New relations between gauge and gravity amplitudes in field and string theory

Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

Stephan Stieberger, MPP München

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I. Amplitude relations

- *relations among **same** amplitudes within **one** theory*

Tree-level N-point QCD amplitude:

$$\mathfrak{A}_N = g_{YM}^{N-2} \sum_{\Pi \in S_{N-1}} \mathrm{Tr}(T^{a_1} T^{a_{\Pi(2)}} \ldots T^{a_{\Pi(N)}})\ A_{YM}(1, \Pi(2), \ldots, \Pi(N))$$

gauge theory: cyclicity, reflection, parity,

Kleiss-Kuijf (KK), Bern-Carrasco-Johansson (BCJ) relations

- *relations between **different** amplitudes within **one** theory*

supersymmetric Ward identities in gauge and gravity theory

- *relations between amplitudes from **different theories***

relations between gauge and gravity amplitudes:
(perturbative) Kawai-Lewellen-Tye (KLT) relations

KLT

$$\begin{array}{ccc} \mathcal{M}_{FT}(1,\ldots,4) & = & s_{12}\, A_{YM}(1,2,3,4)\ \tilde{A}_{YM}(1,2,4,3) \\ \text{graviton amplitude} & = & (\text{gauge amplitude}) \times (\text{gauge amplitude}) \end{array}$$

Supergravity graviton N-point tree-level amplitude:

$$\mathcal{M}_{FT}(1,\ldots,N) = (-1)^{N-3}\, \kappa^{N-2} \sum_{\sigma\in S_{N-3}} A_{YM}(1,\sigma(2,3,\ldots,N-2),N-1,N)$$

$$\times \sum_{\rho\in S_{N-3}} S[\rho|\sigma]\ \tilde{A}_{YM}(1,\rho(2,3,\ldots,N-2),N,N-1)$$

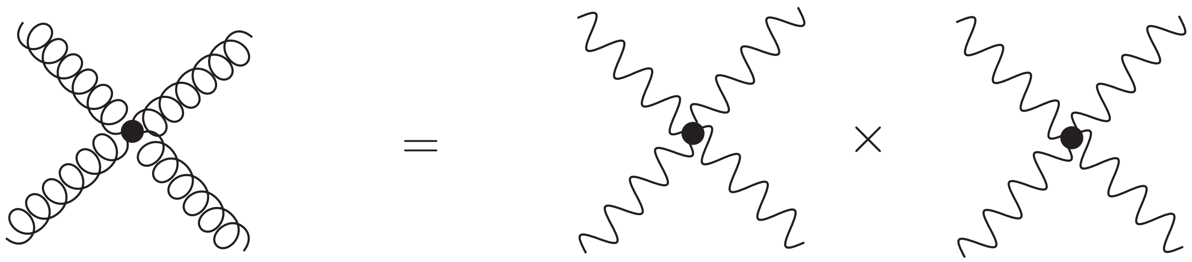
S = KLT kernel

$$\begin{aligned} S[\rho|\sigma] &:= S[\,\rho(2,\ldots,N-2)\,|\,\sigma(2,\ldots,N-2)\,] \\ &= \prod_{j=2}^{N-2} \left(s_{1,j_\rho} + \sum_{k=2}^{j-1} \theta(j_\rho,k_\rho)\, s_{j_\rho,k_\rho} \right) \end{aligned}$$

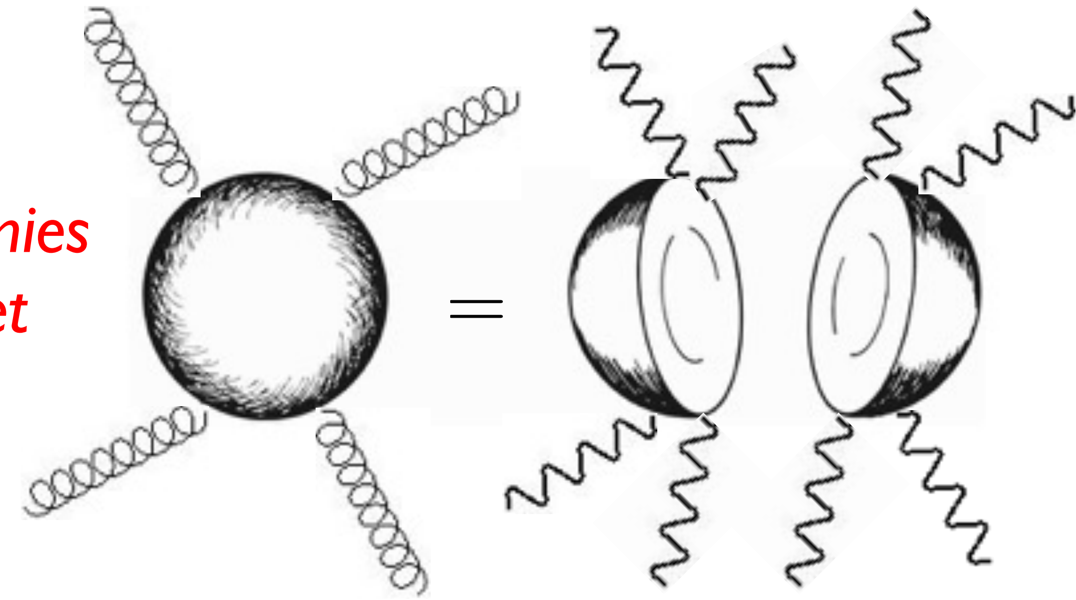
$$s_{ij} = \alpha'(k_i+k_j)^2$$

Bern, Dixon, Perelstein, Rozowsky (1998)

many **relations in field-theory** emerge from
properties of string world-sheet:
monodromy on world-sheet yield KLT, BCJ, … relations



from monodromies of world-sheet



*Structure of string amplitudes has **deep impact** on the **form** and **organization** of quantum field theory amplitudes*

Properties** of scattering amplitudes in both gauge and gravity theories suggest a deeper understanding **from string theory

• *relations among amplitudes from* ***different string vacua***

amplitudes are key players in establishing string dualities

based on: St.St., T.R. Taylor:

- **Closed string amplitudes as single-valued open string amplitudes,** Nucl. Phys. B881 (2014) 269-287, [arXiv:1401.1218]
- **Graviton as a Pair of Collinear Gauge Bosons,** Phys. Lett. B739 (2014) 457-461, [arXiv:1409.4771]
- **Graviton Amplitudes from Collinear Limits of Gauge Amplitudes,** [arXiv:1502.00655]

II. Heterotic gauge amplitudes as single-valued type I gauge amplitudes

Tree-level N-point type I open superstring gauge amplitude:

$$\mathfrak{A}_N^{\rm I} = (g_{YM}^{\rm I})^{N-2} \sum_{\Pi \in S_N/\mathbf{Z}_N} \mathrm{Tr}(T^{a_{\Pi(1)}} T^{a_{\Pi(2)}} \ldots T^{a_{\Pi(N)}})\ A^{\rm I}(\Pi(1), \ldots, \Pi(N))$$

Tree-level N-point heterotic closed string gauge amplitude:

$$\mathfrak{A}_N^{\rm HET} = (g_{YM}^{\rm HET})^{N-2} \sum_{\Pi \in S_N/\mathbf{Z}_N} \mathrm{Tr}(T^{a_{\Pi(1)}} T^{a_{\Pi(2)}} \ldots T^{a_{\Pi(N)}})\ A^{\rm HET}(\Pi(1), \ldots, \Pi(N)) + \mathcal{O}(1/N_c^2)$$

Result: $$\mathcal{A}^{\rm HET}(\Pi) = \mathrm{sv}\left(\mathcal{A}^{\rm I}(\Pi)\right)$$

sv= single-valued projection

What are these amplitudes describing ?

$$\alpha' - \text{expansion}: \quad e^{-\Phi_I}\ \zeta_{n_1,\ldots,n_r}\ \alpha'^l\ \mathrm{Tr}(F^{2+l}) \quad , \quad \sum_{i=1}^{r} n_i = l, \quad l \neq 0,1$$

$$\zeta_{n_1,\ldots,n_r} := \zeta(n_1,\ldots,n_r) = \sum_{0<k_1<\ldots<k_r} \prod_{l=1}^{r} k_l^{-n_l} \quad , \quad n_l \in \mathbf{N}^+ \ , \ n_r \geq 2 \ ,$$

E.g.: $e^{-\Phi_I}\ \zeta_2\ \alpha'^2\ \mathrm{Tr}(F^4)$

Born-Infeld type couplings

sv-projection

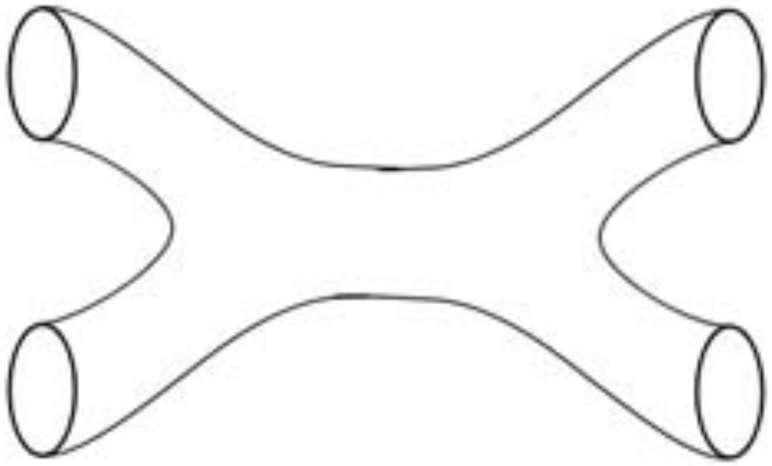
$$\alpha' - \text{expansion}: \quad e^{-2\Phi_H}\ \zeta^{SV}(n_1,\ldots,n_r)\ \alpha'^l\ \mathrm{Tr}(F^{2+l})$$

E.g.: $e^{-2\Phi_H}\ \zeta^{SV}(2)\ \alpha'^2\ \mathrm{Tr}(F^4) = 0$

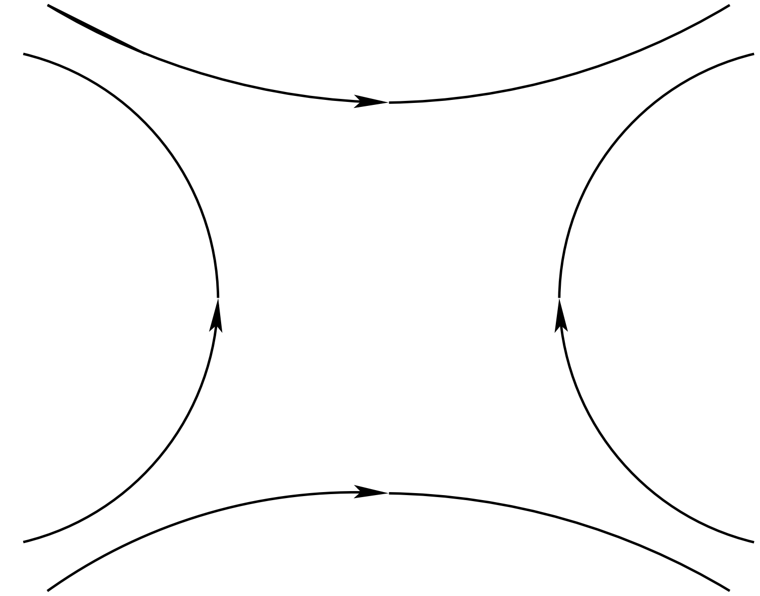
$$\zeta^{SV}(2) = 0$$

no tree-level $\mathrm{Tr}F^4$ term
(consistent with heterotic-type I duality)

e.g. N=4:



$=\ \mathrm{sv}$



$$\int_{\mathbf{C}} d^2z\ \frac{|z|^{2s}\ |1-z|^{2u}}{z\ (1-z)\ \overline{z}} = \mathrm{sv}\left(\int_0^1 dx\ x^{s-1}\ (1-x)^u\right)$$

complex integral on $\mathbf{P}^1\backslash\{0,1,\infty\}$ **iterated real integral on** $\mathbf{RP}^1\backslash\{0,1,\infty$

$$\frac{1}{s}\ \frac{\Gamma(s)\ \Gamma(u)\ \Gamma(t)}{\Gamma(-s)\ \Gamma(-u)\ \Gamma(-t)} = \mathrm{sv}\left(\frac{\Gamma(s)\ \Gamma(1+u)}{\Gamma(1+s+u)}\right)$$

$$s=\alpha'(k_1+k_2)^2$$
$$t=\alpha'(k_1+k_3)^2$$
$$u=\alpha'(k_1+k_4)^2$$

No KLT relations necessary !

KLT: $$\int_{\mathbf{C}} d^2z\ \frac{|z|^{2s}\ |1-z|^{2u}}{z\ (1-z)\ \overline{z}} = \sin(\pi u)\ \left(\int_0^1 x^{s-1}\ (1-x)^{u-1}\right)\ \left(\int_1^\infty x^{t-1}\ (1-x)^u\right)$$

Complex vs. iterated integrals:

$z_{ij} := z_i - z_j$

$$\int_{\mathbf{C}^{N-3}} \left(\prod_{j=2}^{N-2} d^2 z_j \right) \frac{\prod_{i<j}^{N-1} |z_{ij}|^{\alpha' s_{ij}}}{z_{1,\rho(2)}\ z_{\rho(2),\rho(3)} \cdots z_{\rho(N-3),\rho(N-2)}} \frac{1}{\overline{z}_{1,\pi(2)}\ \overline{z}_{\pi(2),\pi(3)} \cdots \overline{z}_{\pi(N-2),N-1}}$$

$$= \ \mathrm{sv} \int_{D(\pi)} \left(\prod_{j=2}^{N-2} dz_j \right) \frac{\prod_{i<j}^{N-1} |z_{ij}|^{\alpha' s_{ij}}}{z_{1,\rho(2)}\ z_{\rho(2),\rho(3)} \cdots z_{\rho(N-3),\rho(N-2)}}$$

$\rho, \pi \in S_{N-3}$

$D(\pi) = \{\ z_j \in \mathbf{R} \mid 0 < z_{\pi(2)} < \ldots < z_{\pi(N-2)} < 1 \}$

This is generalized to ***any*** *closed string amplitude:*

closed string amplitudes as

single-valued open string amplitudes

Multiple zeta-values in superstring theory

Disk integrals: **iterated real integral** on $\mathbf{RP}^1\backslash\{0,1,\infty\}$

Expand w.r.t. α':

$$V_{\rm CKG}^{-1}\int\limits_{z_i<z_{i+1}}\left(\prod_{j=1}^{5}dz_j\right)\prod_{1\leq i<j\leq 5}\frac{|z_{ij}|^{s_{ij}}}{z_{12}z_{23}z_{35}z_{54}z_{41}}$$

$$={\alpha'}^{-2}\left(\frac{1}{s_{12}s_{45}}+\frac{1}{s_{23}s_{45}}\right)+\zeta(2)\left(1-\frac{s_{34}}{s_{12}}-\frac{s_{12}}{s_{45}}-\frac{s_{23}}{s_{45}}-\frac{s_{51}}{s_{23}}\right)+\mathcal{O}(\alpha')$$

Terasoma & Brown: the coefficients of the Taylor expansion of the Selberg integrals w.r.t. the variables s_{ij} can be expressed as linear combinations of MZVs over $\mathbf{Q}$

$$\zeta_{n_1,\ldots,n_r}:=\zeta(n_1,\ldots,n_r)=\sum_{0<k_1<\ldots<k_r}\prod_{l=1}^{r}k_l^{-n_l}\quad,\quad n_l\in\mathbf{N}^+\ ,\ n_r\geq 2\ ,$$

Commutative graded $\mathbf{Q}$ - algebra: $\mathcal{Z}=\bigoplus_{k\geq 0}\mathcal{Z}_k\ ,\quad \dim_{\mathbf{Q}}(\mathcal{Z}_N)=d_N$

with: $d_N=d_{N-2}+d_{N-3},\quad d_0=1,\ d_1=0,\ d_2=1,\ldots$ (Zagier)

w	2	3	4	5	6	7	8	9	10	11	12
$\mathcal{Z}_w$	ζ_2	ζ_3	ζ_2^2	ζ_5 $\zeta_2\ \zeta_3$	ζ_3^2 ζ_2^3	ζ_7 $\zeta_2\ \zeta_5$ $\zeta_2^2\ \zeta_3$	$\zeta_{3,5}$ $\zeta_3\ \zeta_5$ $\zeta_2\ \zeta_3^2$ ζ_2^4	ζ_9 ζ_3^3 $\zeta_2\ \zeta_7$ $\zeta_2^2\ \zeta_5$ $\zeta_2^3\ \zeta_3$	$\zeta_{3,7}$ $\zeta_3\ \zeta_7$ ζ_5^2 $\zeta_2\ \zeta_{3,5}$ $\zeta_2\ \zeta_3\ \zeta_5$ $\zeta_2^2\ \zeta_3^2$ ζ_2^5	$\zeta_{3,3,5}$ $\zeta_2\ \zeta_3^3$ $\zeta_{3,5}\ \zeta_3$ $\zeta_2\ \zeta_9$ ζ_{11} $\zeta_2^2\ \zeta_7$ $\zeta_3^2\ \zeta_5$ $\zeta_2^3\ \zeta_5$ $\zeta_2^4\ \zeta_3$	$\zeta_{1,1,4,6}$ $\zeta_2\ \zeta_{3,7}$ $\zeta_{3,9}$ $\zeta_2^2\ \zeta_{3,5}$ $\zeta_3\ \zeta_9$ $\zeta_2\ \zeta_5^2$ $\zeta_5\ \zeta_7$ $\zeta_2\ \zeta_3\ \zeta_7$ ζ_3^4 $\zeta_2^2\ \zeta_3\ \zeta_5$ $\zeta_2^3\ \zeta_3^2$ ζ_2^6
d_w	1	1	1	2	2	3	4	5	7	9	12

E.g. weight 12 : $\zeta_{5,7} = \frac{14}{9}\ \zeta_{3,9} + \frac{28}{3}\ \zeta_5\ \zeta_7 - \frac{776224}{1576575}\ \zeta_2^6$

- MZVs occur as the values **at unity** of MPs

multiple polylogarithms:

$$\mathcal{L}i_{a_1,\ldots,a_r}(x_1,\ldots,x_r) = \sum_{0<k_1<\ldots<k_r} \prod_{l=1}^{r} \frac{x_l^{k_l}}{k_l^{a_l}}$$

$$\mathcal{L}i_{a_1,\ldots,a_r}(1,\ldots,1) = \sum_{0<k_1<\ldots<k_r} \prod_{l=1}^{r} k_l^{-a_l} = \zeta_{a_1,\ldots,a_r}$$

Single-valued MZVs

$$\zeta_{\rm sv}(n_1,\ldots,n_r)\in\mathbf{R}$$

• special class of MZVs, which occurs as the values **at unity** of SVMPs

$$polylogarithms:\quad \ln(z),\ Li_1(z)=-ln(1-z), Li_a(z), Li_{a_1,\ldots,a_r}(1,\ldots,1,z)$$

SVMPs: **multiple polylogarithms** can be combined
with their complex conjugates
to **remove monodromy** at $z=0,1,\infty$
rendering the function **single-valued** on $\mathbf{P}^1\backslash\{0,1,\infty\}$.

$$\mathcal{L}_2(z)=D(z)=Im\left\{Li_2(z)+\ln|z|\ \ln(1-z)\right\}$$

(Bloch-Wigner dilogarithm)

$$\mathcal{L}_n(z)=Re_n\left\{\sum_{k=1}^{n}\frac{(-\ln(|z|)^{n-k}}{(n-k)!}\ Li_k(z)+\frac{\ln^n|z|}{(2n)!}\right\}\ \text{with:}\quad Re_n=\begin{cases}Im, & n \text{ even}\\ Re, & n \text{ odd}\end{cases}$$

(Zagier)

$$\mathcal{L}_n(1)=Re_n\left\{Li_n(1)\right\}=\begin{cases}0, & n \text{ even}\\ \zeta_n, & n \text{ odd}\end{cases}$$

- coefficients of the Deligne associator W:

(reduced) KZ equation: $$\frac{d}{dz} L_{e_0,e_1}(z) = L_{e_0,e_1}(z) \left(\frac{e_0}{z} + \frac{e_1}{1-z} \right)$$ with generators e_0 and e_1 of the free Lie algebra g

its unique solution can be given as generating series of multiple polylogarithms:

$$L_{e_0,e_1}(z) = \sum_{w \in \{e_0,e_1\}^\times} L_w(z)\ w$$

with the symbol $w \in \{e_0, e_1\}^\times$ denoting a non-commutative word $w_1 w_2 \dots$ in the letters $w_i \in \{e_0, e_1\}$

$$L_1 = 1, \quad L_{e_0^n} = \frac{1}{n!} \ln^n z, \quad L_{e_1^n} = \frac{1}{n!} \ln^n (1-z)$$

Drinfeld associator Z:

$$\zeta(e_1 e_0^{n_1-1} \dots e_1 e_0^{n_r-1}) = \zeta_{n_1,\dots,n_r}$$

$$\zeta(w_1)\zeta(w_2) = \zeta(w_1 ⧢ w_2) \text{ and } \zeta(e_0) = 0 = \zeta(e_1)$$

$$Z(e_0,e_1) := L_{e_0,e_1}(1) = \sum_{w \in \{e_0,e_1\}^\times} \zeta(w)\ w = 1 + \zeta_2\ [e_0,e_1] + \zeta_3\ (\ [e_0,[e_0,e_1]] - [e_1,[e_0,e_1]\ ) + \dots$$

F. Brown (2004) defines generating series of SVMPs:

$$\mathcal{L}_{e_0,e_1}(z) = L_{-e_0,-e_1'}(\overline{z})^{-1}\ L_{e_0,e_1}(z)$$

e_1' determined recursively by fixed–point equation:

$$Z(-e_0,-e_1')\ e_1'\ Z(-e_0,-e_1')^{-1} = Z(e_0,e_1)\ e_1\ Z(e_0,e_1)^{-1}$$

Deligne associator W:

$$W(e_0,e_1) := \mathcal{L}(1) = Z(-e_0,-e_1')^{-1}\ Z(e_0,e_1) = \sum_{w \in \{e_0,e_1\}^\times} \zeta_{\text{sv}}(w)\ w$$

$$W(e_0,e_1) = 1 + 2\ \zeta_3\ ([e_0,[e_0,e_1]] - [e_1,[e_0,e_1]]) + \dots$$

F. Brown (2013)

There is a natural homomorphism:

F. Brown (2013): $\mathrm{sv} : \zeta_{n_1,\ldots,n_r} \longrightarrow \zeta_{\mathrm{sv}}(n_1,\ldots,n_r)$

$$\begin{aligned}
\zeta_{\mathrm{sv}}(2) &= 0 \\
\zeta_{\mathrm{sv}}(2n+1) &= 2\,\zeta_{2n+1} \\
\zeta_{\mathrm{sv}}(3,5) &= -10\,\zeta_3\,\zeta_5
\end{aligned}$$

$$\zeta_{\mathrm{sv}}(3,5,3) = 2\,\zeta_{3,5,3} - 2\,\zeta_3\,\zeta_{3,5} - 10\,\zeta_3^2\,\zeta_5$$

$$\zeta_{\mathrm{sv}}(3,3,5) = 2\,\zeta_{3,3,5} - 5\,\zeta_3^2\,\zeta_5 + 90\,\zeta_2\,\zeta_9 + \frac{12}{5}\,\zeta_2^2\,\zeta_7 - \frac{8}{7}\,\zeta_2^3\,\zeta_5^2$$

Result:

$$\boxed{\mathcal{A}^{\rm HET}(\Pi) = {\rm sv}\left(\mathcal{A}^{\rm I}(\Pi)\right)}$$

unexpected relation between

open and closed string amplitudes

(beyond KLT)

new *string* ***duality***

(to all orders in α' *i.e. beyond BPS)*

- By applying **naively KLT** relations we would **not** have arrived at **these relations**

- **Much deeper connection** between open and closed string amplitudes **than** what is implied by **KLT relations**

- **Full** α'- dependence of **closed string** amplitude is **entirely** encapsulated by **open string** amplitude

- **Any** closed string amplitude can be written as single-valued image of open string amplitude

- **Various connections** between different amplitudes of different vacua can be established

New kind of *duality* relating amplitudes involving
full tower of massive string excitations
(not just BPS states as in most examples of string dualities)

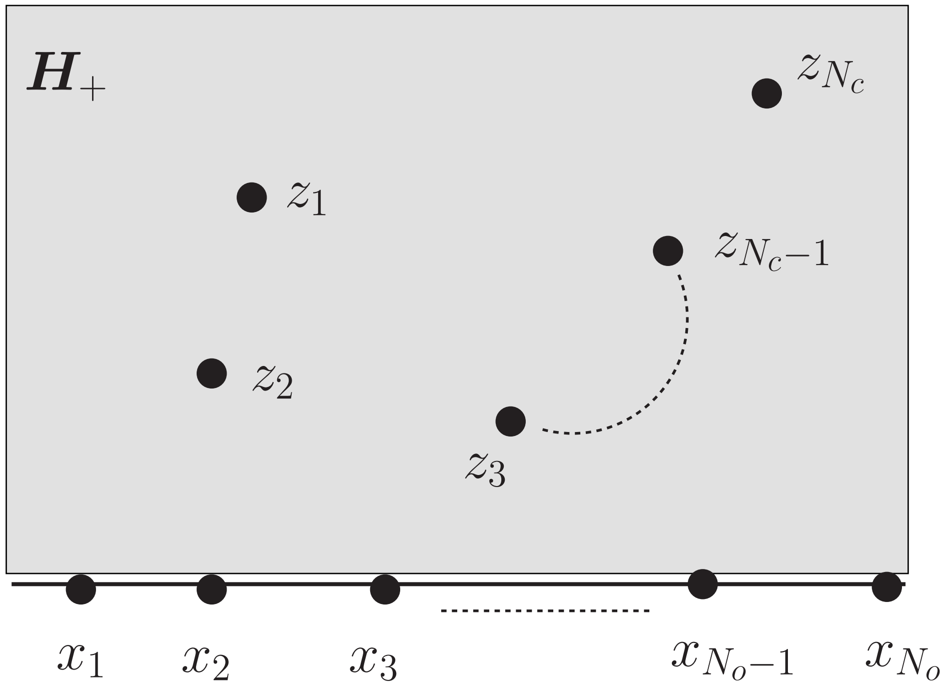
III. Mixed amplitudes in field- and string theory

Mixed amplitudes involving open and closed strings:

"Doubling trick":

- *convert disk correlators to the standard holomorphic ones by extending the fields to the entire complex plane.*

- *integration over positions of world-sheet symmetric closed string states (such as graviton or dilaton) can be extended from the half-plane to the full complex plane*



→ *monodromy problem on the complex plane*

N_o open & N_c closed strings: $2N_o + N_c-$ point **pure** open string amplitude

St.St. arXiv:0907.2211

$N_c = 1$

$$A(1,2,\ \ldots,N\!-\!2;q)$$
$$= \sum_{l=2}^{\lceil \frac{N}{2}\rceil -1} \sum_{i=2}^{l} \sin\left(\pi \sum_{j=i}^{l} s_{j,N-1}\right) A(1,\ldots,i-1,N,i,\ldots,l,N-1,l+1,\ldots,N-2)$$
$$+ \sum_{l=\lceil \frac{N}{2}\rceil}^{N-3} \sum_{i=l+1}^{N-2} \sin\left(\pi \sum_{j=l+1}^{i} s_{j,N-1}\right) A(1,\ldots,l,N-1,l+1,\ldots,i,N,i+1,\ldots,N-2)$$

$(\lceil \frac{N}{2} \rceil - 2)\ (\lfloor \frac{N}{2} \rfloor - 1)$ terms

$s_{i,j} \equiv s_{ij} = 2\alpha' k_i k_j$

relations between
amplitudes involving **open & closed strings** and
pure open string amplitudes

Examples:

$$A(1,2,3;q) = \sin(\pi s_{24})\ A(1,5,2,4,3)\ ,$$
$$A(1,2,3,4;q) = \sin(\pi s_{25})\ A(1,6,2,5,3,4) + \sin(\pi s_{45})\ A(1,2,3,5,4,6)\ ,$$
$$A(1,2,3,4,5;q) = \sin(\pi s_{26})\ A(1,7,2,6,3,4,5) + \sin(\pi s_{36})\ A(1,2,7,3,6,4,5) + \sin[\pi(s_{36}+s_{26})]\ A(1,7,2,3,6,4,5) + \sin(\pi s_{56})\ A(1,2,3,4,6,5,7)$$

(in collinear limit)

take field-theory limit: yields **Einstein-Yang-Mills** for any kinematical configuration

"graviton appears as a pair of collinear gauge bosons"

$$A_{EYM}(1^+,2^+,3^-;q^{--}) = \pi\ s_{24}\ A_{\mathrm{YM}}(1^+,5^-,2^+,4^-,3^-)$$

with SYM amplitude:

$$A_{\mathrm{YM}}(1^+,5^-,2^+,4^-,3^-) = 4\ \frac{[12]^4}{[1q][q3][13][2q]^2}$$

MHV case: Bern, De Freitas, Wong, arXiv:hep-th/9912033
from *squares* of open string amplitudes (*heterotic string*)

IV. Graviton amplitudes from gauge amplitudes

express N-graviton amplitude in **Einstein's gravity**
as collinear limits of
certain linear combinations of **pure SYM amplitudes**
in which each graviton is represented by two gauge bosons

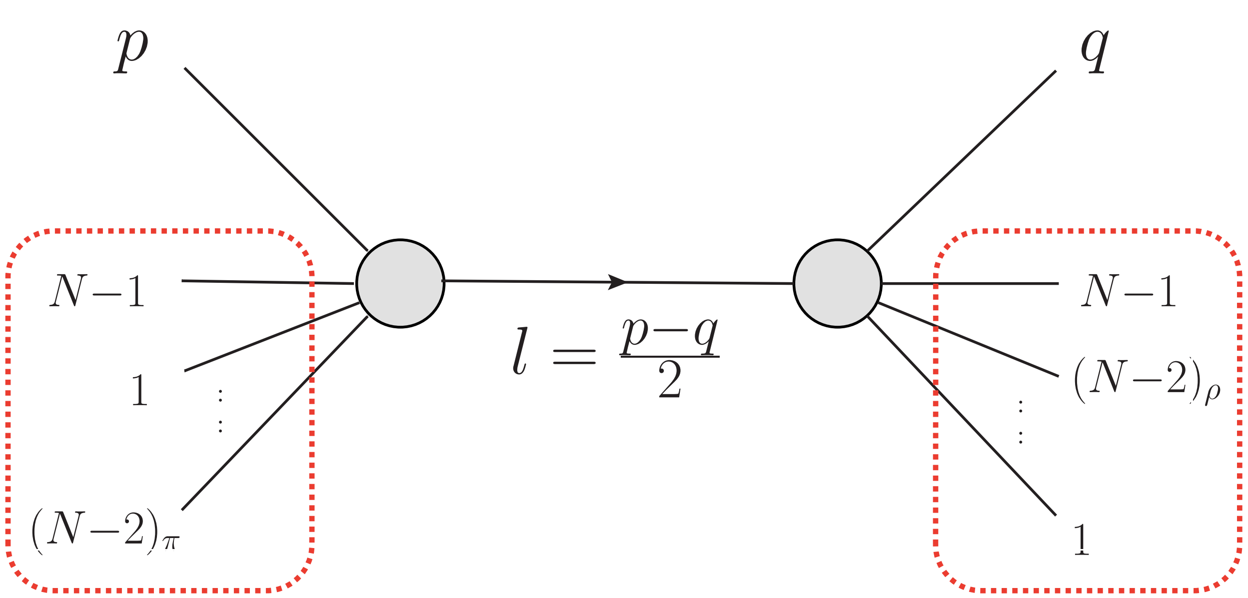
no string theory !
but motivated from string theory

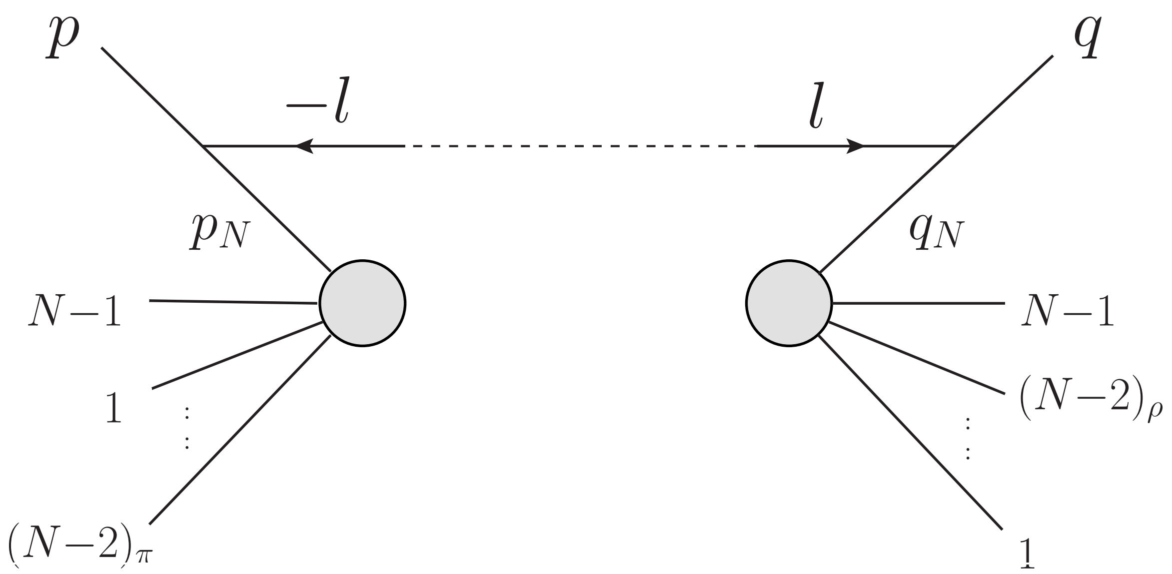
$$A_E[k_1,\lambda_1;\ldots;k_{N-1},\lambda_{N-1};k_N=p+q,\lambda_N=+2]=\lim_{[pq]\to 0}\left(\frac{1}{2x}\right)^4\frac{[pq]}{\langle pq\rangle}s_{pq}^2$$

$$\times\sum_{\pi,\rho\in S_{N-3}}S[\pi|\rho]\;A_{YM}[p,N{-}1,1,\pi(2,3,\ldots,N{-}2),1,\rho(2,\ldots,N{-}2),N{-}1,q]$$

(2N-2 gluons become collinear without producing poles)

Proof: contributions from factorization on **triple** pole $s_{pq}^3 \sim (p-q)^6$





$$A_{YM}[p,N-1,1,\pi(2,3,\ldots,N-2),1,\rho(2,\ldots,N-2),N-1,q] \;\to\; \left(\frac{4}{s_{pq}}\right)^3 \times$$
$$\times\; A_{YM}[p^+,-l^-,-p_N^-] \times A_{YM}[p_N,\mu_N=+1;N-1,1,\pi(2,3,\ldots,N-2)]$$
$$\times\; A_{YM}[1,\rho(2,\ldots,N-2),N-1;q_N,\nu_N=+1] \times A_{YM}[q^-,l^+,-q_N^-]$$

$$\left\{ \begin{array}{l} A_{YM}[p^+,-l^-,-p_N^-] = \frac{x^3}{2}\langle pq\rangle, \\ A_{YM}[q^-,l^+,-q_N^-] = \frac{x}{2}\langle pq\rangle \end{array} \right.$$

yields:

$$A_E[k_1,\lambda_1;\ldots;k_{N-1},\lambda_{N-1};k_N,\lambda_N=+2]$$
$$= \sum_{\pi,\rho\in S_{N-3}} S[\pi|\rho]\; A_{YM}[p_N,\mu_N=+1;N-1,1,\pi(2,3,\ldots,N-2)]$$
$$\times A_{YM}[1,\rho(2,\ldots,N-2),N-1;q_N,\nu_N=+1]$$

Concluding remarks

- **new kind of duality** working beyond usual BPS protected operators
- graviton scattering **unified** into gauge amplitudes
- growing set of **interconnections** between open & closed amplitudes with gauge theory and supergravity amplitudes