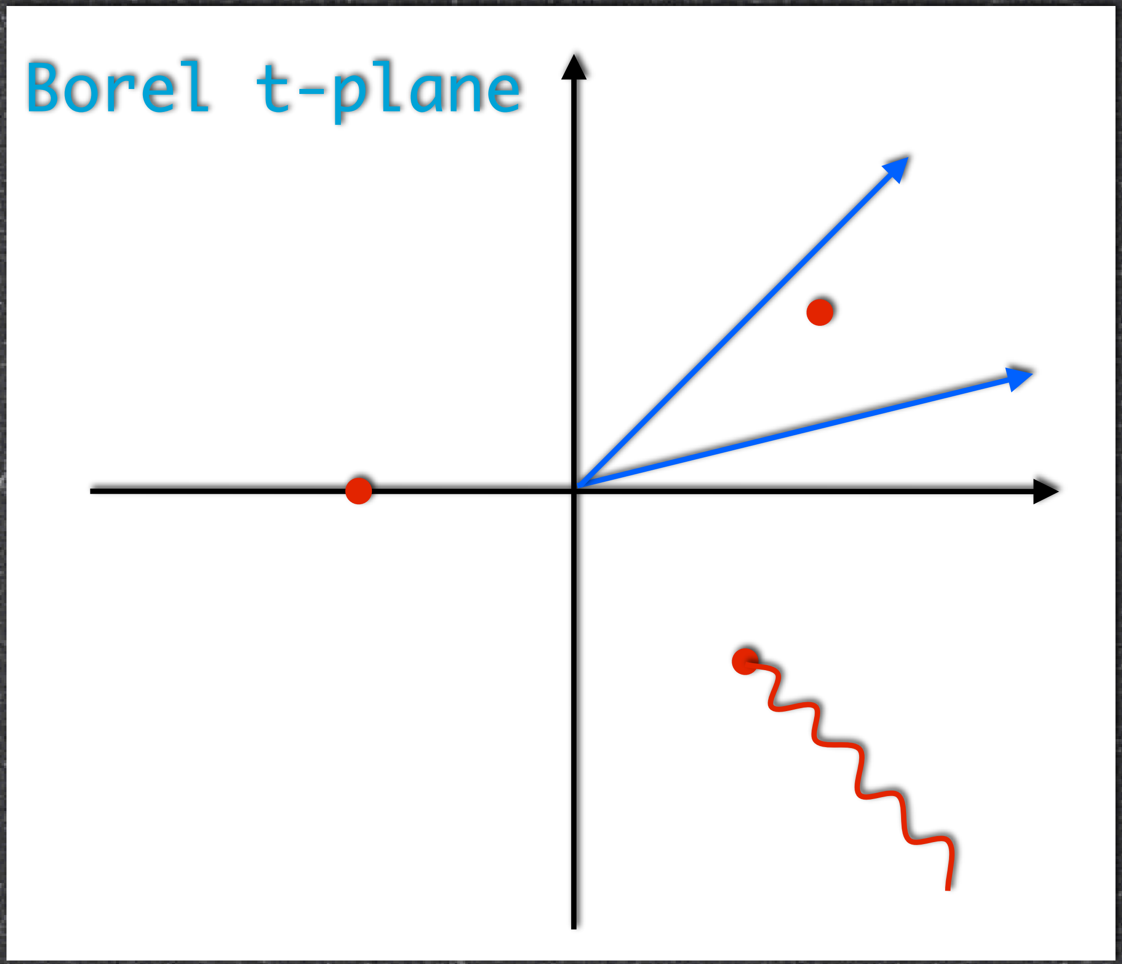
Obtain Analytic Continuation for f(g)

$$\mathcal{S}[f](g) = c_0 + \int_0^{\infty} dt\, e^{-t/g}\, B[f](t)$$

Laplace transform back: Analytic for $\Re(g) > 0$

Different analytic continuations of the SAME physical observable (in pert.theory)

$$\mathcal{S}_\theta[f](g) = c_0 + \int_0^{e^{i\theta}\infty} dt\, e^{-t/g} B[f](t)$$



Different analytic continuations of the SAME physical observable

$$\mathcal{S}_\theta[f](g) = c_0 + \int_0^{e^{i\theta}\infty} dt\, e^{-t/g} B[f](t)$$

Different continuations $\longrightarrow$ Ambiguities

On a Stokes line

$$\mathcal{S}_+[f](g) - \mathcal{S}_-[f](g) \sim 2\pi\, i\, e^{-S/g}$$

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Non-perturbative - non-analytic and Imaginary

Different analytic continuations of the SAME physical observable

On a Stokes line

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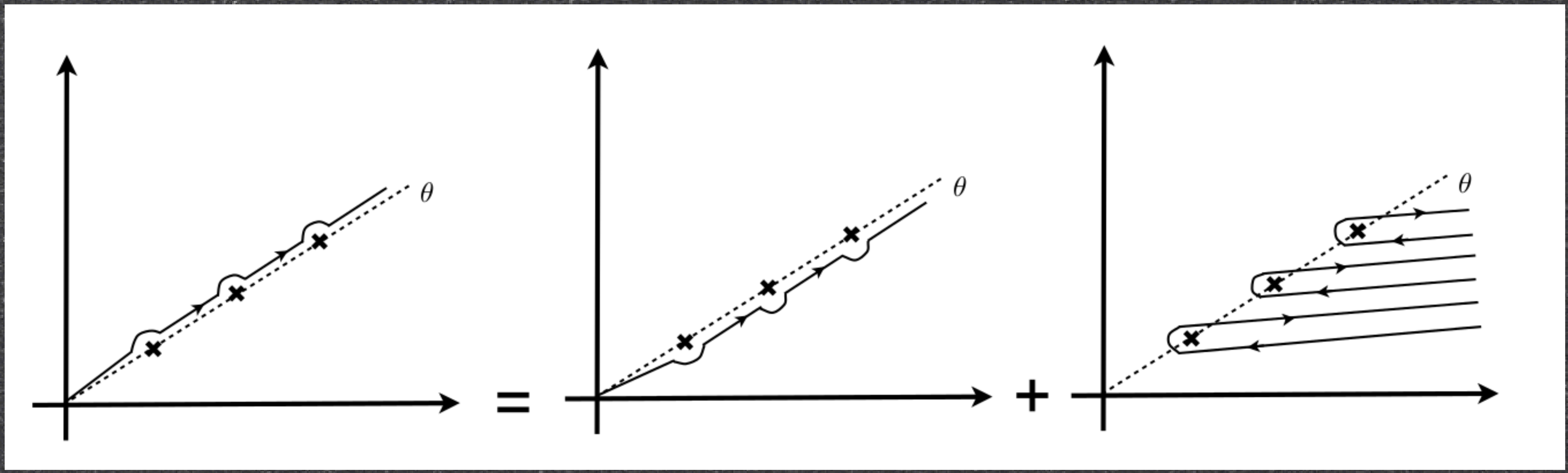

Higher-orders Perturbative Expansion

QM Examples: Ground State Energy $E(g) = \sum_{n=0}^{\infty} c_n \, g^n$

(Stone-Reeves, Bender-Wu)

$\longrightarrow$ Anharmonic Oscillator

$$c_n \sim (-1)^{n+1} \frac{3^n \sqrt{6}}{\pi^{3/2}} \Gamma(n + 1/2)$$

Alternating in signs $\longrightarrow$ Borel transform has Poles on the negative axis

Borel summable

Higher-orders Perturbative Expansion

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⟶ Anharmonic Oscillator ✓

⟶ Cubic Oscillator $\boxed{c_n \sim -(+60)^{n+1} \frac{\Gamma(n+1/2)}{2\pi^{3/2}}}$

Non-Alternating in signs: Poles on the positive axis

Ambiguity

$$E_+(g) - E_-(g) \sim \pm i(\dots)e^{-1/g}$$

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Ambiguity

$E_+(g) - E_-(g) \sim \pm i(\dots)e^{-1/g}$ $\longrightarrow$ Quantum Tunnelling

Higher-orders Perturbative Expansion

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$\longrightarrow$ Anharmonic Oscillator ✓

$\longrightarrow$ Cubic Oscillator ✓

$\longrightarrow$ Double-well $\boxed{c_n \sim (+3)^n \, n!}$

Non-Alternating in signs: Poles on the positive axis

Ambiguity ???

Higher-orders Perturbative Expansion

QM Examples: Ground State Energy $E(g) = \sum_{n=0}^{\infty} c_n\, g^n$

$\longrightarrow$ Double-well $c_n \sim (+3)^n\, n!$

Poles on the positive axis

Ambiguity

$$E_+(g) - E_-(g) \sim \pm i(\dots)e^{-2S_{inst}/g}$$

Instanton/anti-Instanton Events do contribute! $\longrightarrow$ Perturbation theory knows about them!

Bogomolny,Zinn-Justin argument

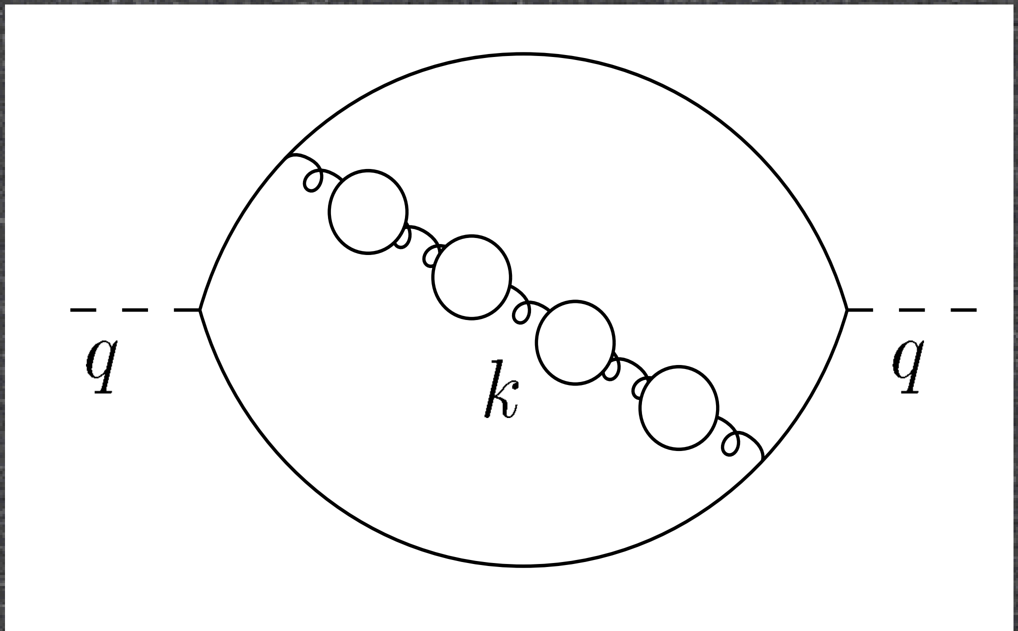
QFT:

IR renormalons problem:

- SU(N) YM on $\mathbb{R}^4$
- $\mathbb{CP}^N$ on $\mathbb{R}^2$
- PCM on $\mathbb{R}^2$

Planar diagrams

non-alternating factorial growth



(Beneke)

Leading ambiguity

$$\pm i\, e^{-2S_{inst}/\beta_0}$$

$\beta_0 \sim N$

QFT:

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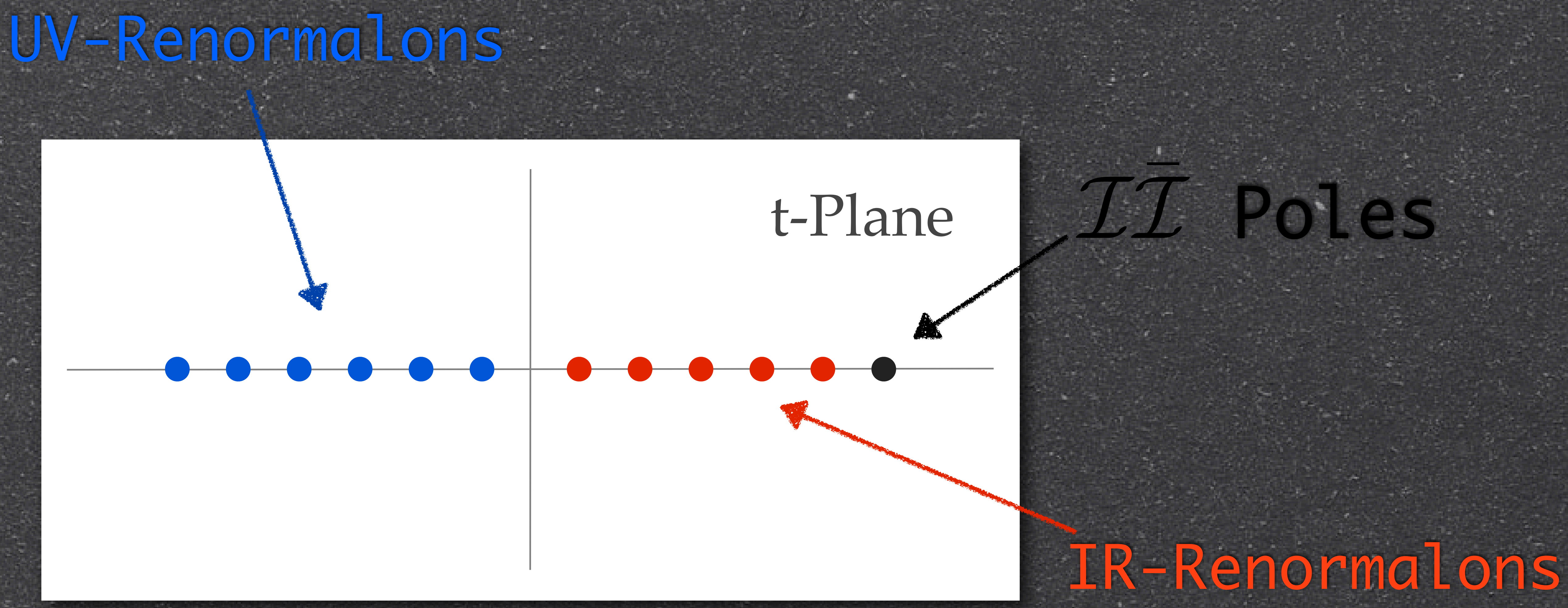


Systematic IR renormalons:

2d Models w/ Exact solutions:

$\swarrow \quad \searrow$

Integrability $\qquad$ Large-N

$\searrow \quad \swarrow$

Expand in small coupling

$$c_n \sim n!\, a^{-n}$$

$$a > 0$$

$$a = O(N^0)$$

Systematic IR renormalons:

2d Models w/ Exact solutions:

PCM

Fateev,Kazakov,Wiegmann

O(N) @ Large N OPE expansion

David, NSVZ, Beneke

O(N) in the resurgent context

Basar,Cherman, Dunne,DD,Unsal (in progress)

Principal Chiral Model: Cherman,DD,Dunne,Unsal

for $\mathbb{CP}^N$ see Argyress,Dunne,Unsal

$$S = \frac{1}{2g^2}\int_M d^2x \, \mathrm{Tr}\, \partial_\mu U \, \partial^\mu U^\dagger \qquad U(x) \in SU(N)$$

- Asymptotically Free $\longrightarrow$ no WZW
- Matrix Large-N model
- Confining/Deconfining "Phase" transition
- can be made SUSY
- IR Renormalons $\longrightarrow$ no Instantons

$$\pi_2(SU(N)) = 0$$

Saddle Points: Unitons

Uhlenbeck ⟶ Full Characterization of soln. to PCM eom

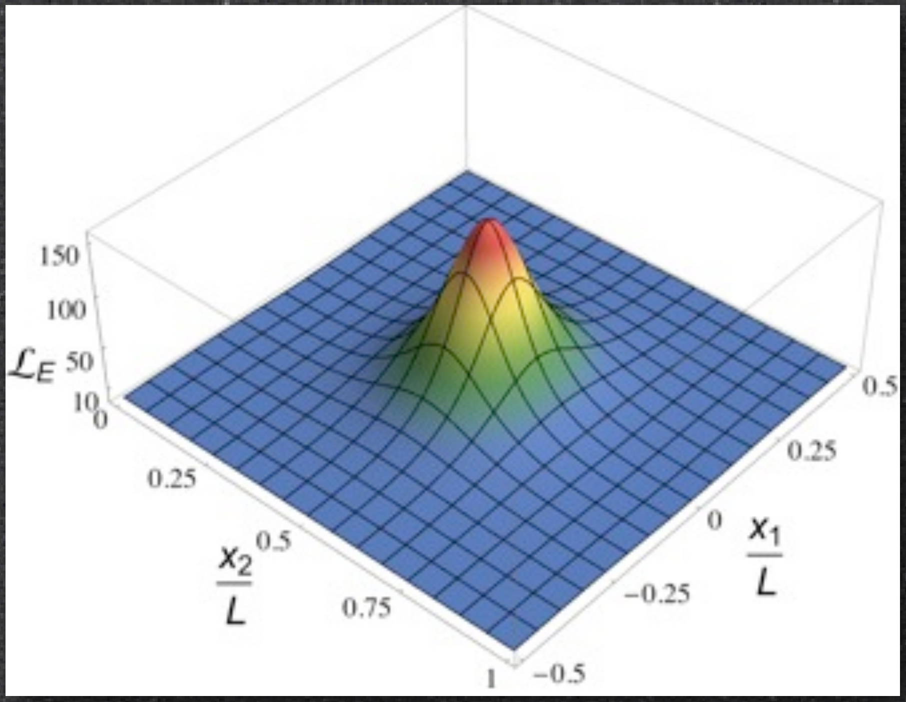
$\mathbb{CP}^{N-1} \subset SU(N)$ geodesic embedding

$U_{uni} = e^{i\pi/N}(1_N - 2P)$ w/ Projector $P_{ij} = \dfrac{v_i\, v_j^\dagger}{v^\dagger \cdot v}$

v_i embedding of $\mathbb{CP}^N$ lumps

$$S_{uni} = 8\pi \times \frac{N}{\lambda}$$

(and integers multiples)



Genuine SADDLES!

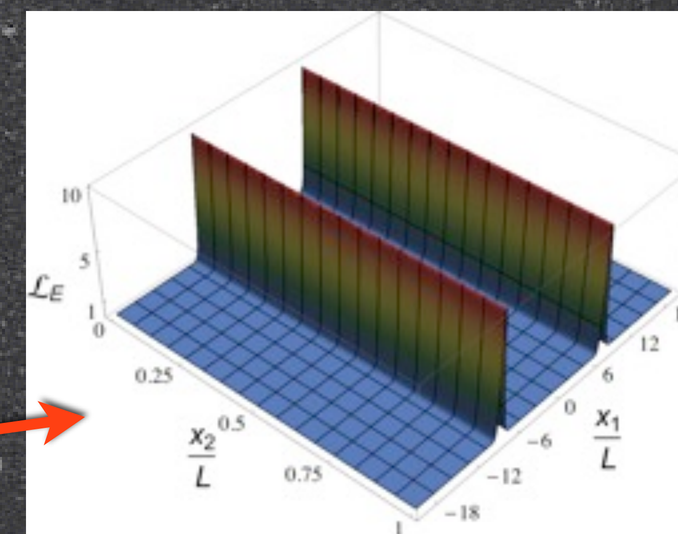
Note: NON-BPS obj. w/ quantized actions!

Compactification w/ a twist:

w/ particular twisted BC $U(t,x+L) = e^{iH_L} U(t,x) e^{iH_R}$

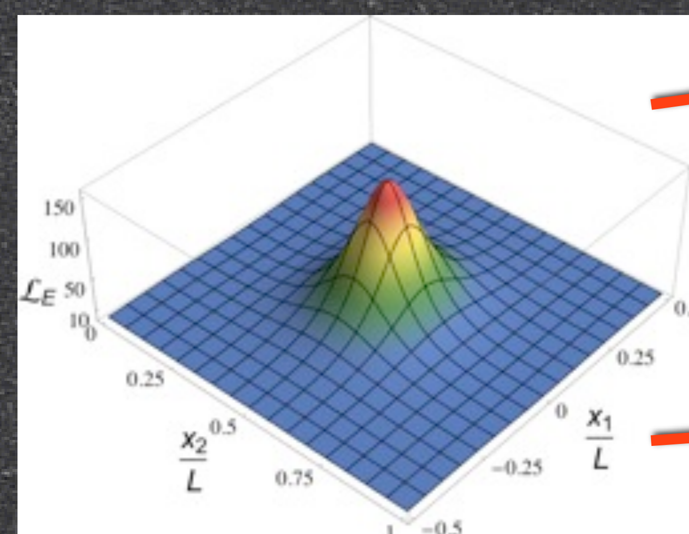
Volume Independence

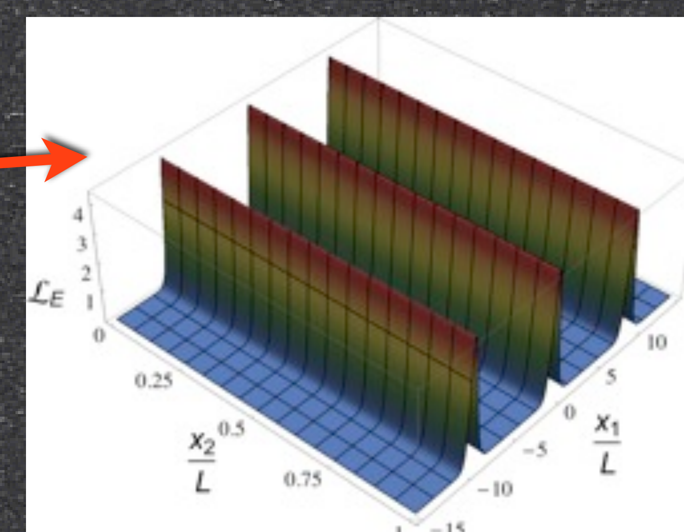
(Armoni-Shifman-Veneziano, Kovtun-Unsal, Yaffe)



on small S^1 Unitons fractionalize into N fundamental objects:

Fractons:





on small S^1 we can use effective QM to study vacuum energy or partition function

Fractons:

take SU(2) in Hopf coords

$$U = \begin{pmatrix} \cos\theta\, e^{i\phi_1} & i\sin\theta\, e^{i\phi_2} \\ i\sin\theta\, e^{-i\phi_2} & \cos\theta\, e^{-i\phi_1} \end{pmatrix}$$

$$\theta(t;t_0) = 2arcCot\left[e^{-\xi(t-t_0)}\right] \qquad \xi = \frac{2\pi}{NL}$$

$$\bar{\theta}(t;t_0) = \pi - 2arcCot\left[e^{-\xi(t-t_0)}\right]$$

$$S_f = \frac{S_{uni}}{N}$$

Correct action to give semiclassical realization of IR renormalons

$$H = \frac{g^2}{4L}P_\theta^2 + \frac{L\xi^2}{g^2}\,\sin^2\theta + (\phi_1, \phi_2\,\text{terms})$$

Fractons:

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Correct action to give semiclassical realization of IR renormalons

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SU(N): N-types of Fractons

→ N-1 simple roots of SU(N)

→ KK-Fracton -> affine root

Same as KK monopoles
in compactified YM
w/ non-trivial holonomies
Lee,Yi and Kraal,van Baal

Fractons:

$$S_f = \frac{S_{uni}}{N}$$

Correct action to give semiclassical realization of IR renormalons

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SU(N): N-types of Fractons

As in QM, the divergence of perturbation theory tells us that FFbar events seem to play a role!

Fracton/Anti-Fractons

Centre of mass

$(t_1 + t_2)/2$

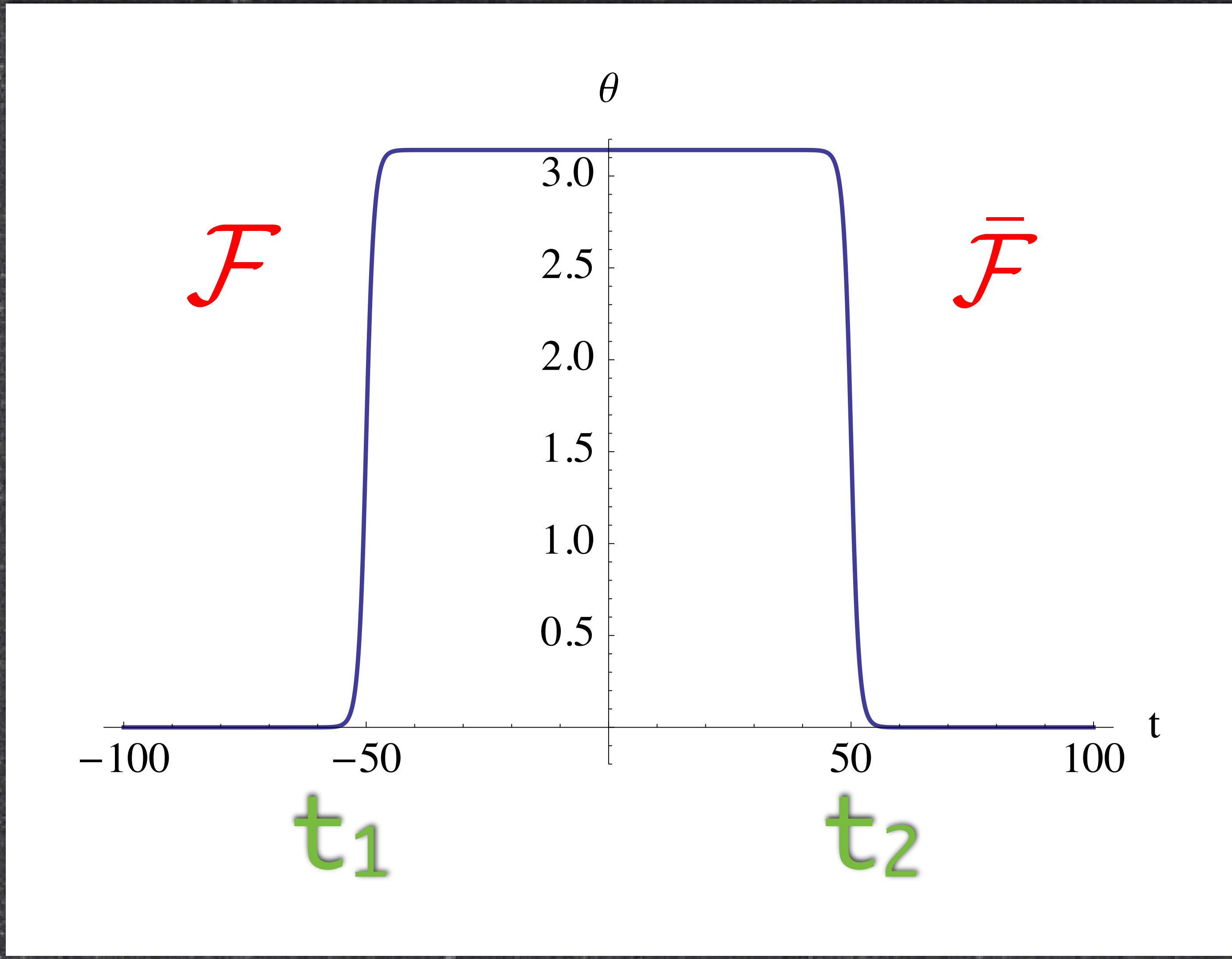
Exact zero-mode

Relative Separation

$\tau = (t_1 - t_2)/2$

Quasi zero-mode

Subtle!



$$I_{\mathcal{F}\bar{\mathcal{F}}} \sim e^{-2S_F} \int_0^\infty d\tau \tau e^{-\left(-1 \times \frac{32\pi}{\lambda} e^{-\tau} + \tau\right)}$$

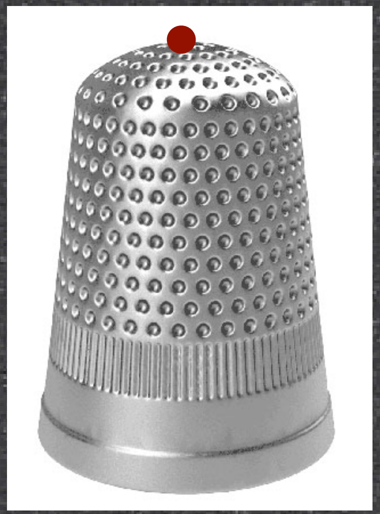
$\longrightarrow$ Support from small separations!

FFbar attract each others

Lefschetz Thimbles: a.k.a. Morse Theory

(Pham, Witten)

Functional steepest descent contours



$$\frac{\partial \phi}{\partial s} = \overline{\frac{\delta S[\phi]}{\delta \phi}}$$

$$\lim_{s \to -\infty} \phi(s) = \phi_{crit}$$

On the thimble $\mathcal{J}_i$

$\longrightarrow$ Im(S) constant

Lattice(Aurora collaboration)

$\longrightarrow$ -Re(S) decreasing

$$\int \mathcal{D}\phi \, e^{-\frac{1}{g^2} S[\phi]} = \sum_{thimbles\, i} N_i \, e^{-\frac{i}{g^2} Im\, S[\phi_{crit}^{(i)}]} \int_{\mathcal{J}_i} \mathcal{D}\phi \, e^{-\frac{1}{g^2} Re\, S[\phi]}$$

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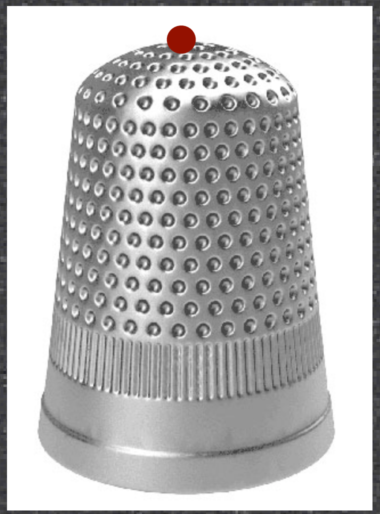

$$\int \mathcal{D}\phi \, e^{-\frac{1}{g^2} S[\phi]} = \sum_{thimbles\, i} N_i \, e^{-\frac{i}{g^2} Im\, S[\phi^{(i)}_{crit}]} \int_{\mathcal{J}_i} \mathcal{D}\phi \, e^{-\frac{1}{g^2} Re\, S[\phi]}$$

- Finite dim integrals — Hyperasymptotics (Berry-Howls)
- Complexified CS — (Witten)
- Complexified QM in phase space — (Witten)
- Complexified Liouville — (Harlow-Maltz-Witten)

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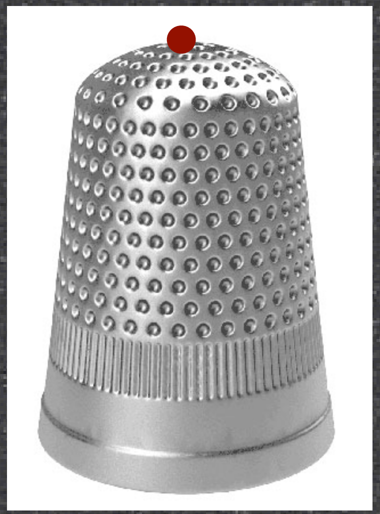
Functional steepest descent contours



$$\int \mathcal{D}\phi\, e^{-\frac{1}{g^2}S[\phi]} = \sum_{thimbles\, i} N_i\, e^{-\frac{i}{g^2}Im\, S[\phi^{(i)}_{crit}]} \int_{\mathcal{J}_i} \mathcal{D}\phi\, e^{-\frac{1}{g^2}Re\, S[\phi]}$$

Stokes phenomenon: intersection numbers N_i can change with phase of parameters

Resurgence: asymptotic expansions about different saddles are closely related. Natural transseries expansion

(Berry-Howls)

FFbar: Critical point at infinity!

as $\lambda \to 0$ all but two directions along thimble become Gaussian,

centre of mass $\longrightarrow$ easy to handle (Morse-Bott)

relative separation $\longrightarrow$ tricky part but effectively reduces the flow to a finite dimensional one

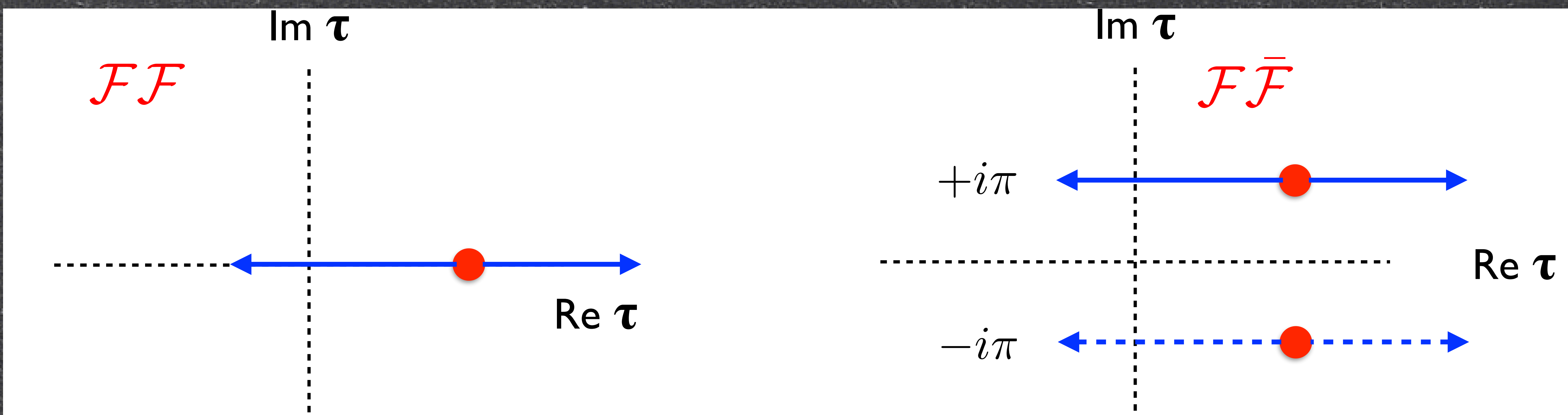
Only "relevant" direction in field space is the separation modulus

FFbar:

Bogomolny,Zinn-Justin-like Argument
(Basar,Cherman,DD,Dunne,Unsal)

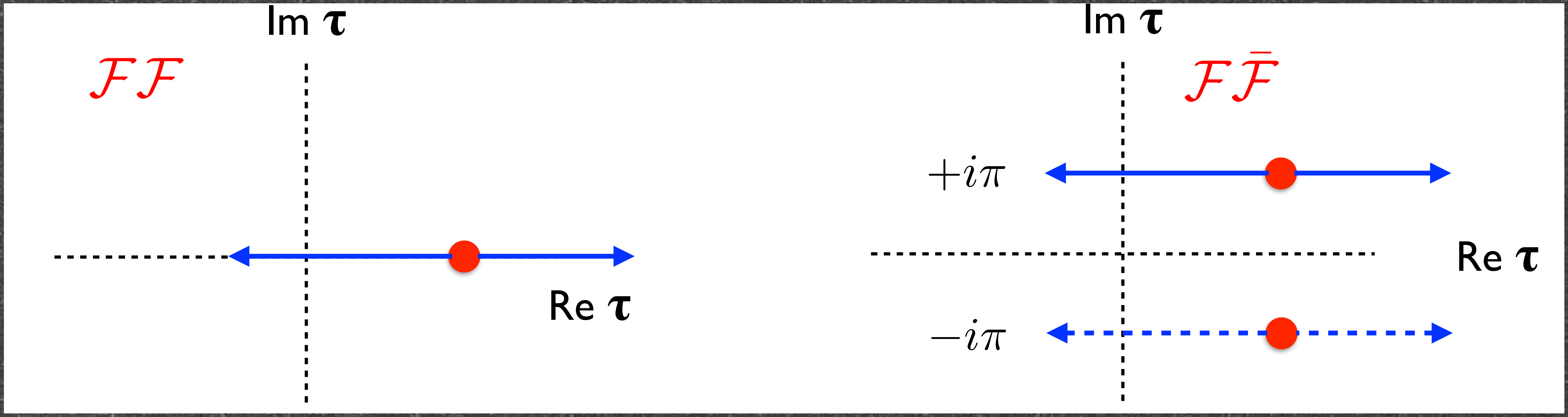
$$I_{\mathcal{F}\bar{\mathcal{F}}} \sim e^{-2S_F} \int_{\cancel{0}}^{\cancel{\infty}} d\tau\, \tau\, e^{-\left(-1 \times \frac{32\pi}{\lambda} e^{-\tau} + \tau\right)}$$

Reduced Flow equations in field space give rise to correct integrations “contours”



Saddle is still at “Infinity” but contours do depend on $\arg\lambda$

FFbar:

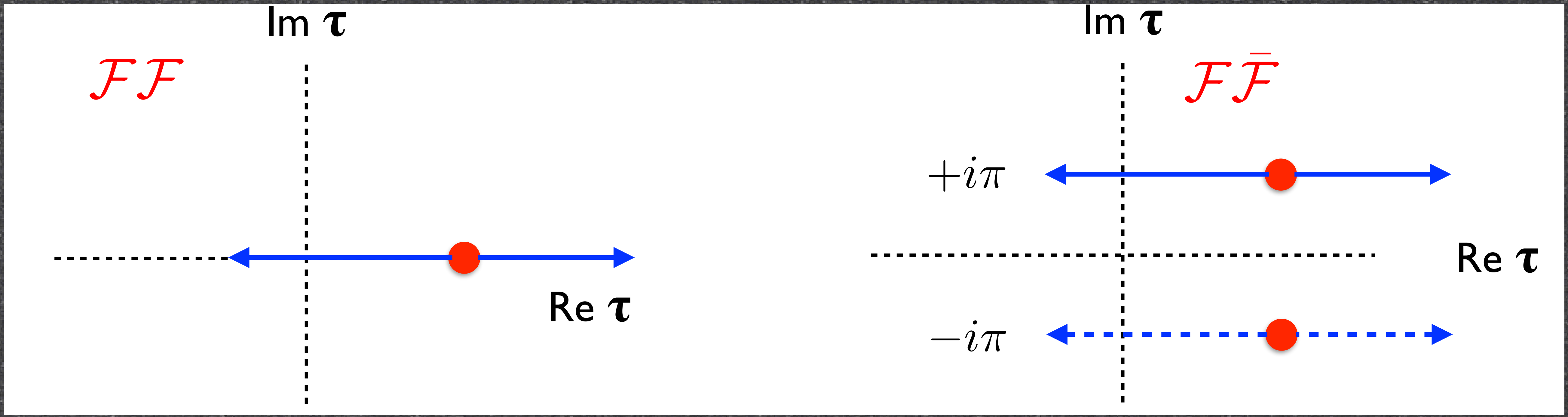


Performing the integral over the correct thimble

$$[\mathcal{F}\bar{\mathcal{F}}]_{\pm} = -\left(-\log\left[\frac{32\pi}{\lambda}\right] - \gamma_E\right) e^{-16\pi/\lambda} \pm i16\pi\, e^{-16\pi/\lambda}$$

According to $\arg\lambda = 0^{\pm}$ $\longrightarrow$ see Airy

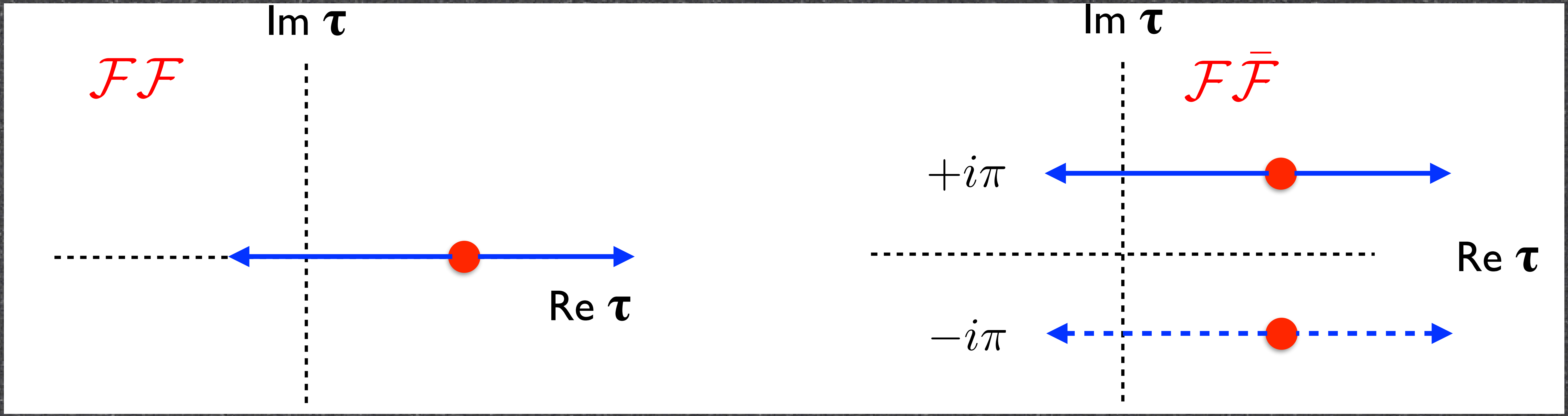
FFbar:



Performing the integral over
the correct thimble

$$[\mathcal{F}\bar{\mathcal{F}}]_{\pm} = -\underbrace{\left(-\log\left[\frac{32\pi}{\lambda}\right] - \gamma_E\right)}_{\text{Same multi-instantons factor from uniform WKB (Dunne-Unsal)}} e^{-16\pi/\lambda} \pm i16\pi\, e^{-16\pi/\lambda}$$

FFbar:



Performing the integral over the correct thimble

Note that the correct thimble lives now in the complexification of the fields space but it still has real dimension = 1

Resumming Pert and Non-Pert:

Leading renormalon ambiguity of perturbation theory cancels against the contour jumps in the thimble decomposition

$$\mathrm{Im}\left[\mathcal{S}_{\pm} E_P(g^2) + [\mathcal{F}\bar{\mathcal{F}}]_{\pm}\right] = 0 \quad \text{up to } O(e^{-4S_F})$$

Path integral is not just perturbation theory!
It is only when we sum over the correct saddles decomposition that we find unique and unambiguous answer!

Borel-Ecalle resummation of a transseries


THE
END
IS
NEAR

Conclusions & Outlook:

* Recovering non-perturbative info from the perturbative expansion (and vice-versa)

* Reinterpret exact quantities as transseries

  Cusp anomalous dimension (Basso-Korchemsky-Kotanski)

* Why phys. observables are resurgent functions? Lefschetz Thimbles and analytic continuation of path integrals

* What if we have only a finite number of terms?

  Padé Approximants

Thanks for Listening!