The Casimir and Cardy Problems in $d = 4$ Quantum Field Theories

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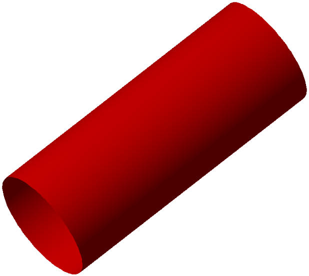
⋆ Closset, Dumitrescu, Festuccia, Seiberg [1205.4142] [1206.5218]
⋆ Closset, Dumitrescu, Festuccia [1309.5876] [1407.2598]
⋆ Di Pietro [1407.6061]
⋆ Assel, Cassani, Di Pietro, Lorenzen, Martelli [1503.05537]

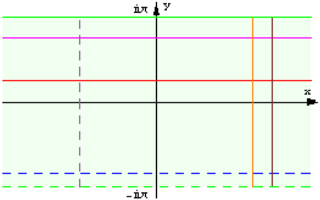
Consider a CFT_2, with central charge c. Let us study the theory on the cylinder $\mathbb{S}^1 \times \mathbb{R}$. The ground state (Casimir) energy is

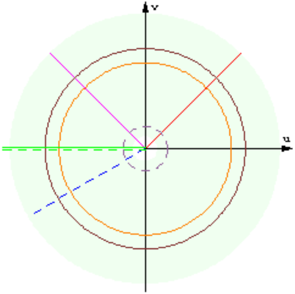
$$E_0 = -\frac{c}{12l} \ .$$

For a free boson it comes from $\sum n = -1/12$.



The general proof relies on the observation that $\mathbb{S}^1 \times \mathbb{R}$ is conformally equivalent to $\mathbb{R}^2$. The energy-momentum tensor transforms with a Schwarzian derivative.





We can study the theory $\mathbb{S}^1 \times \mathbb{S}^1_\beta$ and consider the partition function for large β

$$\lim_{\beta\to\infty} Z(\beta) = e^{-E_0\beta} + ... = e^{\frac{c}{12}\beta} + ...$$

By a modular transformation this can be related to the small β limit of the partition function. One finds the Cardy formula:

$$\lim_{\beta\to 0} Z(\beta) = e^{\frac{(2\pi)^2 c}{12\beta}} + ...$$

We can now consider a $d = 4$ CFT and put it on $\mathbb{S}^3 \times \mathbb{R}$, which is conformally flat. We can then compactify the Euclidean time direction to be $\mathbb{S}^1_\beta$. We can ask the same two questions:

- What is the large β limit? (Casimir)
- What is the small β limit? (Cardy)

The conformal anomalies of $d = 4$ CFT are known to be just a, c

$$T^{\mu}_{\mu} \sim aE_4 + cW^2$$

and these fix the analog of the Schwarzian derivative in $d = 4$. Following this idea one finds

$$E_0 = \frac{3}{4}a \ .$$

However, people have done explicit computations of the ground state energy E_0 and got different answers using different regularizations [Birrell, Davies ; Brown, Cassidy.....]

One should realize that E_0 is actually **unphysical** in $d = 4$. It depends on the regularization scheme. The point is that we can add

$$\delta S \sim b \int d^4x \sqrt{g} R^2 \ . \qquad T^{\mu}_{\mu} = ... + b \Box R$$

This can be viewed as a modification of the Schwarzian derivative. No such modification exists in $d = 2$.

This clearly modifies the ground state energy by $\delta E_0 \sim b/r$. The parameter b can be tuned to any desirable value.

The Casimir energy maps under the AdS/CFT duality to the ADM mass of AdS_5 [Balasubramanian-Kraus]. This is thus scheme dependent.

Below we will see that with SUSY it becomes physical! It would be great to compare with an AdS_5 computation.

So one of the points below would be a proof that, for supersymmetric CFTs, the parameter b **disappears**! Therefore, the Casimir energy becomes physical again. And the ground state energy **is not** $\frac{3}{4}a$. Rather, what we find is

$$E_0 = \frac{4}{27r}(a+3c)\ .$$

This is of order N^2. The partition function at large β therefore behaves like $e^{-\frac{4\beta}{27r}(a+3c)}$.

$$\lim_{\beta\to\infty} Z(\beta) = e^{-\frac{4\beta}{27r}(a+3c)}$$

We will also consider the small β limit of a $d=4$ CFT on $\mathbb{S}^3\times\mathbb{S}^1_\beta$. We can understand this by a local action for the background fields in $d=3$:

$$\lim_{\beta\to 0} Z_{\mathbb{S}^3\times\mathbb{S}^1_\beta} = e^{\frac{a'}{\beta^3}l^3+\frac{b'}{\beta}l+\dots}$$

$$S_{d=3}=\int d^3x\sqrt{g}\left(a'T^3+b'TR+\dots\right) \ ,$$

where R is the Ricci scalar. Usual high-temperature expansion.

In examples we find that a', b' are not given by any combination of anomalies. For example, a', b' depend on the coupling constants.

Below we will see that in supersymmetric theories $a'=0$ and

$$b'\sim(a-c)$$

More precisely, we will see that in supersymmetric theories

$$\lim_{\beta\to 0} Z(\beta) = \lim_{\beta\to 0} Tr_{\mathcal{H}}((-1)^F e^{-\beta H}) = e^{\frac{(4\pi)^2}{3\beta}(c-a)}$$

This is very much reminiscent of the Cardy formula. Does this suggest some kind of generalized modular invariance in $d=4$? maybe...

Summary: The Casimir and Cardy problem appear to be natural in supersymmetric theories in $d = 4$. To study them we need to understand supersymmetry in curved space. This is a nice story, which has lots of other applications.

Supersymmetric Field Theories have a conserved charge Q_α such that it squares to a translation

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \ .$$

Supersymmetry transformations are generated by

$$\delta \equiv \zeta^\alpha Q_\alpha + \bar\zeta_{\dot\alpha} \bar{Q}^{\dot\alpha}$$

where ζ_α is a **constant** spinor

$$\partial_\mu \zeta_\alpha = 0 \ .$$

The spinors that survive in curved space satisfy some generalized Killing spinor equation of the form [Festuccia-Seiberg...]

$$(\nabla_\mu - A_\mu^R)\zeta = \sigma_{\mu\nu}\epsilon^{\nu\alpha\beta\gamma}\partial_\alpha B_{\beta\gamma}\zeta \ .$$

with A^R a one-form gauge field and $B_{\mu\nu}$ a Kalb-Ramond gauge field. The (g, A^R, B) multiplet is known as the "new minimal" multiplet [Sohnius, West].

For a general choice of these background fields, the coupling to curved space breaks supersymmetry. No solutions exist.

It turns out that a necessary and sufficient condition to preserve at least one supercharge is that $\mathcal{M}_4$ is *Hermitian* and the G-bundle is *holomorphic*. [Closset, Dumitrescu,Festuccia,ZK, Seiberg]

A particularly interesting case to consider is

$$\mathbb{S}^3 \times \mathbb{S}^1$$

This is a Hermitian manifold, and there is a two-complex-dimensional moduli space of complex structures. One can also introduce holomorphic gauge bundles.

Let us specify a four-manifold $\mathcal{M}_4$ with some complex structure $J^2 = -1$, a Hermitian metric $g_{i\bar{j}}$, and some holomorphic G-bundle A_μ^G. So we have

$$Z_{\mathcal{M}_4}[J_i^j, \overline{J}_{\bar{i}}^{\bar{j}}, g_{i\bar{j}}, A_\mu^G ...]$$

The $\cdots$ stand for additional parameters, e.g. coupling constants.

The Partition Function

Several general properties of the partition function $Z_{\mathcal{M}_4}[J_i^j, \overline{J}_{\bar{i}}^{\bar{j}}, g_{i\bar{j}}, A_\mu^G, ...]$: [Closset, Dumitrescu, Festuccia, ZK]

- Given the complex structure $J^2 = -1$, the partition function is *independent* of the Hermitian metric $g_{i\bar{j}}$.
- The dependence on the complex structure moduli is *holomorphic*.
- The partition function depends *holomorphically* on the moduli of the holomorphic G-bundle.
- The partition function is independent of small variations of the coupling constants.

Let us therefore consider $\mathbb{S}^3 \times \mathbb{S}^1$ in more detail. We can choose the round metric,

$$ds^2 = d\theta^2 + (d\mathbb{S}^3)^2 \ .$$

To preserve supersymmetry we need to turn on some flux of $B_{\mu\nu}$ through the $\mathbb{S}^3$ as well as to turn on $A^R = \frac{1}{r} d\theta$.

The flux through $\mathbb{S}^3$ is not crucial but the flat A^R gauge field is a chemical potential. This chemical potential must exist in order to preserve SUSY.

There are four supercharges, which transform under $SU(2)_L \times SU(2)_R$ in

$$(1/2, 0) \oplus (1/2, 0)$$

The superalgebra is

$$\{Q_\alpha, Q^\dagger_{\dot\alpha}\} = \sigma^0_{\alpha\dot\alpha}(H - \frac{1}{r}R) + \frac{1}{r}\sigma^i_{\alpha\dot\alpha}J^i_L$$

In the limit of $r \to \infty$ we recover the flat space algebra. Let us specialize to one particular supercharge Q_2,

$$\{Q_2, Q^\dagger_{\dot 2}\} = H - \frac{1}{r}R + \frac{1}{r}J^3_L$$

We can interpret the partition function as a trace over the Hlibert space, $\mathcal{H}$, on $\mathbb{S}^3$,

$$Z(\beta) = Tr_{\mathcal{H}}((-1)^F e^{-\beta H})$$

From the superalgebra we see that we only receive contributions from states on $\mathbb{S}^3$ that satisfy

$$H - \frac{1}{r}R + \frac{1}{r}J_L^3 = 0 \ .$$

These are in short representations.

We start from the small β limit. Then, we get a local action in $d = 3$ (at least as far as negative powers of β are concerned). We therefore need to supersymmetrize the $3d$ action on $\mathbb{S}^3$

$$S_{d=3} = \int d^3x\sqrt{g}\left(a' T^3 + b' TR + ...\right) \ ,$$

(where R is the Ricci scalar) such that it enjoys $\mathcal{N} = 2$ supersymmetry.

$$S_{d=3} = \int d^3x\sqrt{g}\left(a' T^3 + b' TR + \ldots\right) \ ,$$

- $a' = 0$ because the cosmological constant is not compatible with new-minimal supergravity.
- The supersymmetric Einstein-Hilbert term is in the same multiplet as a Chern-Simons term $A \wedge dG$ (G being the KK graviphoton). The coefficient of a Chern-Simons term cannot depend on coupling constants. We can compute it in free field theory. This gives
$$b' \sim (a - c) \ .$$

Therefore, the coefficient of R is protected and we can make a prediction about the asymptotics of the number of states on $\mathbb{S}^3$:

$$\lim_{\beta\to 0} Tr_{\mathcal{H}}((-1)^F e^{-\beta H}) = e^{\frac{(4\pi)^2}{3\beta}(c-a)}$$

If $c > a$ there is a proliferation of bosonic BPS states. Otherwise, there is a dramatic cancelation with exponential precision. SUSY does not imply any such cancelations at the kinematical level. This may explain why theories with $a > c$ are so hard to construct.

Let us now consider the opposite, Casimir, limit, i.e. $\beta \to \infty$.

We explained that it is unphysical in the general case. Let us prove that it is universal once we add supersymmetry.

In the limit $\beta \to \infty$ it is natural to reduce on the three-sphere in $\mathbb{S}^3 \times \mathbb{R}$. Then, we need to compute the ground state energy in Quantum Mechanics.

The Quantum Mechanics that we get has four supercharges and R-symmetry group $SU(2) \times U(1)$. The algebra contains for example

$$\{Q_2, Q_2^\dagger\} = H - \frac{1}{r} R + \frac{1}{r} J_L^3$$

Clearly the vacuum has $J_L^3 |VAC\rangle = 0$, for otherwise, it would not be unique.

If we just had QM with finitely many degrees of freedom we could not fix E_0 because we can add a normal ordering constant to $H \to H + c/r$ and $R \to R + c$

Shifting the R-charge by a constant $R \to R + c$ corresponds to adding a Chern-Simons term in Quantum Mechanics

$$\delta S = c \int dt A_0^R$$

However, the allowed counter-terms must descend from counter-terms in four dimensions. The only possible four-dimensional term from which it could descend is

$$\int d^4x \sqrt{g}\; A_\mu^R \epsilon^{\mu\nu\rho\sigma} \partial_\nu B_{\rho\sigma}$$

which gives a dependence on r which is different than what we need.

This therefore proves that the Casimir energy on $\mathbb{S}^3$ is physical in supersymmetric theories. (A different proof can be constructed using the classification of counter-terms in [Assel, Cassani, Martelli].)

The relation to the Chern-Simons term in Quantum Mechanics shows that the Casimir energy is independent of coupling constants.

From the superalgebra we see that $H = \frac{1}{r}R$ in the vacuum. So instead of computing the energy we can compute the R-charge of the vacuum.

A single fermion contributes to the Chern-Simons coupling $\kappa \int dt\ A$

$$\kappa_{ir} - \kappa_{uv} = -\frac{q}{2} sgn(M)\ .$$

In QM we cannot determine κ_{uv}. But here we know $\kappa_{uv} = 0$ because the theory comes from $d = 4$.

In the reduction to quantum mechanics, one encounters long multiples, chiral multiplets, and Fermi multiplets. Long multiplets always have fermions of opposite mass. So the R-charge of the vacuum only comes from chiral and Fermi multiplets. One finds

$$2rE_0 = \lim_{t\to 0} \left(\sum_l (l+1)(l+r) e^{-t(l+r)} \right) - (r \to 2-r)$$

$$= \frac{8}{27}(a + 3c)\ .$$

In AdS/CFT, E_0 is interpreted as the (ADM) mass of AdS_5. In general, the ADM mass of AdS_5 is not meaningful.

But our discussion shows that for supersymmetric theories it should be meaningful. It would be interesting to compute it and compare with the prediction

$$E_0 = \frac{4}{27r}(a + 3c)$$

We have also computed the Casimir energy for the most general complex structure on the cylinder $S^3 \times \mathbb{R}$.

Another obvious extension would be to compute the Casimir energy on $\mathcal{M}_3 \times \mathbb{R}$ with arbitrary $\mathcal{M}_3$ that is Seifert. This looks doable.

Thank you for your attention