A menagerie of ambitwistor string theories

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With David Skinner. arxiv:1311.2564, and various collaborations with Tim Adamo, Eduardo Casali, Yvonne Geyer, Arthur Lipstein, Ricardo Monteiro & Kai Roehrig, 1312.3828, 1404.6219, 1405.5122, 1406.1462, CGMMRS 150?.????.

[Cf. also Cachazo, He, Yuan arxiv:1306.2962, 1306.6575, 1307.2199, 1309.0885, 1412.3479]

Amplitude formulae for massless theories.

Theorem (Cachazo, He, Yuan 2013,2014)

Tree-level massless amplitudes in d-dims are integrals/sums

$$\mathcal{M}(1,\ldots,n)=\delta^d\left(\sum_i k_i\right)\int_{\mathbb{CP}^{1\,n}}\frac{I^l(\epsilon^l_i,k_j)I^r(\epsilon^r_i,k_j)}{\mathrm{Vol}\,\mathrm{SL}(2,\mathbb{C})}\prod_i{}'\bar\delta(k_i\cdot P(\sigma_i))d\sigma_i$$

where $I^l(\epsilon^l_i,k_i,\sigma_i)$ and $I^r(\epsilon^r_i,k_i,\sigma_i)$ depend on the theory.

- polarizations ϵ^l_i for spin 1, $\epsilon^l_i\otimes\epsilon^r_i$ for spin-2 ($k_i\cdot\epsilon_i=0\ldots$).
- Introduce skew $2n\times 2n$ matrices $M=\begin{pmatrix}A & C\\ -C^t & B\end{pmatrix}$,

$$A_{ij}=\frac{k_i\cdot k_j}{\sigma_i-\sigma_j},,\quad B_{ij}=\frac{\epsilon_i\cdot\epsilon_j}{\sigma_i-\sigma_j},\quad C_{ij}=\frac{k_i\cdot\epsilon_j}{\sigma_i-\sigma_j},\qquad \text{for } i\neq j$$

  and $A_{ii}=B_{ii}=0$, $C_{ii}=\epsilon_i\cdot P(\sigma_i)$.
- For YM, $I^l=Pf'(M)$, $I^r=\prod_i\frac{1}{\sigma_i-\sigma_{i-1}}$.
- For GR $I^l=Pf'(M^l)$, $I^r=Pf'(M^r)$

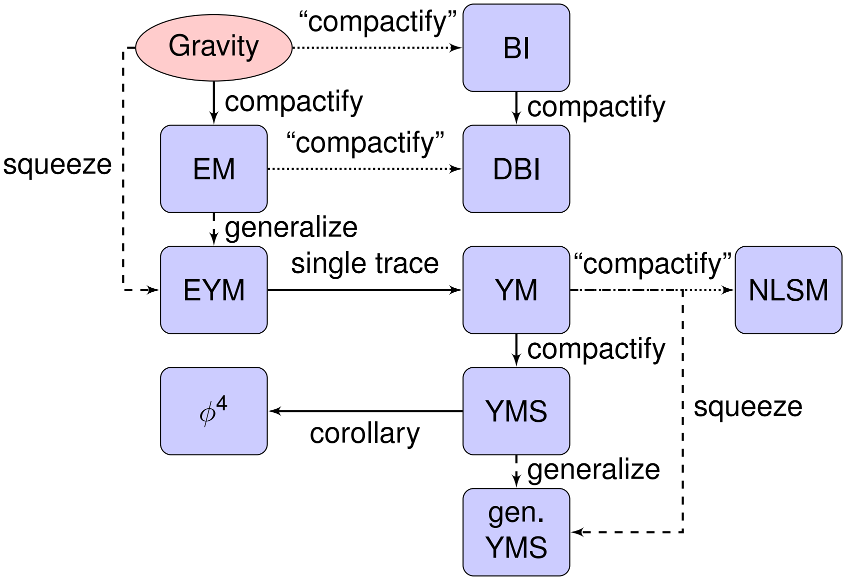

Figure: Theories studied by CHY and operations relating them.

Complexify real space-time $M_{\mathbb{R}} \rightsquigarrow M$, and null covectors P.

$\mathbb{A} :=$ space of complex null geodesics with scale of P.

- $\mathbb{A} = T^*M|_{P^2=0}/\{D_0\}$ where $D_0 := P \cdot \nabla =$ geodesic spray.
- D_0 has Hamiltonian P^2 wrt symplectic form $\omega = dP_\mu \wedge dx^\mu$.
- $\theta = P_\mu dx^\mu$ symplectic potential, $\omega = d\theta$, descends to $\mathbb{A}$.

Theorem (LeBrun 1983)

The complex structure on $\mathbb{A}/$scalings determines M and conformal metric g. Correspondence is stable under deformations of the complex structure of $P\mathbb{A}$ that preserve θ.

From deformations of $\mathbb{A}$ to the scattering equations

- θ determines complex structure on $P\mathbb{A}$ via $\theta \wedge d\theta^{d-2}$. So:
- Deformations of complex structure $\leftrightarrow [\delta\theta] \in H^1_{\bar\partial}(P\mathbb{A}, L)$.

Proposition

For $\delta g_{\mu\nu} = \mathrm{e}^{ik\cdot x}\epsilon_\mu\epsilon_\nu$ *on flat space-time*

$$\delta\theta = \bar\delta(k\cdot P)\delta g_{\mu\nu}P^\mu P^\nu = \bar\delta(k\cdot P)\mathrm{e}^{ik\cdot X}(\epsilon\cdot P)^2 \,.$$

Ambitwistor repn $\Rightarrow \bar\delta(k\cdot P) \Rightarrow$ scattering equs.

$\Sigma =$ Riemann surface, and complexified (M, g).

Bosonic ambitwistor string action:

- $X : \Sigma \to M, \quad P \in X^* T^* M \otimes K$

$$S_B = \int (P \cdot \bar{\partial} X - e\, P^2/2)\,.$$

  with $e \in \Omega^{0,1} \otimes T$, where $K = \Omega^{1,0}_{\Sigma}$ and $T = T^{1,0}\Sigma$.
- e enforces $P^2 = 0$,
- P^2 generates gauge freedom: $\delta(X, P, e) = (\alpha P, 0, 2\bar{\partial}\alpha)$.

So target is:
Ambitwistor space: $\mathbb{A} = T^* M|_{P^2=0}/\{\text{gauge}\}$.

Quantization of bosonic ambitwistor string

Gauge fix $e = 0 \rightsquigarrow$ ghosts $(\tilde{b}, \tilde{c}) \in (K^2, T)$ plus usual $(b, c) \in (K^2, T)$ for diffeos

$$S_{\text{ghost}} = \int b\bar{\partial}c + \tilde{b}\bar{\partial}\tilde{c}\,, \qquad Q = \int cT + \tilde{c}P^2\,.$$

- Integrated vertex ops = perturbations of action $\leftrightarrow \delta g$.
- Action is $\int \theta = \int P \cdot \bar{\partial}X$ so integrated vertex operator is

$$\mathcal{V}_i = \int_\Sigma \delta\theta(\sigma_i) = \int_\Sigma \bar{\delta}(k_i \cdot P(\sigma_i))\, \mathrm{e}^{ik\cdot X(\sigma_i)}(\epsilon_i \cdot P(\sigma_i))^2\,.$$

- Quantum consistency implies field equations:

$$\{Q, \mathcal{V}_i\} = 0 \quad \Leftrightarrow \quad k^2 = 0, \quad k^\mu \epsilon_{\mu\nu} = 0\,.$$

Fixed vertex ops $\leftrightarrow$ quotient by $G = \mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^3 \rightsquigarrow$ amplitude as path-integral

$$\mathcal{M}(1, \ldots, n) = \int \frac{D[X, P, \ldots]}{\text{Vol } G}\, \mathrm{e}^{iS} \prod_{i=1}^{n} \mathcal{V}_i\,.$$

- Take $e^{ik_i\cdot X(\sigma_i)}$ factors into action to give

$$S = \frac{1}{2\pi}\int_\Sigma P\cdot\bar\partial X + 2\pi\sum_i ik\cdot X(\sigma_i)\,.$$

- Gives field equations $\bar\partial X = 0$ and,

$$\bar\partial P = 2\pi\sum_i ik\delta^2(\sigma-\sigma_i)\,.$$

- Solutions $X(\sigma)=X=\text{const.}$, $P(\sigma)=\sum_i\frac{k_i}{\sigma-\sigma_i}d\sigma$.

Thus path-integral reduces to

$$\mathcal{M}(1,\dots,n)=\delta^d(\sum_i k_i)\int_{(\mathbb{CP}^1)^{n-3}}\frac{\prod_i{}'\bar\delta(k_i\cdot P)\,(\epsilon_i\cdot P(\sigma_i))^2}{\text{Vol } G}$$

We see $P(\sigma)$ appearing and scattering equations.

Unfortunately: amplitudes for $S\sim\int_M R+R^3$.

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Worldsheet matter

- Decorate null geodesics with spin vectors, vectors for internal degrees of freedom & other holmorphic CFTs.
- Take
$$S = S_B + S^l + S^r$$
where S^l, S^r are some worldsheet matter CFTs.
- Total vertex operators given by
$$v^l v^r \bar{\delta}(k \cdot P)\, \mathrm{e}^{ik\cdot X}$$
with v^l, v^r worldsheet currents from S^l, S^r resp..
- Amplitudes become
$$\mathcal{M}(1,\ldots,n) = \delta^d\Big(\sum_i k_i\Big) \int_{(\mathbb{CP}^1)^{n-3}} I^l I^r \frac{\prod_i{}' \bar{\delta}(k_i \cdot P)}{\text{Vol } G}$$
where I^l, I^r are worldsheet correlators of v^ls, v^rs resp..
- In good situations, Q-invariance and discrete symmetries (GSO) rule out unwanted vertex operators.

Spinning light rays and worldsheet SUSY, S_Ψ

Let Ψ^μ be spin 1/2, fermion on Σ, RNS spin vector:

$$S_\Psi = \int g_{\mu\nu}\Psi^\mu\bar\partial\Psi^\nu - \chi P_\mu\Psi^\mu$$

$\chi \rightsquigarrow$ constraints $P\cdot\Psi = 0 \rightsquigarrow$ worldline susy

$$D = \Psi\cdot\frac{\partial}{\partial X} + P\cdot\frac{\partial}{\partial\Psi}\,, \qquad \{D,D\} = P\cdot\frac{\partial}{\partial X}\,.$$

- Gauge fix $\chi = 0 \rightsquigarrow$ bosonic ghosts $(\beta,\gamma)\in(K^{3/2},T^{1/2})$.
- Extend BRST operator with $Q_\Psi = \int\gamma P\cdot\Psi$.
- For v^l or v^r, must replace $\epsilon\cdot P$ by

$$u = \delta(\gamma)\epsilon\cdot\Psi \quad \text{(fixed)}, \qquad v = \epsilon\cdot P + \epsilon\cdot\Psi k\cdot\Psi\,.$$

- Worldsheet correlator $\langle u_1 u_2 v_3\ldots v_n\rangle = Pf'(M)$ for I^l or I^r.
- GSO symmetry $(\Psi,\gamma,\beta)\to(-\Psi,-\gamma,-\beta)\Rightarrow$ need u or v.

Free Fermions and current algebras

- Free 'real' Fermions $\rho^a \in \mathbb{C}^m \otimes K^{1/2}$

$$S_\rho = \int_\Sigma \delta_{ab}\rho^a\bar{\partial}\rho^b \,, \qquad a = 1, \dots n,$$

- These generate current algebra e.g., if $\mathfrak{g}$ is Lie algebra with structure constants f^{abc}, $m = \dim \mathfrak{g}$ then

$$j^a = f^{abc}\rho^b\rho^c \,, \qquad j^a(\sigma)j^b(0) = \frac{k\delta^{ab}}{\sigma^2} + \frac{f^{abc}j^c}{\sigma} + \dots .$$

Here level $k = C$.

- Current algebra gives Vertex ops

$$v = t \cdot j \,.$$

- Correlators give 'Parke-Taylor' factors + unwanted multi-trace terms

$$\langle v_1 \dots v_n \rangle = \frac{\mathrm{tr}(t_1 \dots t_n)}{\sigma_{12}\sigma_{23} \dots \sigma_{n1}} + \dots$$

where $\sigma_{ij} = \sigma_i - \sigma_j$.

Comb system [Casali-Skinner]

- To avoid multitrace terms, take current algebra level $k = 0$.
- But, then there is no trace $\langle v_1 \dots v_n \rangle = 0$.
- Gauge fermionic spin 3/2 current to give two fixed vertex operators to end chain of structure contants 'comb'.
- Use fermions $\tilde{\rho}^a, \rho^a \in \mathfrak{g} \otimes K^{1/2}$, bosons $q^a, y^a \in \mathfrak{g} \otimes K^{1/2}$

$$S_{CS} = \int_\Sigma \tilde{\rho}_a \bar{\partial} \rho^a + q_a \bar{\partial} y^a + \chi \, \mathrm{tr} \rho \left(\frac{[\tilde{\rho}, \rho]}{2} + [q, y] \right) .$$

- Gauge fix $\chi = 0 \rightsquigarrow$ ghosts (β, γ)
- Gives vertex operators

$$u = \delta(\gamma) t \cdot \rho \quad \tilde{u} = \delta(\gamma) t \cdot \tilde{\rho} \,, \qquad v = t \cdot [\rho, \rho] \,, \quad \tilde{v} = [\tilde{\rho}, \rho] + [q, y] \,.$$

- To be nontrivial, correlator must have just one untilded VO

$$\langle u_1 \tilde{u}_2 \tilde{v}_3 \dots \tilde{v}_n \rangle = \mathcal{C}(1, \dots, n) := \frac{\mathrm{tr}(t_1 [t_2, [t_3, \dots [t_{n-1}, t_n] \dots])}{\sigma_{12} \cdots \sigma_{n1}} .$$

Above lead essentially to original models:

- $(S^l, S^r) = (S_{\tilde{\Psi}}, S_{\Psi}) \rightsquigarrow$ type II gravity,
- $(S^l, S^r) = (S_{CS}, S_{\Psi}) \rightsquigarrow$ heterotic with YM,
- $(S^l, S^r) = (S_{CS}, S_{CS}) \rightsquigarrow$ bi-adjoint scalar.

The first two are critical in $d = 10$ (with appropriate gauge group in heterotic case).
The latter two come with unphysical gravity.

S_{CS} improves on current algebras in avoiding multi-trace terms.

Combined matter systems

S_{Ψ_1,Ψ_2} $= S_{\Psi_1} + S_{\Psi_2}$ two worldsheet susy's for S^l or S^r. This is maximum. It gives VO currents

$$u = \delta(\gamma_1) k \cdot \Psi_2 \,, \qquad v = k \cdot \Psi_1 k \cdot \Psi_2 \,.$$

$S_{\Psi,\rho}$ $= S_\Psi + S_\rho$ combines 'real' Fermions with susy, $\rightsquigarrow$ VO currents as usual for S_Ψ and

$$u_t = \delta(\gamma) t \cdot \rho \,, \qquad v_t = k \cdot \Psi t \cdot \rho \,.$$

$S_{\Psi,CS}$ $= \int_\Sigma \Psi \cdot \bar\partial \Psi + \tilde\rho_a \bar\partial \rho^a + q_a \bar\partial y^a + \chi \left(P \cdot \Psi + \mathrm{tr}\rho \left(\frac{[\tilde\rho,\rho]}{2} + [q,y] \right) \right) .$

With ghosts etc., the VO currents are those for S_Ψ and

$$\begin{aligned} \tilde u_t &= \delta(\gamma) t \cdot \tilde\rho \,, \qquad u_t = \delta(\gamma) t \cdot \rho \,, \\ \tilde v_t &= k \cdot \Psi t \cdot \tilde\rho + t \cdot ([\tilde\rho, \rho] + [q, y]) \,, \quad v_t = k \cdot \Psi t \cdot \rho + t \cdot [\rho, \rho] \,. \end{aligned}$$

GSO now reverses signs of all fields in matter system.

With $(S^l, S^r) = (S_{\Psi,CS}, S_{\tilde{\Psi}})$ we obtain Einstein-T^*YM.

- Graviton vertex operators now come from

$$u^l u^r = \delta(\gamma)\epsilon\cdot\Psi\,\delta(\tilde{\gamma})\tilde{\epsilon}\cdot\tilde{\Psi}\,, \quad v^l v^r = \epsilon\cdot(P+\Psi k\Psi)\,\tilde{\epsilon}\cdot(P+\Psi k\cdot\Psi)\,.$$

  Worldsheet correlators give $Pf'(M)Pf'(\widetilde{M})$.
- Two types of gluon VOs from $u^l_t u^r, v^l_t v^r$ and $\tilde{u}^l_t u^r, \tilde{v}^l_t v^r$.
- With gluons, worldheet correlator gives sum of

$$\mathcal{C}(T_1)\ldots\mathcal{C}(T_r)Pf'(\Pi)Pf'(\tilde{M})$$

  where $(T_1, \ldots, T_r)$ is a partition of gluons, Π from CHY.

Theory is linear YM on full YM background i.e., T^*YM + gravity

$$\int_M R_{NS}\,\mathrm{dvol} + \,\mathrm{tr}(A\wedge D_{\tilde{A}}{}^*F_{\tilde{A}})$$

with $A \leftrightarrow u^l_t u^r, v^l_t v^r$, $\tilde{A} \leftrightarrow \tilde{u}^l_t u^r, \tilde{v}^l_t v^r$.
Can replace S_{CS} with anomalous S_{YM} with single gluon type.

Ambitwistor strings with combined matter

S^l \ S^r	S_Ψ	S_{Ψ_1,Ψ_2}	$S^{(\tilde{m})}_{\rho,\Psi}$	$S^{(\tilde{N})}_{CS,\Psi}$	$S^{(\tilde{N})}_{CS}$
S_Ψ	E				
S_{Ψ_1,Ψ_2}	BI	Galileon			
$S^{(m)}_{\rho,\Psi}$	*EM* $\mathrm{U}(1)^m$	DBI	*EMS* $\mathrm{U}(1)^m \times \mathrm{U}(1)^{\tilde{m}}$		
$S^{(N)}_{CS,\Psi}$	EYM	ext. DBI	*EYMS* $\mathrm{SU}(N) \times \mathrm{U}(1)^{\tilde{m}}$	*EYMS* $\mathrm{SU}(N) \times \mathrm{SU}(\tilde{N})$	
$S^{(N)}_{CS}$	YM	Nonlinear σ	*EYMS* $\mathrm{SU}(N) \times \mathrm{U}(1)^{\tilde{m}}$	*gen. YMS* $\mathrm{SU}(N) \times \mathrm{SU}(\tilde{N})$	*Biadjoint Scalar* $\mathrm{SU}(N) \times \mathrm{SU}(\tilde{N})$

Table: Theories arising from the different choices of matter models.

Models from different geometric realizations of $\mathbb{A}$

We can start with other formulations of null superparticles

- Pure spinor version (Berkovits) $S = \int P \cdot \bar\partial X + p_\alpha \bar\partial \theta^\alpha + \ldots$.
- Green-Schwarz version:

$$S = \int P \cdot \bar\partial X + P_\mu \gamma^\mu_{\alpha\beta} \theta^\alpha \bar\partial \theta^\beta \, .$$

- $d = 4$, Twistor-strings of Witten, Berkovits & Skinner

$$\mathbb{A} = \{(Z, W) \in \mathbb{T} \times \mathbb{T}^* | Z \cdot W = 0\} / \{Z \cdot \partial_Z - W \cdot \partial_W\}$$

$$S = \int_\Sigma W \cdot \bar\partial Z + a \, Z \cdot W$$

- In 4d have full ambitwistor representation [Geyer, Lipstein, M. 1404.6219]

$$S = \int_\Sigma Z \cdot \bar\partial W - W \cdot \bar\partial Z + aZ \cdot W$$

1. Not same as twistor string, (Z, W) spinors on world sheet.
2. Valid for any amount of supersymmetry.
3. New simpler 4d formulae with reduced moduli.

Relation to null infinity, BMS and soft gravitons

Geyer,Lipstein & M. 1406.1462 (cf. Adamo, Casali & Skinner 1405.5122).

All real null geodesics intersect $\mathscr{I}^{\pm}$ so $\mathbb{A}_{\mathbb{R}} = T^*\mathscr{I}^{\pm}$; so

$$\mathbb{A} = T^*\mathscr{I}^{+}_{\mathbb{C}} \cup T^*\mathscr{I}^{-}_{\mathbb{C}} \qquad \text{glued over } \mathbb{A}_{\mathbb{R}} \, .$$

and

$$\text{Vertex op} = \delta\theta = \bar\partial(\text{infinitesimal diffeo } T * \mathscr{I}^{-} \to T^*\mathscr{I}^{+}) \, .$$

The diffeos intertwine with BMS transformations.

Soft gravitons: $k \to 0$ then diffeo $\to$ supertranslation.

Gives Strominger/Weinberg soft graviton thm as Ward identity.

Theorem (Strominger/Weinberg)

Weinberg soft theorem: as $k_n \to 0$

$$\mathcal{M}(1, \dots, n) \to \mathcal{M}(1, \dots, n-1) \sum_{i=1}^{n-1} \frac{(\epsilon_n \cdot k_i)^2}{k_n \cdot k_i} \, ,$$

follows from supertranslation equivariance.

Summary & Outlook

Chiral $\alpha' = 0$ ambitwistor strings, use LeBrun's correspondence & give theories underlying CHY formulae new & old.

- NS sector of type II sugra extends to Ramond as in RNS string via spin-operator from bosonizing Ψs. [Adamo, Casali, Skinner] How do these work in new theories?
- Incorporates colour/kinematics Yang-Mills/gravity ideas.
- Many new critical models; criticality gives extension to loops. [Adamo, Casali, Skinner]
  - At genus g, P is a 1-form and acquires dg zero-modes.
  - These are the loop momenta for g-loops.
  - Very likely gives correct answer for loop processes [Casali-Tourkine].
- Quantization ties scattering of null geodesics into that for gravitational waves.

Thank You