Entanglement and Bootstrap

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Eurostrings 2015 @ Cambridge University

Tomography from Entanglement

based on arXiv:1412.1879 with J. Lin, M. Marcolli, B. Stoica

Analytic Bootstrap Bounds

to be published, with Hyungrok Kim and Petr Kravchuk

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Tension between **Quantum** and **Gravity**

Non-Local Entanglement $\Leftrightarrow$ **Local Geometry**

Holographic Expectations:

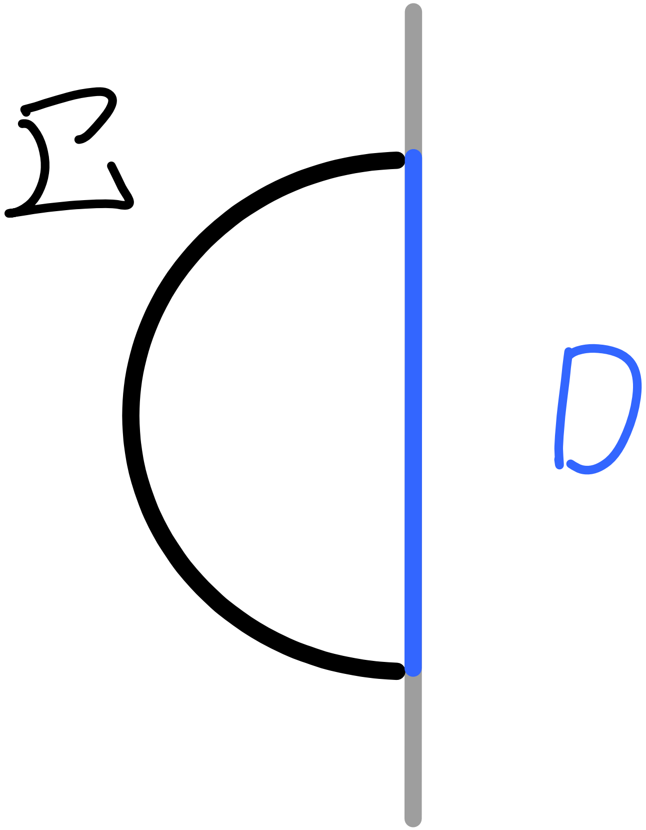
☆ Bulk Locality Emerges from Boundary Entanglement

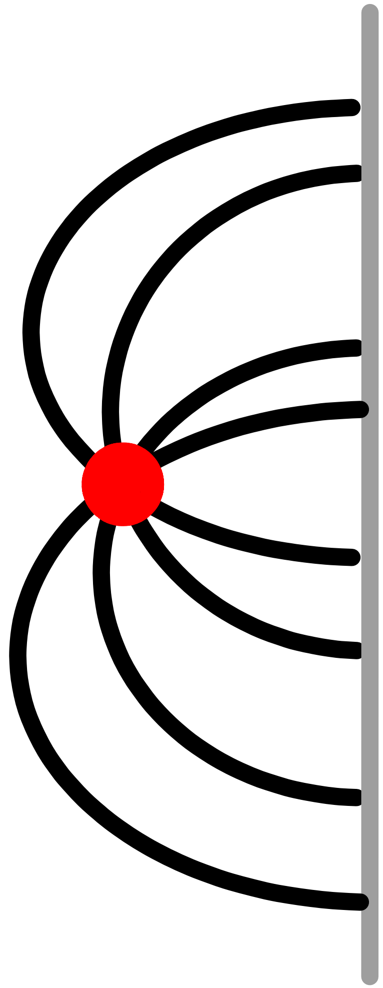
☆ Bulk-Boundary Relation is Non-Local

We will show how these expectations are realized in a specific setup.

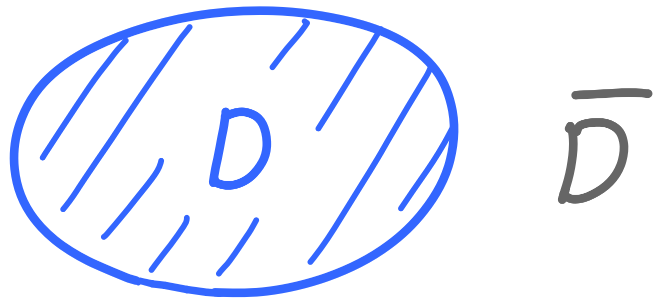
Main Result:

Ryu-Takayanagi formula for **boundary entanglement** can be inverted to diagnose **local data in the bulk**.





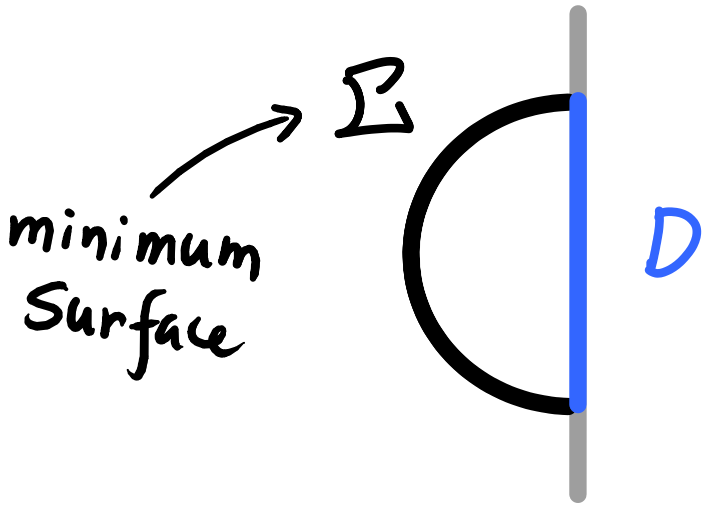
Entanglement Entropy



$|\psi\rangle$: State in CFT

$$\Rightarrow \rho = \mathrm{tr}_{\mathcal{H}_{\bar{D}}} |\psi\rangle\langle\psi|$$

$$S_E = -\mathrm{tr}_{\mathcal{H}_D}\, \rho \log \rho$$



Ryu-Takayanagi Formula

$$S_E = \frac{\mathrm{Area}(\Sigma)}{4G_N}$$

Relative Entropy

$$|\psi_0\rangle,\ |\psi_1\rangle \in \mathcal{H}_{CFT} \Rightarrow \rho_0,\ \rho_1$$

$$S(\rho_1|\rho_0) = \mathrm{tr}[\rho_1 \log(\rho_1/\rho_0)]$$

- measure of distinguishability
- $S(\rho_1|\rho_0) \geqq 0$
- monotonic in |domain|
- not symmetric in ρ_0 and ρ_1

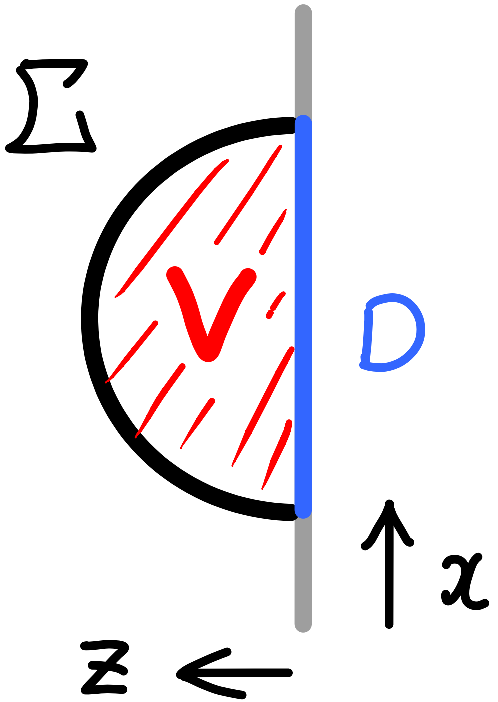
We find: For $|\psi_0\rangle$: vacuum $\Leftrightarrow$ pure AdS_{d+1}

$|\psi_1\rangle$: any state in CFT_d

For small disk D $\quad (E\,R \ll 1)$

radius of D.

energy of $|\psi_1\rangle$



$$S(\rho_1|\rho_0) = \frac{8\pi^2 G_N}{R}$$

$$\times \int_V (R^2 - z^2 - x^2)\, \varepsilon \sqrt{g}\, dz\, d^{d-1}x$$

Bulk Energy Density

$$S(\rho_1|\rho_0) = \frac{8\pi^2 G_N}{R} \int_V (R^2 - z^2 - x^2)\, \varepsilon \sqrt{g}\, dz\, d^{d-1}x$$

Bulk Energy Density

Entropy Inequalities ⇔ Positive Energy Conditions

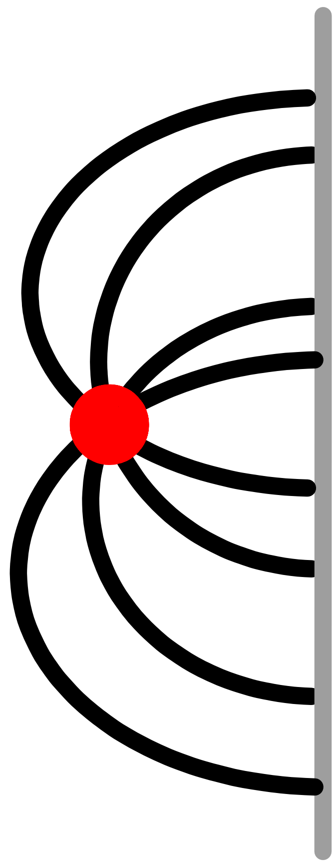
$$S \geq 0, \quad \frac{\partial}{\partial R} S \geq 0$$

$$\Leftrightarrow \int_V [R^2 \pm (z^2 + x^2)]\, \varepsilon \sqrt{g}\, dz\, d^{d-1}x \geq 0$$

$$S(\rho_1 | \rho_0) = \frac{8\pi^2 G_N}{R} \int_V (R^2 - z^2 - x^2)\, \varepsilon \sqrt{g}\, dz\, d^{d-1}x$$

Bulk Energy Density

The relation can be inverted by the Radon transform to express the **bulk energy density** by the **entanglement data on the boundary**.

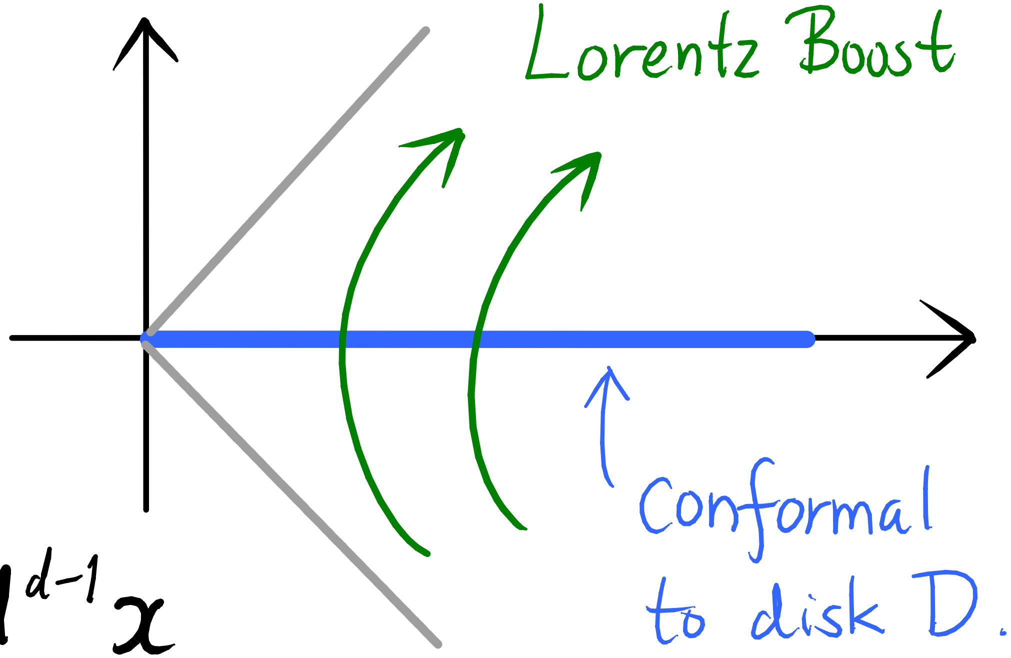


Holographic Expression for Relative Entropy

$$S(\rho_1 | \rho_0) = \underbrace{\mathrm{tr}\, \rho_1 \log \rho_1}_{\text{Ryu-Takayanagi}} - \underbrace{\mathrm{tr}\, \rho_1 \log \rho_0}_{\text{modular Hamiltonian}}$$

modular Hamiltonian

$$\rho_0 \propto e^{-H_{mod}}$$



$$H_{mod} = \frac{\pi}{R} \int_D (R^2 - |\vec{x}|^2)\, T^t{}_t \, d^{d-1}x$$

Relative Entropy in terms of Bulk Metric

$$ds^2 = \frac{\ell_{AdS}^2}{z^2}\left[dz^2 + (\eta_{\mu\nu} + h_{\mu\nu})\,dx^\mu dx^\nu\right]$$

$$\Delta\langle H_{mod}\rangle \equiv \mathrm{tr}\,\rho_1 H_{mod} - \mathrm{tr}\,\rho_0 H_{mod}$$

$$= \frac{\ell_{AdS}^{d-1}\cdot d}{16 G_N R}\int_D (R^2 - |\vec{x}|^2)\, h_{ij}\,\eta^{ij}\, z^{-d}\, d^{d-1}x$$

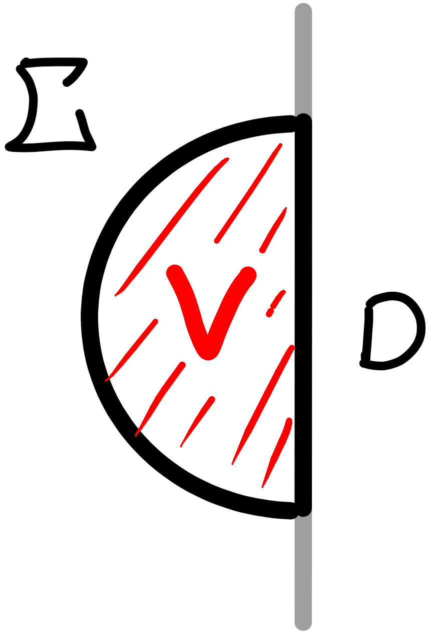
$$\Delta S_E \equiv -\mathrm{tr}\,\rho_1 \log\rho_1 + \mathrm{tr}\,\rho_0\log\rho_0$$

$$= \frac{\ell_{AdS}^{d-1}}{8 G_N R}\int_\Sigma (R^2\eta^{ij} - x^i x^j)\, h_{ij}\, z^{-d}\, d^{d-1}x$$

Using Wald's formalism, one can find a $(d-1)$-form χ such that

$$\Delta \langle H_{mod} \rangle = \int_D \chi$$

$$\Delta S_E = \int_\Sigma \chi$$



$$S(\rho_1 | \rho_0) = \int_D \chi - \int_\Sigma \chi$$

$$= \int_V d\chi$$

$$d\chi \propto R_{tt} - \frac{1}{2} g_{tt} + \Lambda g_{tt}$$

(works w/ higher derivatives, too)

$$S(\rho_1 | \rho_0) = \Delta \langle H_{mod} \rangle - \Delta S_E$$

$$= \int_V d\chi$$

$$d\chi \propto R_{tt} - \frac{1}{2} g_{tt} R + \Lambda g_{tt}$$

To leading order, $\Delta \langle H_{mod} \rangle = \Delta S_E$

small number : ER ← radius of D

↑ energy of $|\psi_1\rangle$

First Law of Entanglement

⇔ Linearized Einstein Equations

Blanco, et al., arXiv:1305.3182; Lashkari, et al., arXiv:1308.3716; Faulkner, et al., arXiv:1312.7856

Consider backreaction from bulk matter $t_{\mu\nu}$.

$$S(\rho_1 | \rho_0) = \Delta \langle H_{mod} \rangle - \Delta S_E$$

$$= \int_V dx$$

$\mathcal{E}$: bulk energy density

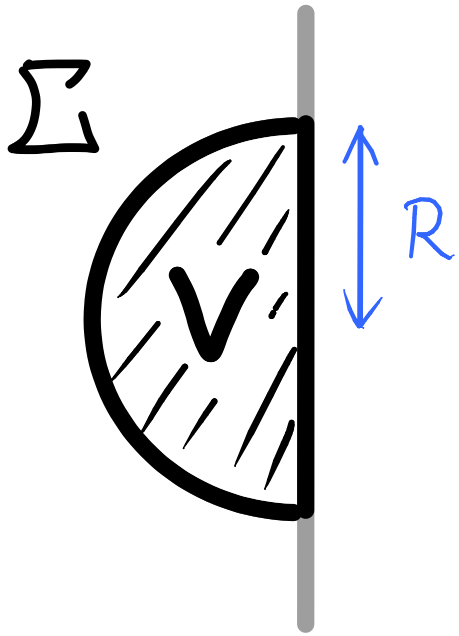
$$\mathcal{E} = t^t{}_t$$

$$= \frac{8\pi^2 G_N}{R} \int_V (R^2 - z^2 - x^2)\, t^t{}_t \sqrt{g}\, dz\, d^{d-1}x$$

$$\geq 0$$

Boundary Entropy Inequalities

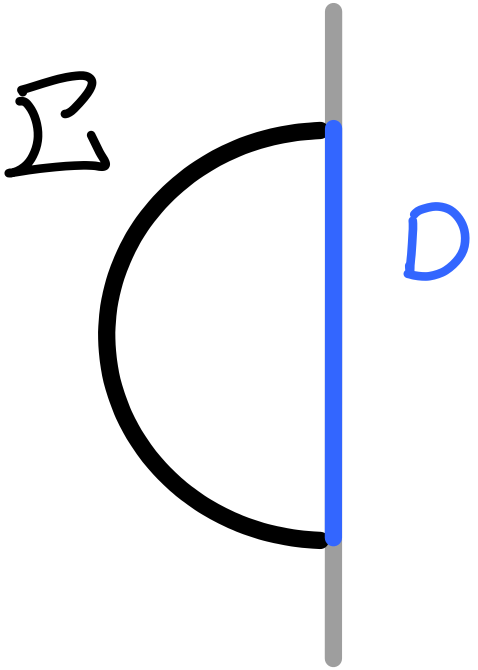
$\Leftrightarrow$ Bulk Energy Conditions



$$\left(\frac{\partial}{\partial R} + \frac{1}{R}\right) S(\rho_1 | \rho_0) = 16\pi^2 G_N \int_{V} \varepsilon \sqrt{g_V}$$

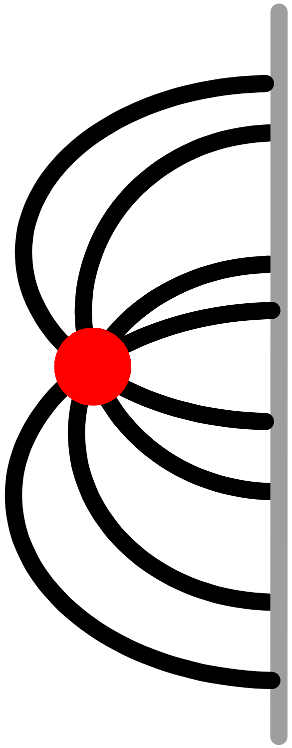
$$\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} - \frac{1}{R^2}\right) S(\rho_1 | \rho_0) = 16\pi^2 G_N \int_{\Sigma} \varepsilon \sqrt{g_\Sigma}$$

This relation can be inverted to express the bulk energy density by the relative entropy.



Radon Transform:

$$\int_{\Sigma} \varepsilon \sqrt{g_{\Sigma}} \Rightarrow S(\rho_1|\rho_0)$$



Inverse Radon transform:

$$\int \left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} - \frac{1}{R^2} \right) S(\rho_1|\rho_0)$$

$$\Rightarrow \varepsilon(z, x)$$

Summary

☆ Bulk stress tensor near boundary can be diagnosed by boundary entanglement entropy.

☆ Entropy inequalities on the boundary are (integrated) positive energy conditions in the bulk.

☆ To do: Go deeper in the bulk interior.

Analytic Bootstrap Bounds

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to be published, with Hyungrok Kim and Petr Kravchuk

$$G(z) = \langle \phi(0)\,\phi(z)\,\phi(1)\,\phi(\infty) \rangle$$

scalar 4-point in CFT_d

$$= \begin{cases} \sum_{\mathcal{O}} C_{\phi\phi\mathcal{O}}^2 \, z^{\Delta_{\mathcal{O}} - 2\Delta_0} & (1) \\[2em] \sum_{\mathcal{O}_{p(\text{rimary})}} C_{\phi\phi\mathcal{O}_p}^2 \, z^{-2\Delta_0} F_{\Delta_{\mathcal{O}}, l_{\mathcal{O}}}(z) & (2) \end{cases}$$

We will discuss (1), but the results can be generalized to (2).

$$G(z) = \int_0^\infty z^{\Delta - 2\Delta_0}\, S(\Delta)\, d\Delta$$

We want to use the crossing symmetry,

$$G(1-z) = G(z),$$

to find bounds on

$$B_z(x) \equiv \frac{1}{G(z)}\, z^{\Delta - 2\Delta_0}\, S(\Delta)$$

"Branching Ratio"

Consider $z = 1/2$ (general z , later).

- By definition, $\displaystyle\int_0^\infty B_{1/2}(\Delta)\, d\Delta = 1$

- Crossing Symmetry for z : real

$$\Longleftrightarrow \int_0^\infty [\Delta - 2\Delta_0]^{2k+1} B_{1/2}(\Delta)\, d\Delta = 0,$$

where $[x]^n \equiv x(x-1)\cdots(x-n+1)$

Duality of Linear Optimization

$$\max(\vec{c}\cdot\vec{x}),$$

subject to $A\vec{x} = \vec{b}$, $\vec{x} \geqq \vec{0}$

$$\Updownarrow$$

$$\min(\vec{b}\cdot\vec{y})$$

subject to $A^T\vec{y} \geqq \vec{c}$

$$\max \left(\int_{\Delta-\varepsilon}^{\Delta+\varepsilon} B_{1/2}(\Delta')\, d\Delta' \right)$$

$\max(\vec{c}\cdot\vec{x})$

subject to

$$\begin{cases} \circ \ \displaystyle\int_0^\infty [\Delta - 2\Delta_0]^{2k+1} B_{1/2}(\Delta)\, d\Delta = 0 \\ \circ \ \displaystyle\int_0^\infty B_{1/2}(\Delta)\, d\Delta = 1 , \end{cases}$$

$A\vec{x} = \vec{b}$

and $B_{1/2}(\Delta) \geq 0$ $\leftarrow$ $\vec{x} \geq \vec{0}$

Dual Problem:

$$P(\Delta) = 1 + \sum_k [\Delta - 2\Delta_0]^{2k+1} \lambda_k$$

$$\min\left(\frac{1}{P(\Delta)}\right),$$

$$\text{subject to } P(\Delta') \geqq 0,$$

$$\Delta' \in (0, \infty)$$

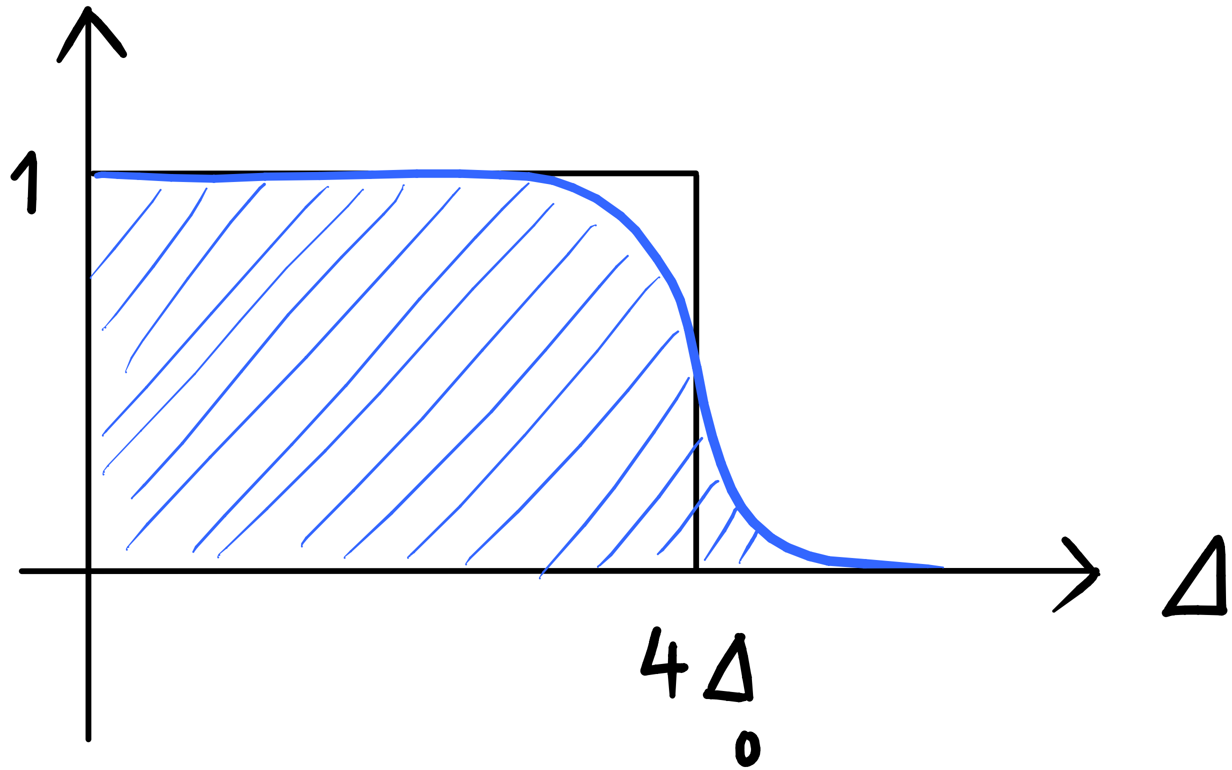
Optimization subject to

$$B_z(x) \equiv \frac{1}{G(z)} z^{\Delta - 2\Delta_0} S(\Delta)$$

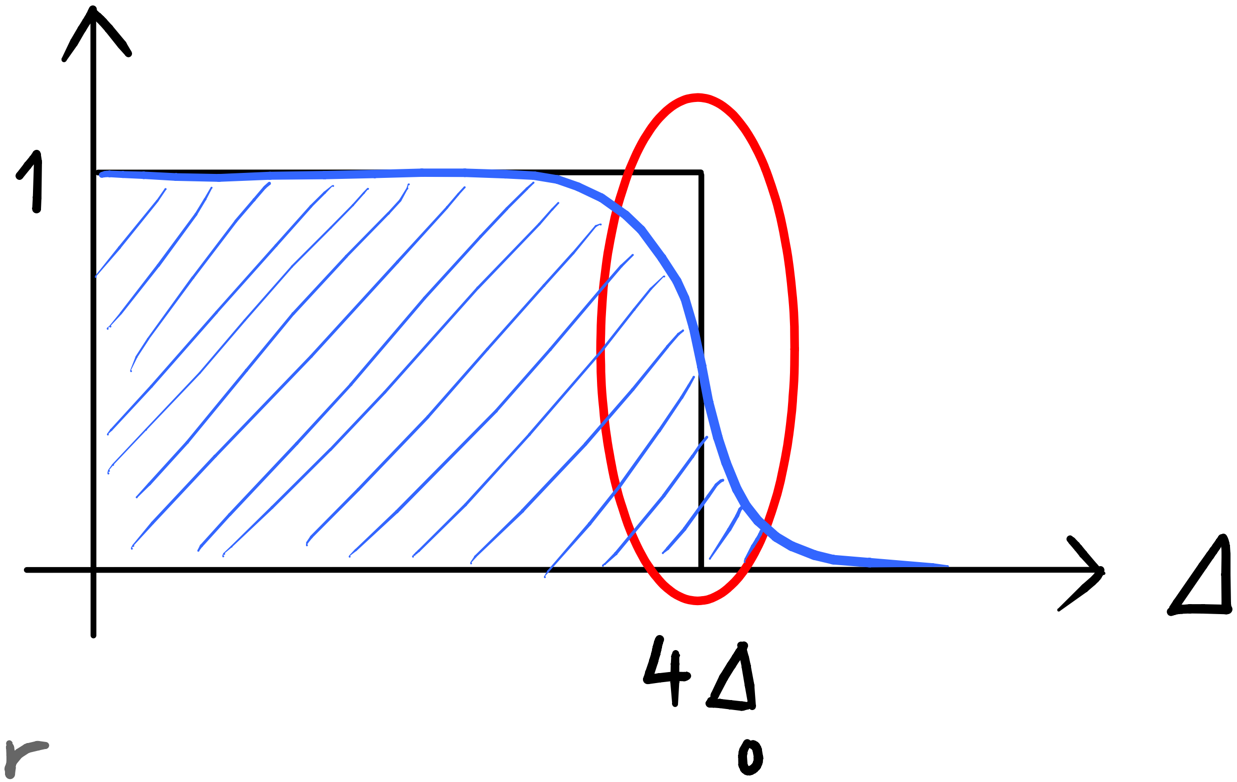
$$\int_0^\infty [\Delta - 2\Delta_0]^{2k+1} B_{1/2}(\Delta)\, d\Delta = 0$$

$$\int_0^\infty B_{1/2}(\Delta)\, d\Delta = 1 \,, \quad B_{1/2}(\Delta) \geq 0$$

$$\int_\Delta^\infty B_{1/2}(\Delta')\, d\Delta'$$



$$\int_{\Delta}^{\infty} B_{1/2}(\Delta')\, d\Delta'$$



For $\Delta_0 \gg 1$,

the threshold behavior

is well-described by the **Chebyshev polynomials**:

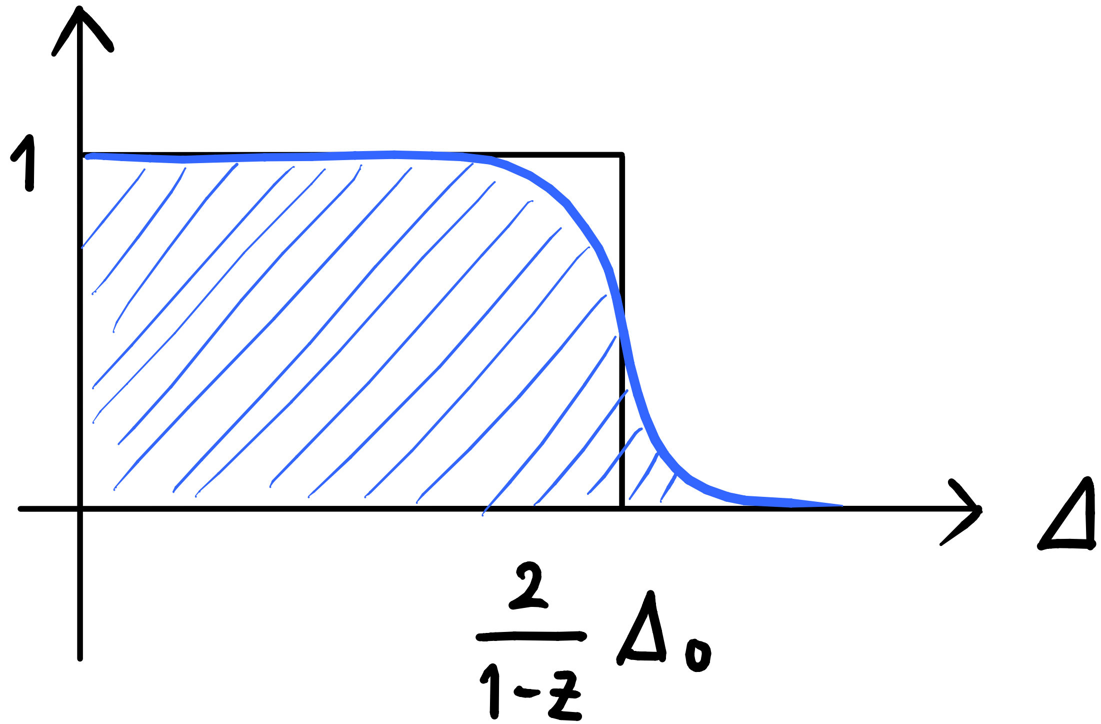
$$\int_{\Delta}^{\infty} B_{1/2}(\Delta')\, d\Delta' \leq \frac{2}{1 + T_{2n+1}\left(\frac{\Delta}{2\Delta_0} - 1\right)}$$

$$n \ll \sqrt{\Delta_0}$$

For general z : real,

$$\int_{\Delta}^{\infty} B_z(\Delta')\, d\Delta' \le \frac{2}{1 + T_{2m+1}\left((1-z)\frac{\Delta}{\Delta_0} - 1 \right)}$$

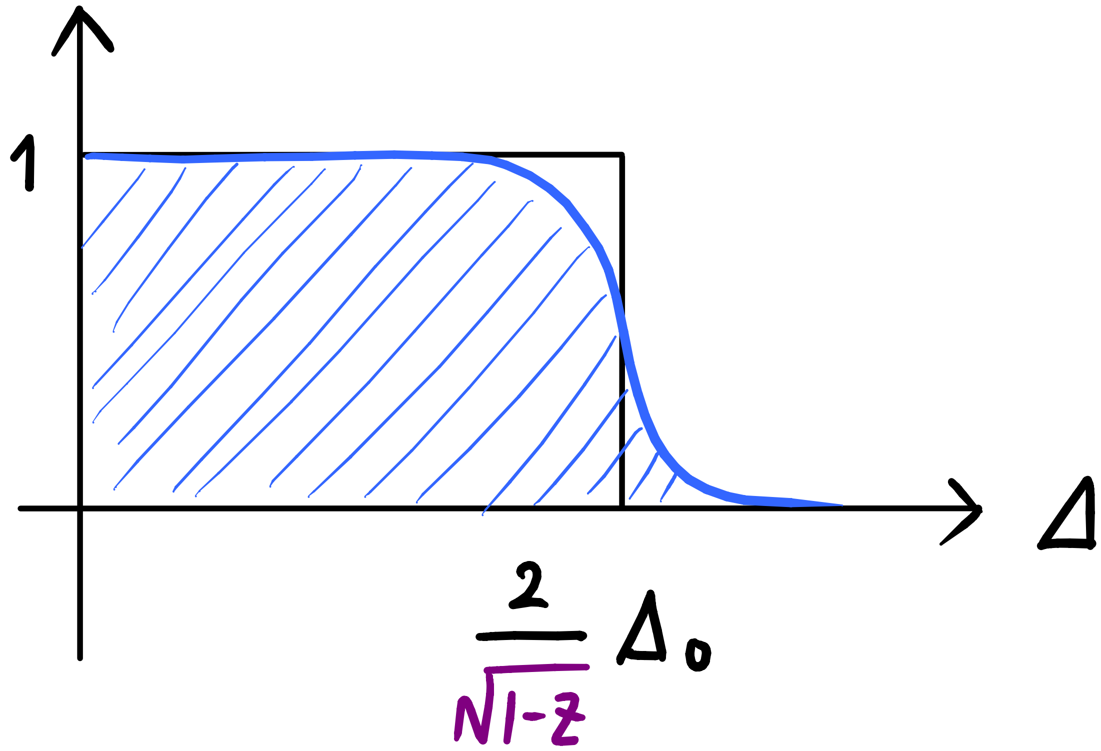
$$\int_{\Delta}^{\infty} B_z(\Delta')\, d\Delta'$$



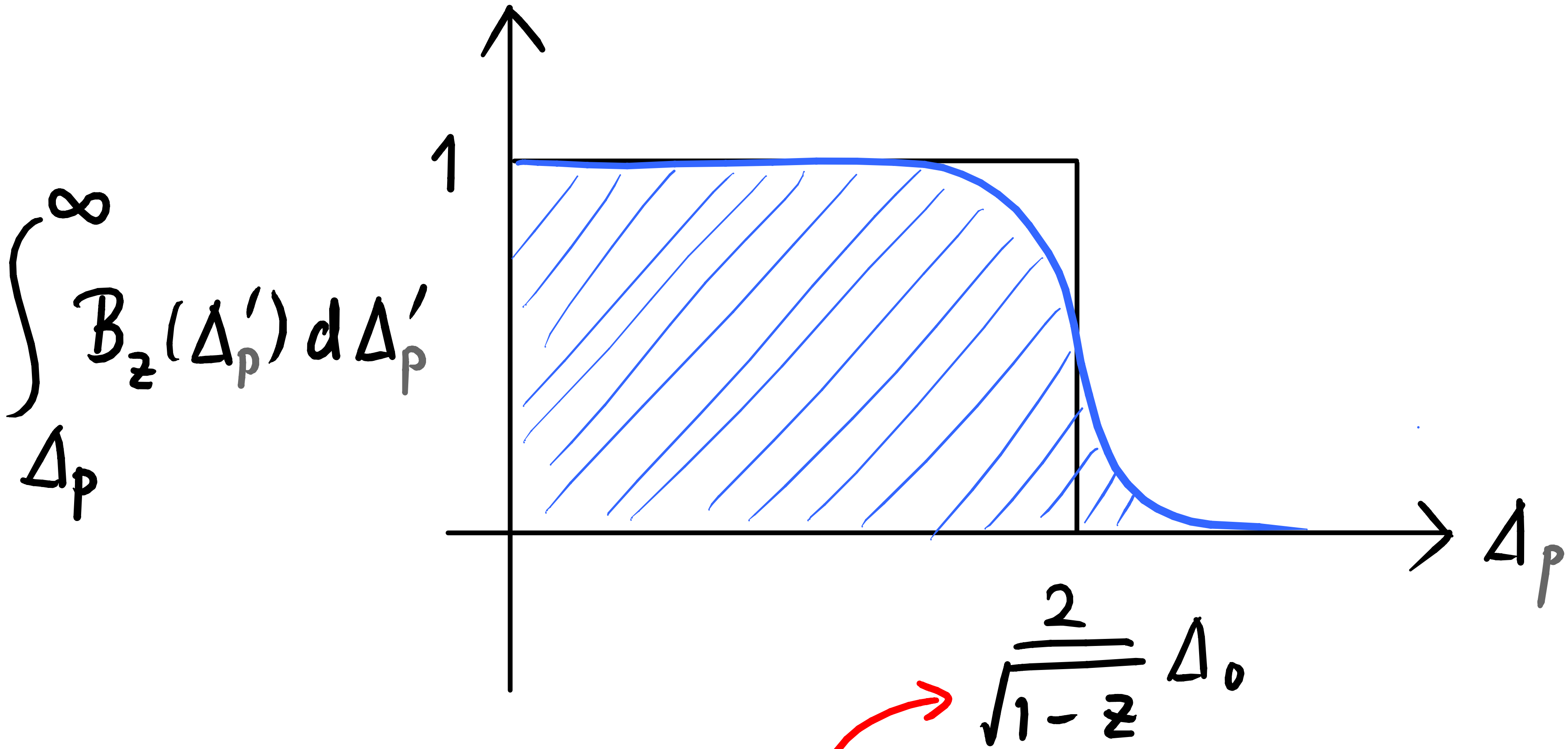
For $\Delta_{p(\text{rimary})}$ with conformal blocks,

$$\int_{\Delta_p}^{\infty} B_z(\Delta_p') \, d\Delta_p' \leq \frac{2}{1 + T_{2n+1}\left(\sqrt{1-z}\,\frac{\Delta_p}{\Delta_0} - 1\right)}$$

$$\int_{\Delta}^{\infty} B_z(\Delta') \, d\Delta'$$



$B_z(\Delta_p)$: Branching Ratio, $\phi_{\Delta_0}(z)\,\phi_{\Delta_0}(0) \to \phi_{\Delta_p}(0)$

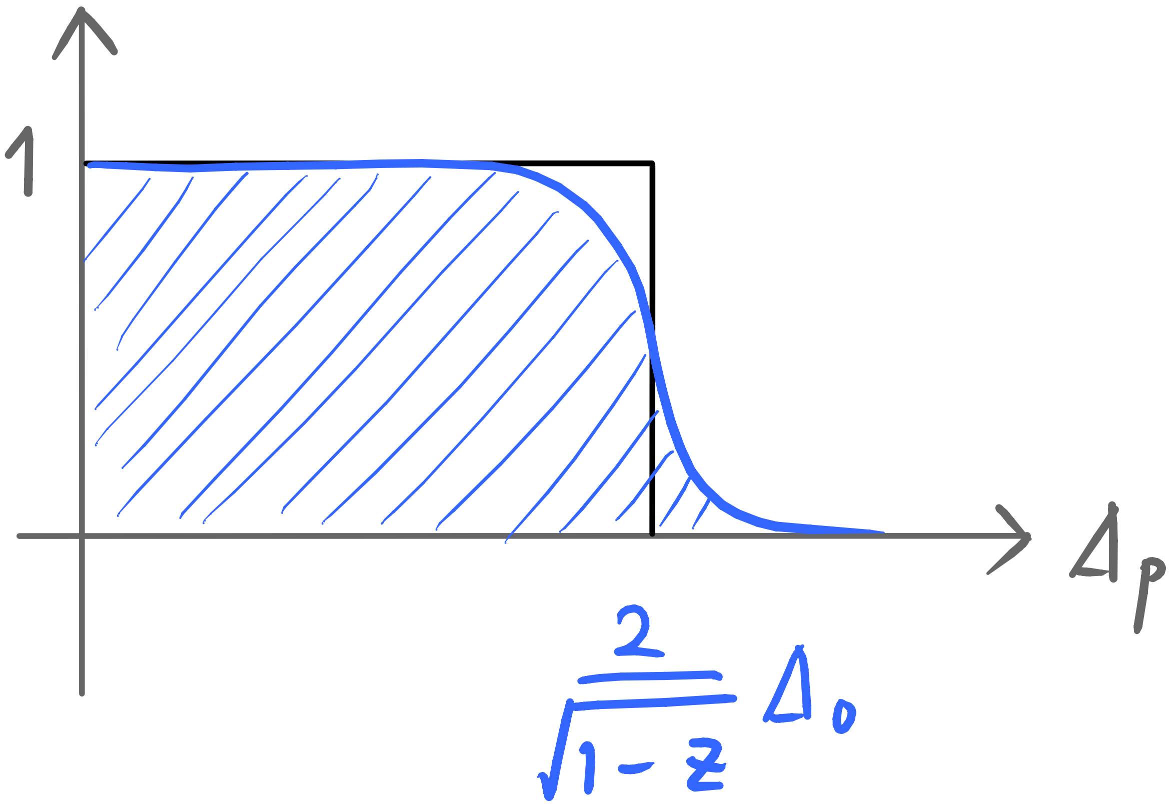


Agrees with mean field theory
= free field in AdS

Crossing symmetry

$$G(z) = G(1-z) \implies B_z(\Delta_p) \sim B_{1-z}(\tilde{\Delta}_p),$$

$$\Delta_p = \frac{2}{\sqrt{1-z}}\Delta_0 - \sqrt{\frac{z}{1-z}}\,\tilde{\Delta}_p \lesssim \frac{2}{\sqrt{1-z}}\Delta_0$$



Chebyshev polynomials bound corrections.

We can also estimate

$$\int_{\Delta-\varepsilon}^{\Delta+\varepsilon} B_{1/2}(\Delta')\,d\Delta' \leq \frac{2^{\Delta+1}}{1+\dfrac{\Gamma(\Delta-2\Delta_0+1)\,\Gamma(2\Delta_0)}{\Gamma\left(\frac{\Delta+3}{2}\right)\Gamma\left(\frac{\Delta-1}{2}\right)}}$$

This applies for finite Δ and Δ_0

with $\Delta > 4\Delta_0 > 2$.

For $\Delta_0 \gg 1$, it is related to 1208.6449

by Pappadopulo, Rychkov, Espin, Rattazzi

Summary

The linear optimization of the bootstrap constraints shows that the main contributions to

$$\phi_{\Delta_0}(z)\,\phi_{\Delta_0}(0) \to \phi_{\Delta_p}(0)$$

are from $\Delta_p < \dfrac{2}{\sqrt{1-z}}\,\Delta_0$,

with the analytic bounds on the tails.