Thank the Organizers

Construction of an Umbral module

With John Duncan

J. Harvey, Eurostrings 2015

OUTLINE

1. Moonshine, Monstrous and Umbral
2. A free fermion trick
3. The 3E8 example of Umbral Moonshine
4. Indefinite lattice theta functions
5. A Vertex Operator Algebra for indefinite lattices
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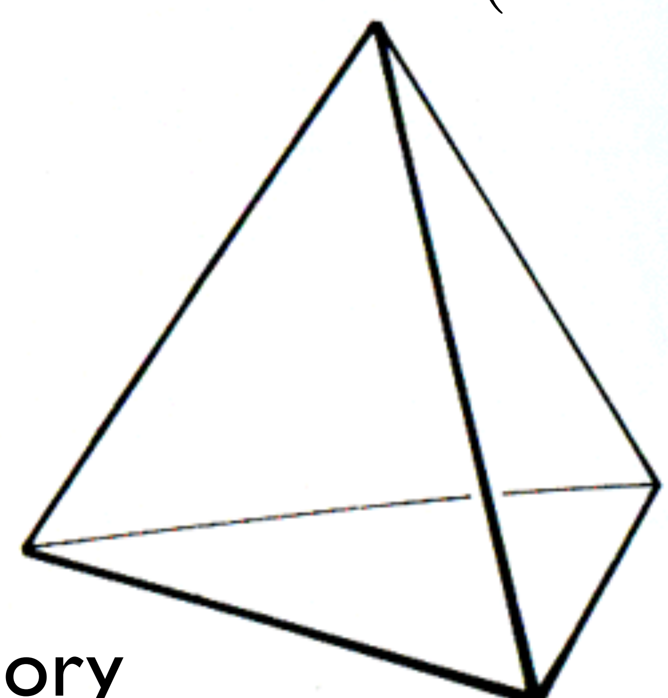
7. What does it mean for string theory?

M O O N S H I N E



M (Mock)ModularForms
G
Groups (Finite)
S
String theory
A
Algebras (VOA)

The original example of this structure is Monstrous Moonshine

G The Monster sporadic group

M The weight zero modular function

$$J(\tau) = q^{-1} + 196884q + \cdots , \quad q = e^{2\pi i\tau}$$

A Vertex Operator Algebra (OPE of chiral Vertex operators)

S Bosonic or Heterotic String on an asymmetric orbifold background $(\mathbb{R}^{24}/\Lambda_L)/(\mathbb{Z}/2)$

We now have Mathieu Moonshine and its extension to Umbral Moonshine. We know the extension for two of these objects:

$G:$ $\mathrm{M24}$ or more generally $G^X = Aut(L^X)/W^X$

$M:$ A weight 1/2 mock modular form whose coefficients count the multiplicities of massive N=4 SCA characters in the elliptic genus of K3

Eguchi, Ooguri, Tachikawa

$$H^{(2)}(\tau) = 2q^{-1/8}\left(-1 + 45q + 231q^2 + 770q^3 + \cdots\right)$$

or more generally, a weight 1/2, m-1 component vector-valued mock modular form $H_r^X(\tau)$

Here L^X is one of the 23, rank 24 Niemeier lattices and $m = m(X)$ is its Coxeter number.

X	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4D_4$	A_6^4	$A_7^2D_5^2$
ℓ	2	3	4	5	6	7	8
G^X	M_{24}	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$GL_2(3)$	$SL_2(3)$	Dih_4
$\bar{G}^X$	M_{24}	M_{12}	$AGL_3(2)$	$PGL_2(5)$	$PGL_2(3)$	$PSL_2(3)$	2^2

X	A_8^3	$A_9^2D_6$	$A_{11}D_7E_6$	A_{12}^2	$A_{15}D_9$	$A_{17}E_7$	A_{24}
ℓ	9	10	12	13	16	18	25
G^X	Dih_6	4	2	4	2	2	2
$\bar{G}^X$	Sym_3	2	1	2	1	1	1

X	D_4^6	D_6^4	D_8^3	$D_{10}E_7^2$	D_{12}^2	$D_{16}E_8$	D_{24}
ℓ	6+3	10+5	14+7	18+9	22+11	30+15	46+23
G^X	$3.Sym_6$	Sym_4	Sym_3	2	2	1	1
$\bar{G}^X$	Sym_6	Sym_4	Sym_3	2	2	1	1

X	E_6^4	E_8^3
ℓ	12+4	30+6,10,15
G^X	$GL_2(3)$	Sym_3
$\bar{G}^X$	$PGL_2(3)$	Sym_3

Are there analogs of the VOA algebraic structure and/or a string theoretic interpretation of these structures?

More specifically, is there an explicit construction of the infinite-dimensional G^X modules implied by these constructions along with an explicit action of G^X ?

That is, the M24 mock modular form suggests that there exists an infinite set of vector spaces

$$K^{(2)}_{-1/8}, K^{(2)}_{7/8}, K^{(2)}_{15/8}, \cdots$$

of dimension 2, 2x45, 2x231, …that provide representations of M24 and should be combined into

$$K^{(2)} = \bigoplus_{n=0}^{\infty} K^{(2)}_{n-1/8}$$

In Monstrous Moonshine we have

$$V^\natural = \bigoplus_{n \geq -1} V_n$$

$$\mathrm{dim} V_n = c(n), \quad J(q) = \sum_{n=-1}^{\infty} c(n) q^n$$

with V_n the vector space of states of conformal dimension $n - c/24$ in the Monster CFT.

We would like an analog of the Frenkel-Lepowsky-Meurman construction of the Monster CFT for Umbral Moonshine.

We know the modules exist for Mathieu Moonshine (Gannon) and in fact for all the cases of Umbral Moonshine (Duncan, Griffin&Ono), but proving they exist is not the same as giving an explicit and natural construction. Such a construction may point us in some interesting new directions in string theory and CFT if history is any guide.

I will present an explicit construction for one of the simplest cases of Umbral Moonshine with $L^X = E_8^3$ and $G^X = S^3$, the permutation group on three objects that appears in arXiv:1412.8191 [math.RT]

This paper is written in mathy VOAese and may be difficult to read for string theorists (like me), my goal here is to advertise the main ideas and ingredients while suppressing the complicated details.

A free fermion trick

Monstrous moonshine started with observations about the weight zero modular J function and the decomposition of its coefficients into dimensions of Monster irreps:

$$J(\tau) = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

$$196883 + 1 \qquad\qquad 21296876 + 196883 + 1$$

But note that moonshine for the Monster is more obvious for the weight 1/2 modular form

$$\eta(\tau)J(\tau) = q^{1/24}\left(q^{-1} - 1 + 196883q + 21296876q^2 + 842609326q^3 + \cdots\right)$$

dimensions of Monster irreps

The reason is simple, Virasoro characters for $c>1$ involve a factor of $1/\eta(\tau)$ so this is essentially a decomposition into Virasoro characters.

We can interpret this in terms of a c=24+1/2 CFT.

Consider the Ramond sector of a free Majorana fermion

$$\{\psi_n, \psi_m\} = 2\delta_{n+m,0}, \quad n, m \in \mathbb{Z}$$

Physicists often want there to be a fermion number operator $(-1)^F$ obeying $\{(-1)^F, \psi_n\} = 0, \quad \text{all } n$ along with $\left((-1)^F\right)^2 = 1, \quad \psi_0^2 = 1$ and the smallest representation of this Clifford algebra is two-dimensional.

For example in Ginsparg's CFT notes one finds

$$\sigma_z | \pm \frac{1}{16} \rangle = \pm | \pm \frac{1}{16} \rangle$$

$$(-1)^F = \sigma_z (-1)^{\sum \psi_{-n} \psi_n}, \quad \psi_0 = \sigma_x (-1)^{\sum \psi_{-n} \psi_n}$$

But Ginsparg also notes that it is not strictly necessary to have a well defined $(-1)^F$ operator in CFT and one can then work with a single ground state which can either be

$$|v_{tw}^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|+\frac{1}{16}\rangle \pm |-\frac{1}{16}\rangle\right)$$

If we denote by $A_{tw}^{\pm}$ the Hilbert space built by acting with fermion creation operators on these ground states then it is easy to see that

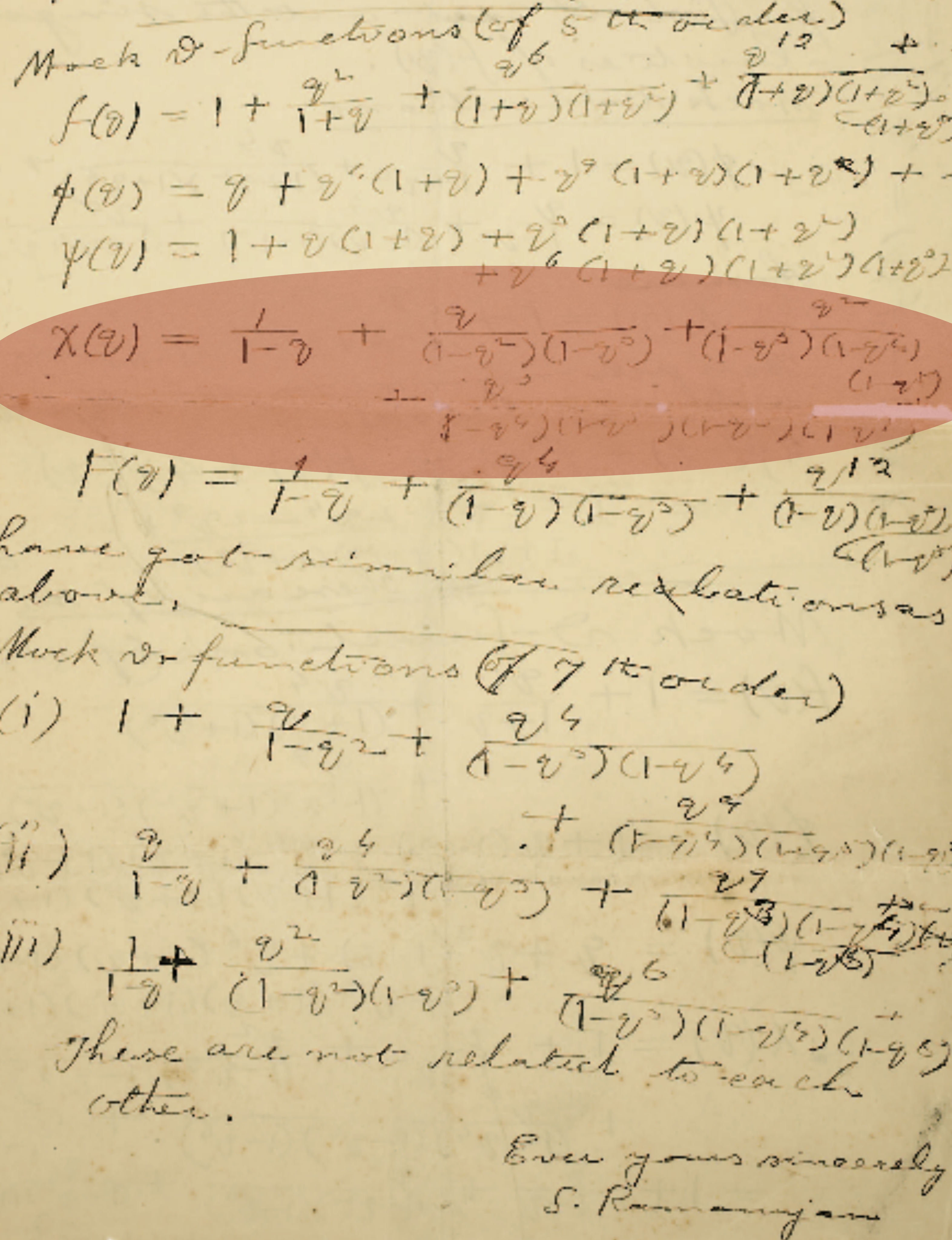
$$Tr_{A_{tw}^{\pm}}\psi_0 q^{L_0-c/24} = \pm q^{1/24}\prod_{n>0}(1-q^n) = \pm\eta(\tau)$$

Thus moonshine for the Monster is even more evident if we work in the c=24+1/2 CFT that is the tensor product of the Monster CFT with a c=1/2 free fermion CFT dealt with as on the previous slide.

We will use this “trick” below, taking the tensor product of a c=3 theory with a c=1/2 free Ramond sector fermion without $(-1)^F$.

Let's now turn to Umbral Moonshine and the example based on the E_8^3 Niemeier lattice.

Although m=30, there are only two independent components for the vector valued mock modular form. One of our results is that these two components are given in terms of Ramanujan's fifth order mock theta functions. This result was anticipated in our (Cheng, Duncan & Harvey) paper on Umbral Moonshine, but here we are able to prove the identification.


Mock ϑ-functions (of 5th order)

$f(q) = 1 + \frac{q^2}{1+q} + \frac{q^6}{(1+q)(1+q^2)} + \frac{q^{12}}{(1+q)(1+q^2)(1+q^3)} + \cdots$

$\phi(q) = q + q^4(1+q) + q^9(1+q)(1+q^2) + \cdots$

$\psi(q) = 1 + q(1+q) + q^3(1+q)(1+q^2) + q^6(1+q)(1+q^2)(1+q^3)$

$\chi(q) = \frac{1}{1-q} + \frac{q}{(1-q^2)(1-q^3)} + \frac{q^2}{(1-q^3)(1-q^4)(1-q^5)} + \frac{q^3}{(1-q^4)(1-q^5)(1-q^6)(1-q^7)}$

$F(q) = \frac{1}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^{12}}{(1-q)(1-q^3)(1-q^5)}$

have got similar relations as above.

Mock ϑ-functions (of 7th order)

(i) $1 + \frac{q}{1-q^2} + \frac{q^4}{(1-q^3)(1-q^4)} + \frac{q^9}{(1-q^4)(1-q^5)(1-q^6)}$

(ii) $\frac{q}{1-q} + \frac{q^4}{(1-q^2)(1-q^3)} + \frac{q^9}{(1-q^3)(1-q^4)(1-q^5)} + \cdots$

(iii) $\frac{1}{1-q} + \frac{q^2}{(1-q^2)(1-q^3)} + \frac{q^6}{(1-q^3)(1-q^4)(1-q^5)} +$

These are not related to each other.

Ever yours sincerely
S. Ramanujan

χ_1 from Ramanujan's last letter to Hardy in 1920 introducing mock theta functions.

He introduced 17 examples of mock theta functions in this letter. Most of them have reappeared in Umbral Moonshine.

$$H^X_{1A,r}(\tau) = \begin{cases} \pm 2q^{-1/120}\left(\chi_0(q) - 2\right), & \text{if } r = \pm1, \pm11, \pm19, \pm29, \\ \pm 2q^{71/120}\chi_1(q), & \text{if } r = \pm7, \pm13, \pm17, \pm27. \end{cases}$$

In terms of the Pochammer symbol $(x;q)_n = \prod_{k=0}^{n-1}(1 - xq^k)$

$$\chi_0 = \sum_{n\geq 0} \frac{q^n}{(q^{n+1};q)_n} = 1 + q + q^2 + 2q^3 + q^4 + 3q^5 + \cdots$$

$$\chi_1 = \sum_{n\geq 0} \frac{q^n}{(q^{n+1};q)_{n+1}} = 1 + 2q + 2q^2 + 3q^3 + 3q^4 + 4q^5 + \cdots$$

and we have identities due to Zwegers:

$$2 - \chi_0(q) = \frac{q^{1/12}}{\eta(q)^2}\left(\sum_{k,l,m\geq 0} + \sum_{k,l,m<0}\right)(-1)^{k+l+m}q^{(k^2+l^2+m^2)/2+2(kl+lm+mk)+(k+l+m)/2},$$

$$\chi_1(q) = \frac{q^{1/12}}{\eta(q)^2}\left(\sum_{k,l,m\geq 0} + \sum_{k,l,m<0}\right)(-1)^{k+l+m}q^{(k^2+l^2+m^2)/2+2(kl+lm+mk)+3(k+l+m)/2},$$

The last expression is rather suggestive, but also odd. The numerator looks like a theta function for a 3-dimensional lattice, but summed only over part of the lattice. And the partition function for 3 bosons on a lattice should be of the form

$$\frac{\Theta_L(\tau)}{\eta(\tau)^3}$$

but the expression only has $\eta(\tau)^2$ in the denominator.

The resolution involves two ingredients. First, tensor with a free Ramond sector fermion so that

$$\eta(\tau) \times \frac{1}{\eta(\tau)^3} = \frac{1}{\eta(\tau)^2}$$

Second, interpret the sum in terms of indefinite theta functions.

In Zwegers 2002 reanalysis of mock modular forms he discussed three constructions, two of which have played a prominent role in recent moonshine developments:

Appell-Lerch sums which are closely related to massless characters of the N=4 SCA and thus appear in the decomposition of the elliptic genus of K3 into N=4 characters.

Meromorphic Jacobi forms and their decomposition into Polar and Free parts and growth conditions (Dabholkar, Murthy and Zagier)

$$\psi(\tau, z) = \psi^P(\tau, z) + \psi^F(\tau, z)$$

Meromorphic in z, typically single or double poles

Polar part, double center Black Holes

Single center, immortal Black Holes

For weight 1, index m $\psi^F(\tau, z) = \sum_r H_r(\tau)\theta_{m,r}(\tau, z)$

Weight 1/2 vector-valued mock modular form

This formula of Zwegers suggests we utilize his third construction in terms of theta functions attached to cones in lattices of indefinite signature.

$$\theta(\tau) = \sum_{p\in\Lambda} q^{p^2/2} \longrightarrow \theta_{c_1,c_2}(\tau, \bar{\tau}) = \sum_{p\in\Lambda} \rho_{c_1,c_1}(\tau, \bar{\tau}) q^{p^2/2}$$

theta function for lattice with positive definite quadratic form

theta function for indefinite form with a non-holomorphic convergence factor for negative norm points

This sort of construction has appeared recently in the physics literature in BPS state counting problems: Manschot, Cardoso, Cirafici, Jorge, Nampuri, Pioline.

The holomorphic part of this theta function involves a sum over cones within the light-cones of the lattice where the points have positive norm

For our explicit construction we consider the lattice

$$L = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 \quad \text{with} \quad \langle e_i, e_j \rangle = 2 - \delta_{i,j}$$

with an obvious Sym_3 symmetry that acts by permuting the basis vectors e_i . We also consider the Sym_3 invariant vector

$$\rho = (e_1 + e_2 + e_3)/5 \in L^*$$

$$\lambda = ke_1 + le_2 + me_3, \quad \langle \lambda, \rho \rangle = k + l + m$$

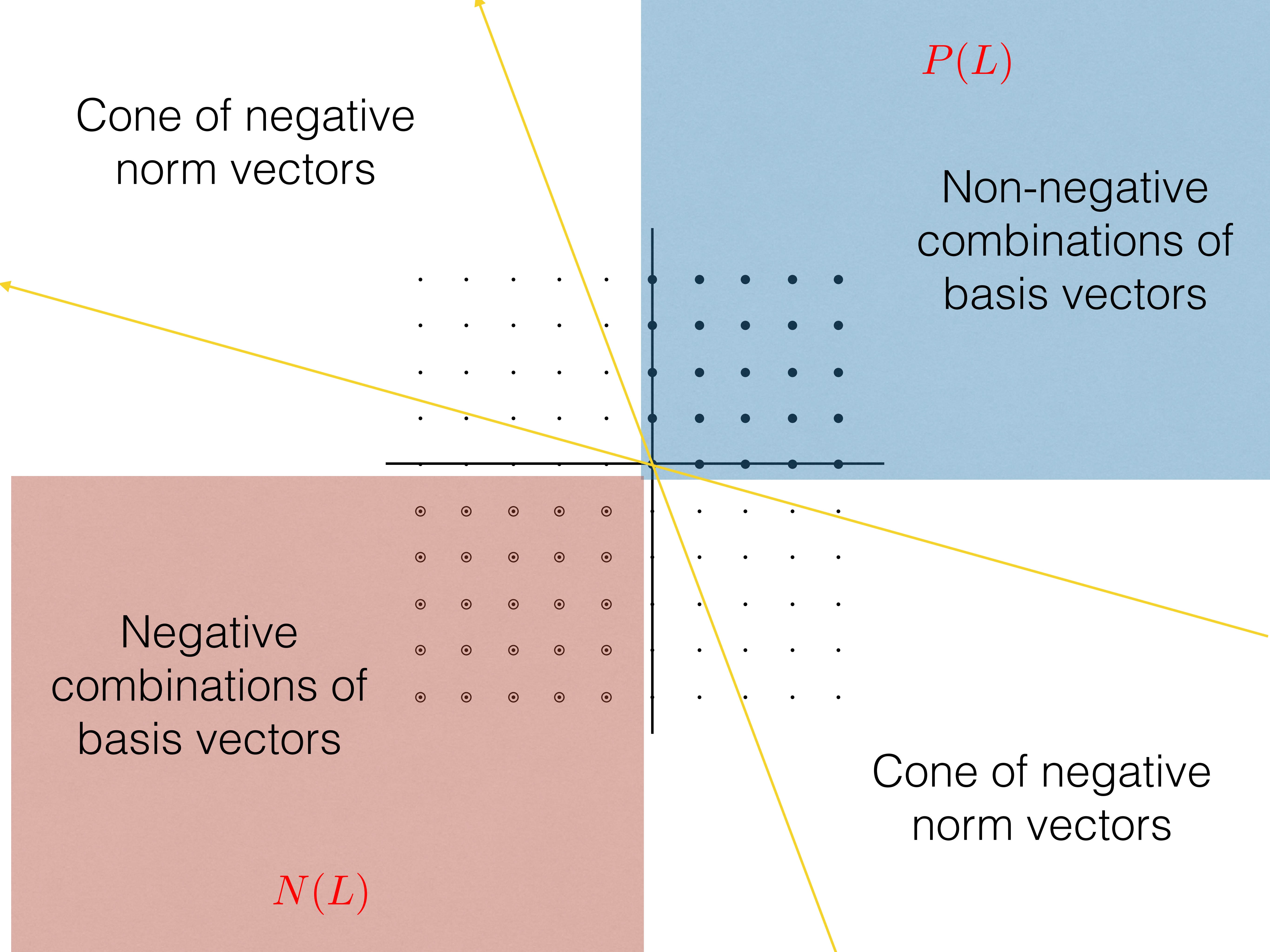
and the action of $g_{\rho/2}(\lambda) = (-1)^{k+l+m}$ which divides the lattice into even and odd parts according to the sign of $g_{\rho/2}(\lambda)$

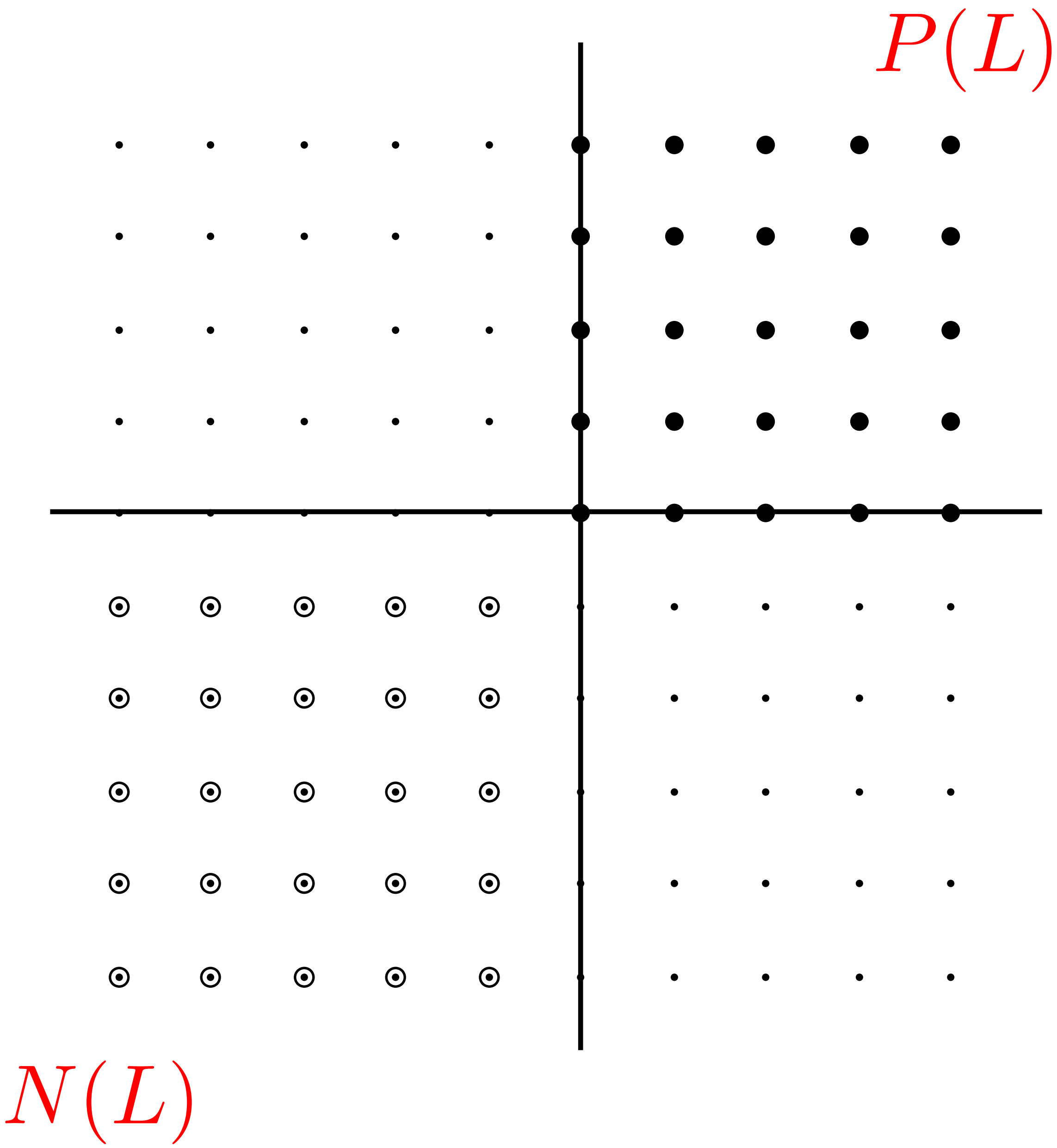
One can associate a $c = 3$ bosonic Virasoro algebra and vertex operators to this lattice in the usual way with vertex operators constructed out of products of

$$\partial_z^n X^i \qquad\qquad e^{i\lambda\cdot X}$$

Note however that if we include lattice vectors that do not have positive norm we will not have finite dimensional vector spaces at a fixed eigenvalue of L_0

On the other hand, if we restrict to (a subspace of) vectors with positive definite norm we don't have the usual OPE of Vertex Operators since addition of vectors with positive norm can give vectors with negative norm. Our solution is to restrict to (a subspace of) vectors with positive definite norm, but to modify the addition law for vectors and hence the OPE.


$P(L)$
Cone of negative norm vectors
Non-negative combinations of basis vectors
Negative combinations of basis vectors
Cone of negative norm vectors
$N(L)$



$$\lambda \tilde{\mp} \mu = \lambda + \mu \in P \qquad \lambda, \mu \in P$$

$$\lambda \tilde{\mp} \mu = -\lambda + \mu \in N \qquad \lambda \in P, \ \mu \in N$$

$$\lambda \tilde{\mp} \mu = \lambda - \mu \in N \qquad \lambda \in N, \ \mu \in P$$

$$\lambda \tilde{\mp} \mu = -\lambda - \mu \in P \qquad \lambda, \mu \in N$$

The operation $\tilde{\mp}$ defines a “commutative monoid” (like an Abelian group but without inverses) which is enough to define an algebra on lattice vectors in $P(L) \cup N(L)$ and by extension an algebra of vertex operators.

Now we put all these pieces together to obtain a VOA with c=3+1/2 and verify that certain trace functions agree exactly with those of the $X = E_8^3$ example of Umbral Moonshine. That is the $H^X_{1A,r}$ and their relatives twisted by order 2 or 3 elements in Sym_3 can all be written in the form

$$\mathrm{tr}_{V^{\pm}_{tw,a}} g g_{\rho/2} \psi_0 q^{L_0 - c/24}$$

Free fermion plus 3d lattice theory with vectors in D, shifted by a multiple of the vector rho

action of S3 on the lattice

+/- weighting on even/odd lattice vectors

Verifying this and checking the shadows and uniqueness requires a lot of detailed messing around with results in Zwegers Ph.D thesis.

What does it mean for string theory?

I have no idea.

Really, I have no idea.

BPS states have indefinite charge lattices

Narain compactifications have indefinite momentum/winding lattices

The Umbral moonshine mock modular forms are very special and rigid and have deep ties to number theory. They started with the study of K3 surfaces. We really ought to understand what is going on.

Thank the Audience