Superconformal quantum mechanics and the discrete light-cone quantisation of $\mathcal{N}=4$ SUSY Yang-Mills

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Background

Non-perturbative quantum field theory is almost as mysterious now as it ever has been. The possibility of reducing the problem to finite-dimensional quantum mechanics is therefore a very attractive proposition! In favourable circumstances, this may be achievable using discrete light-cone quantisation (DLCQ), reviewed in the bottom-right.

A rare case where non-perturbative physics appears somewhat tractable is that of $\mathcal{N}=4$ supersymmetric Yang-Mills theory. In recent years, substantial evidence has emerged that the theory is integrable in the planar limit and that the spectrum of the dilatation operator may be computable. However, this raises as many questions as it solves:

- How far can we push integrability?
- Can we give a first-principles derivation?
- Can we solve the spectral problem non-perturbatively?

Of course, answering any of these questions would also be of great interest in the context of the AdS/CFT correspondence.

Our aim is to address these questions in DLCQ. Using matrix theory, the quantum mechanics arising from the DLCQ of $\mathcal{N}=4$ supersymmetric Yang-Mills theory with gauge group $SU(N)$, in the sector with K units of light-like momentum, has been known for some time.

This is the starting point of our description

It is an $\mathcal{N}=(4,4)$ supersymmetric σ-model on the moduli space $\mathcal{M}_{K,N}$ of K $SU(N)$ Yang-Mills instantons on $\mathbb{R}^2 \times T^2_\tau$, in the limit area$(T^2) \to 0$. Here τ is both the complex structure of T^2 and the Yang-Mills coupling.

This moduli space has at least three useful descriptions which are reviewed below. A significant part of our work concerns using these different descriptions to gain a much more explicit understanding, and perhaps even solution, of the σ-model.

In particular, the expectation from DLCQ is that the model has an $SU(1,1|4)$ superconformal invariance. Seeing this action directly in the σ-model is a highly non-trivial problem.

Further information

1. N. Dorey & A. Singleton, *Superconformal quantum mechanics and the discrete light-cone quantisation of $\mathcal{N}=4$ SUSY Yang-Mills*, *JHEP* **1502** (2015) 067, [arXiv:1409.8440]
2. N. Dorey & A. Singleton, *Instantons, integrability and discrete light-cone quantisation*, [arXiv:1412.5178]
3. A. Singleton, *Superconformal quantum mechanics and the exterior algebra*, *JHEP* **1406** (2014) 131, [arXiv:1403.4933]

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The moduli space 1

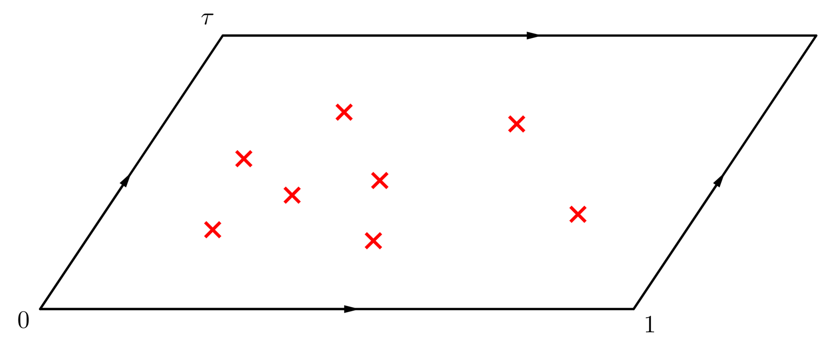
The instanton moduli space $\mathcal{M}_{K,N}$ is a hyper-Kähler manifold of real dimension 4(KN-N+1) := 4r, so it admits quantum mechanics with $\mathcal{N}=(4,4)$ supersymmetry. It is given by the Higgs branch of a brane system:

	t	$\mathbb{R}^2$	T^2_τ	transverse $\mathbb{R}^5$
$N\times$ D4	•	•	•	
$K\times$ D0	•			

The moduli space 2

Using T-duality on the previous picture gives an equivalent description of $\mathcal{M}_{K,N}$, with K D2 branes wrapping the dual torus and N localised D2 branes. The Higgs branch is described by a Hitchin system on the dual torus with impurities at N marked points.

$$\partial_{\bar z}\phi(z,\bar z) + [A_{\bar z}, \phi] = 2\pi i \sum_{i=1}^{N} Q_i \tilde{Q}_i \delta^{(2)}(z - Z_i)$$



The moduli space 3

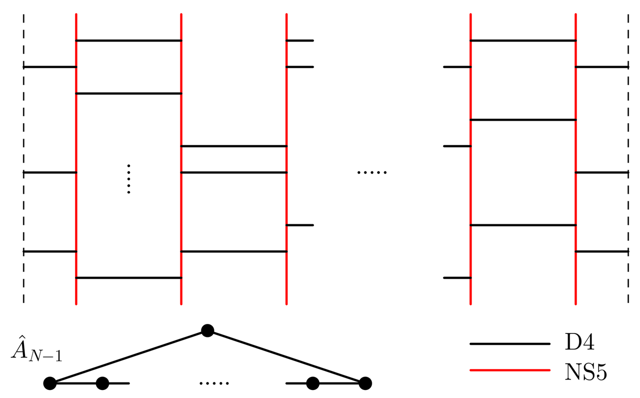
The third picture is obtained via 3d mirror symmetry. This gives a description of $\mathcal{M}_{K,N}$ as the Coulomb branch of a 4d $\mathcal{N}=2$ SCFT compactified on $\mathbb{R}^3 \times S^1_R$. This picture is especially useful as it exhibits $\mathcal{M}_{K,N}$ as a fibre bundle.

The base is the Coulomb branch of the field theory on $\mathbb{R}^4$. In particular, it is a scale-invariant special Kähler manifold of complex dimension r. This means there are complex coordinates $a^I : I = 1, \dots KN - N + 1$ with respect to which the metric is given by a holomorphic prepotential $\mathcal{F}(a)$ of degree 2:

$$ds^2 = \sum_{I,J} \mathrm{Im}\tau_{IJ} da^I d\bar{a}^J, \qquad \tau_{IJ} = \frac{\partial^2 \mathcal{F}}{\partial a^I \partial a^J}, \qquad a^I \frac{\partial}{\partial a^I}\mathcal{F} = 2\mathcal{F}.$$

The fibre is a $2r$-torus with complex structure τ_{IJ} and size inversely proportional to the compactification radius R. It disappears in the $R \to \infty$ limit relevant to our problem.

The $4d$ SCFT is the $\hat{A}_{N-1}$ quiver with gauge group $G = U(1) \times \prod_{j=1}^{N} SU(K)$. The prepotential on its Coulomb branch is known exactly. In fact, this gives the full hyper-Kähler structure in the large R limit.



Headlines

The Coulomb branch description of $\mathcal{M}_{K,N}$ in the large R limit gives us the hyper-Kähler structure, and hence the supersymmetries. Adding in the scale invariance of the base $4d$ Coulomb branch gives:

Our main result

A concrete realisation of the DLCQ of $\mathcal{N}=4$ SUSY Yang-Mills, including:

- A precise description of the target space geometry as the base $4d$ Coulomb branch.
- Explicit expressions for all $SU(1,1|4)$ generators in terms of known functions.

A similar σ-model can be obtained from any scale-invariant special Kähler geometry, including the Coulomb branch of any $4d$ $\mathcal{N}=2$ SCFT. The Lagrangian is

$$\mathcal{L} = \mathrm{Im}\tau_{IJ}\dot{a}^I\dot{\bar{a}}^J + i\mathrm{Im}\tau_{IJ}\bar{\psi}^J_A D_t \psi^{IA} - \frac{1}{2}R_{I\bar{J}K\bar{L}}\psi^{IA}\bar{\psi}^{\bar{J}}_A\psi^{KB}\bar{\psi}^{\bar{L}}_B$$
$$-\frac{1}{12}\mathrm{Re}\left(\epsilon_{ABCD}G_{IJKL}\psi^{IA}\psi^{JB}\psi^{KC}\psi^{LD}\right) + \mathrm{c.c},$$

where ψ^{IA} are fermions in the **4** of $SU(4)$, and τ, R and G are all determined by the prepotential. The chiral term is novel and unique to special Kähler geometry.

There are very strong hints that special sectors of our model are integrable. Open questions under investigation include:

- How can we deal with singularities of $\mathcal{M}_{K,N}$?
- What does our integrability have to do with planar $\mathcal{N}=4$?
- Can we solve the spectral problem, for instance using Bethe Ansatz techniques?
- Analogous considerations should work for the six-dimensional (2,0) theory. Can we count the BPS states?

What is discrete light-cone quantisation?

The reduction of a field theory on a null circle. Explicitly, choose light-cone coordinates $x_\pm = x_0 \pm x_1$, treat x_+ as time and identify $x_- \sim x_- + 2\pi R_-$.

Remarkably, this procedure leads to a finite-dimensional model. In the sector with K units of momentum around the circle, $P_+ = K/R_-$, we get quantum mechanics with $\mathcal{O}(K)$ degrees of freedom.

We expect to preserve any symmetries of the original theory which commute with P_+. For a superconformal theory in d dimensions, we generically keep $SO(2,1) \times SO(d-2) \times R$ and half the supersymmetries. For $\mathcal{N}=4$ SUSY Yang-Mills we get $SU(1,1|4)$.