Eurostrings 2015, Cambridge, UK

Scattering Inequalities

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1312.2007, 1410.0354, 1412.8478

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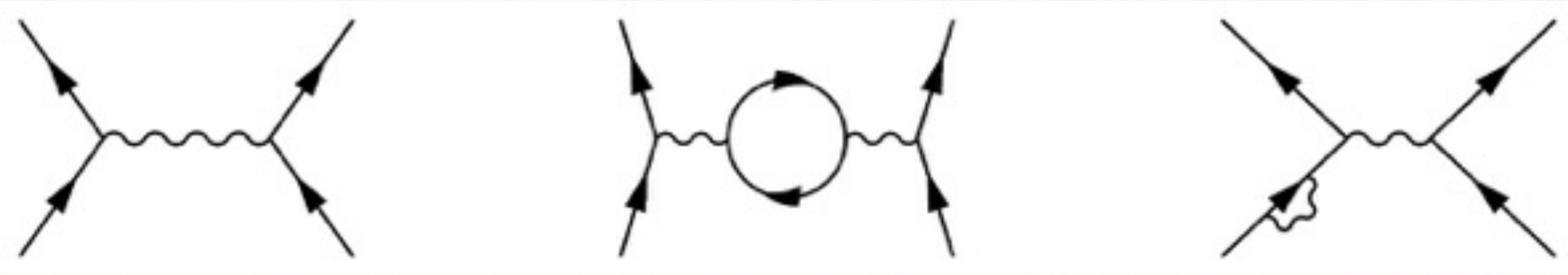
1412.8584, in progress

Scattering Amplitudes

- Basic objects in Quantum Field Theory (QFT)
- Predictions for colliders: cross-sections
- My motivation: new ideas in QFT

Perturbative QFT

- Loop expansion



- <u>Integrand</u>: rational function before integration

$I(\ell_j, k_i, s_i)$ sum of Feynman diagrams

$$\Omega = d^4\ell_1 \dots d^4\ell_L \, I(\ell_j, k_i, s_i) \qquad A = \int_{\ell_j \in \mathbb{R}} \Omega$$

Integrand form

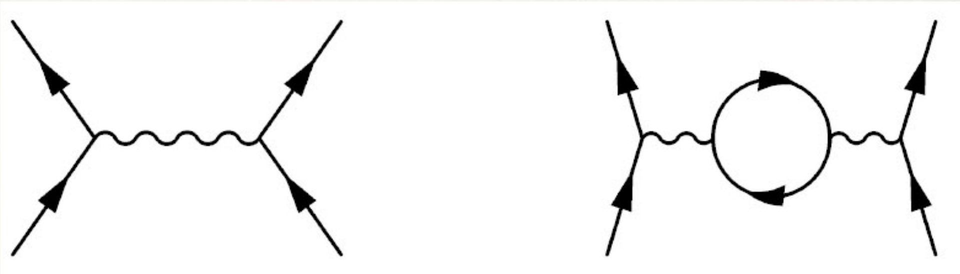
Integrand

- Ideal object to study: finite, well-defined
- Fixed by principles of QFT
- Qualitative information about the final amplitudes
  - Collinear limits: IR divergencies
  - Poles at infinity: UV structure
  - Types of singularities: transcendental properties

Feynman diagrams

- Gauge redundancy: off-shell virtual particles

- Two principles manifest:

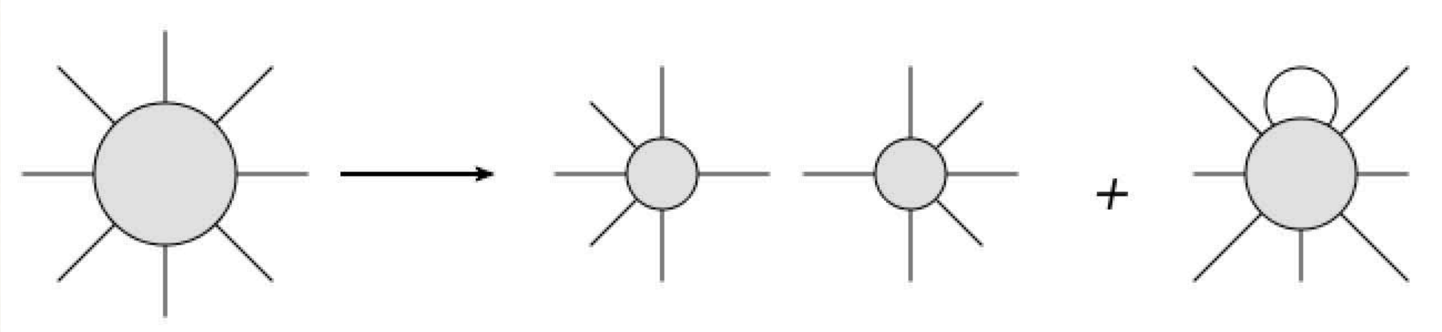


I) Locality: particles interact point-like

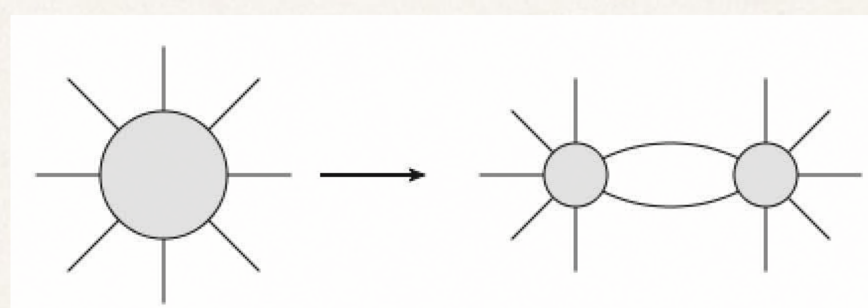
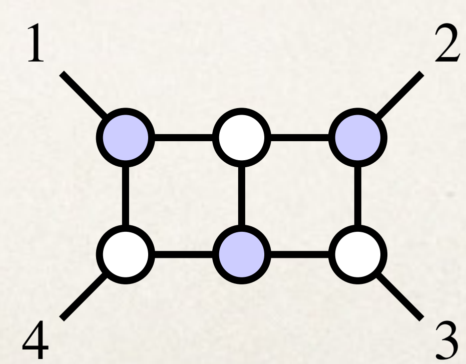
Amplitude: only poles $\frac{1}{P^2} \to \infty$ $P = \sum_{i \in \sigma} p_i$

II) Unitarity: sum of probabilities is 1

Amplitude: factorization

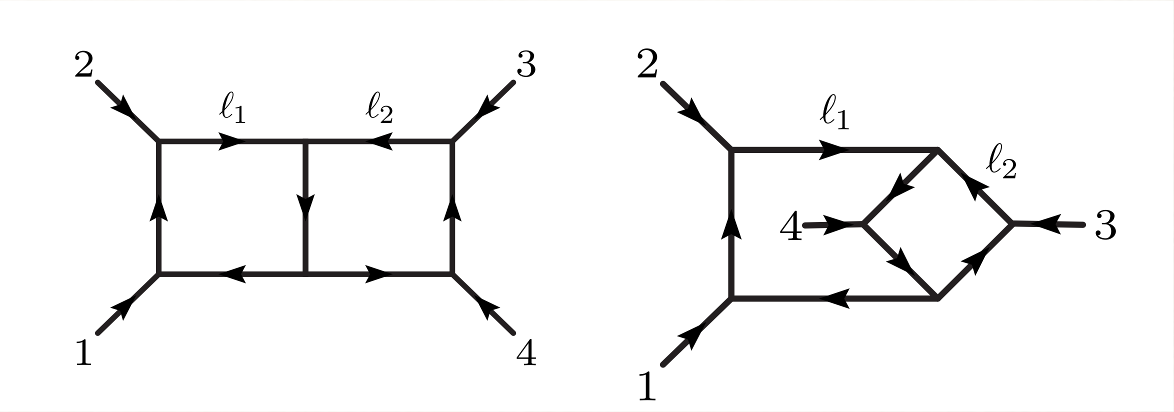


Modern methods

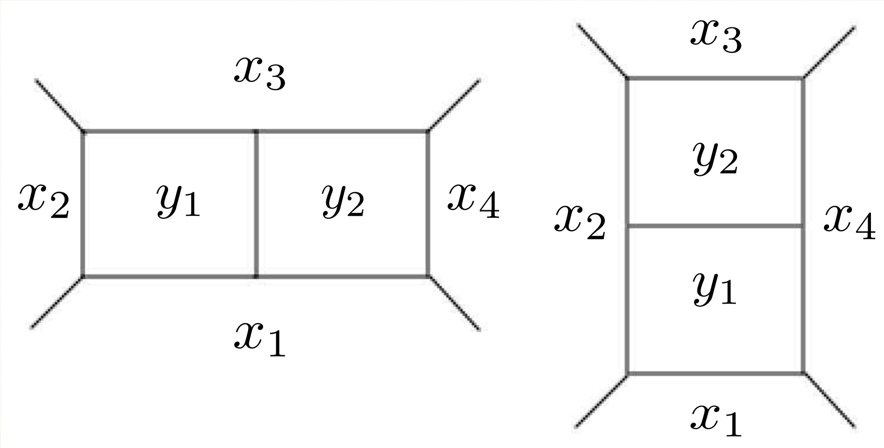
- Re-express the integrand in the basis of integrals
- Fix coefficients using cuts $$I = \sum_j c_j \, I_j$$
- Unitarity cuts:  $$\ell^2 = (\ell + Q)^2 = 0$$
- Maximal cuts, leading singularities: 

Planar limit

- The integrand defined as a sum of diagrams
  - No global loop momenta
  - Each diagram: its own labels



- Planar limit: dual variables



$$k_1 = (x_1 - x_2) \quad k_2 = (x_2 - x_3)$$
$$\ell_1 = (x_3 - y_1) \quad \ell_2 = (y_2 - x_3)$$
etc

Global labels

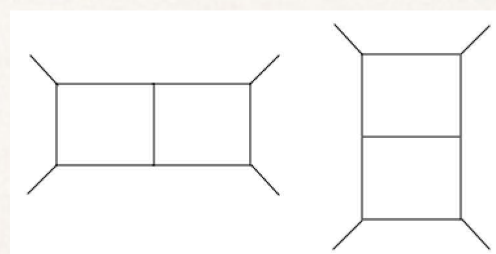
Integrand: <u>single</u> function

Conditions on the amplitude

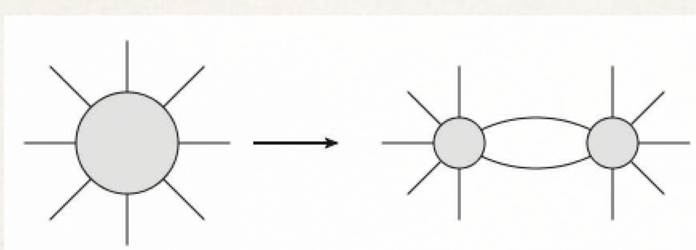
Standard methods

- Planar diagrams

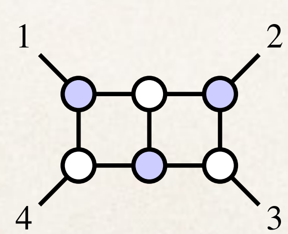
Locality + Planarity



- Match physical cuts/singularities



Unitarity

$\mathrm{Cut}(I) =$ 

Construction not known in general

Alternative

- Same set of conditions
- Packaged in a different way

?

Complete set known

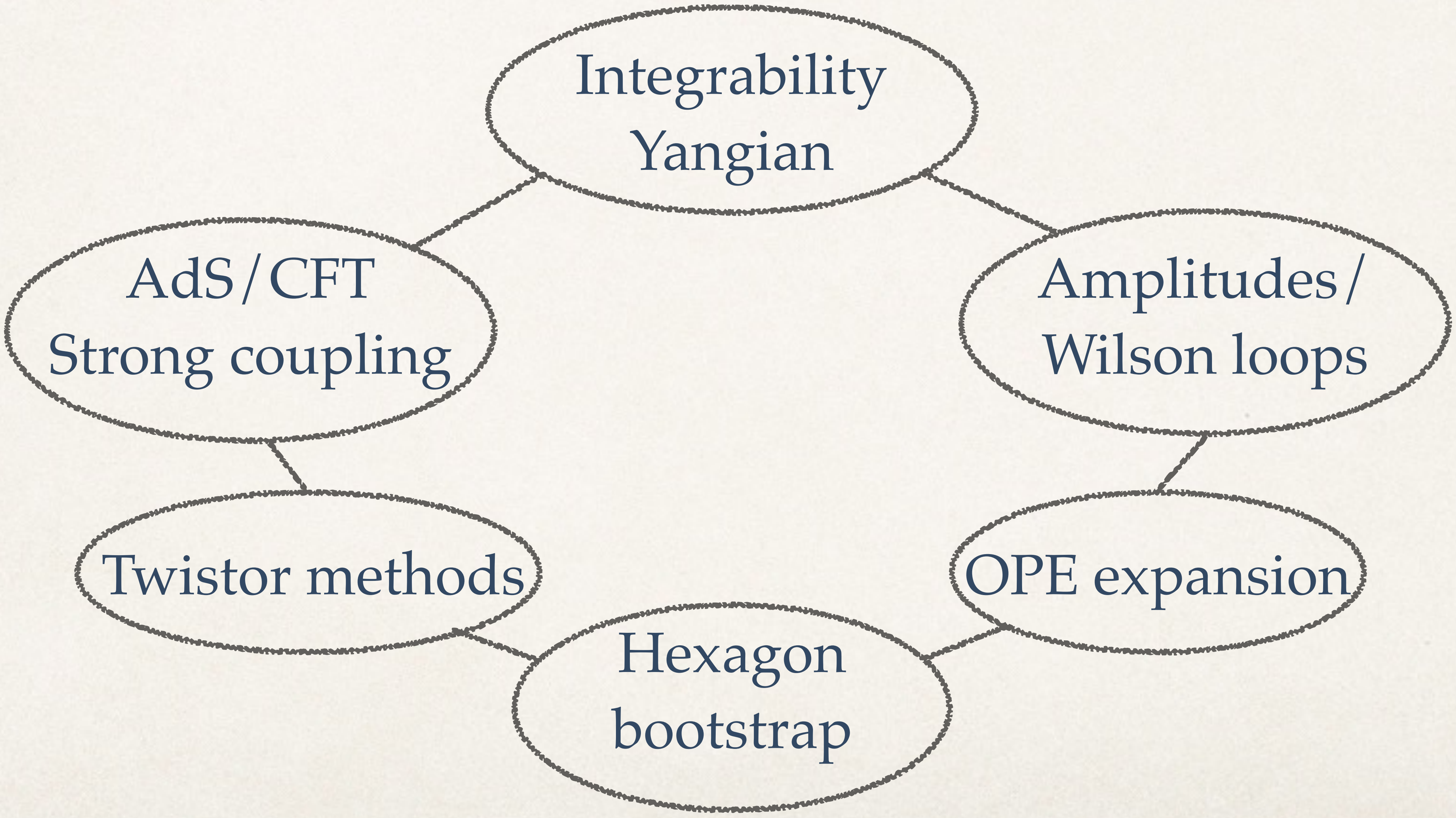
Maximally supersymmetric Yang-Mills theory in planar limit

(Brink-Scherk-Schwarz 1977)

- "Simplest Quantum Field Theory"
- Conformal + dual conformal, convergent series
- Toy model for QCD
  - Tree-level amplitudes identical
  - Loop amplitudes simpler, no confinement
- Past: new methods for amplitudes originated here

Many faces of the theory

- Useful playground for many theoretical ideas



Integrand in planar N=4 SYM

- Superamplitudes $\mathcal{I}_{n,\ell}$

$$\mathcal{I}_{n,\ell} = \sum_k \tilde{\eta}^{4k} I_{n,k,\ell}$$

k : number of negative gluons

- Dual conformal symmetry
  (Drummond, Henn, Korchemsky, Sokatchev 2006)
  - Integral basis: no triangle subdiagrams
    = no poles at infinity momentum
- Recursion relations using on-shell diagrams
  (Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT 2012)
  - Logarithmic singularities

$$\Omega \sim \frac{dx}{x} \quad \text{near} \quad x = 0$$

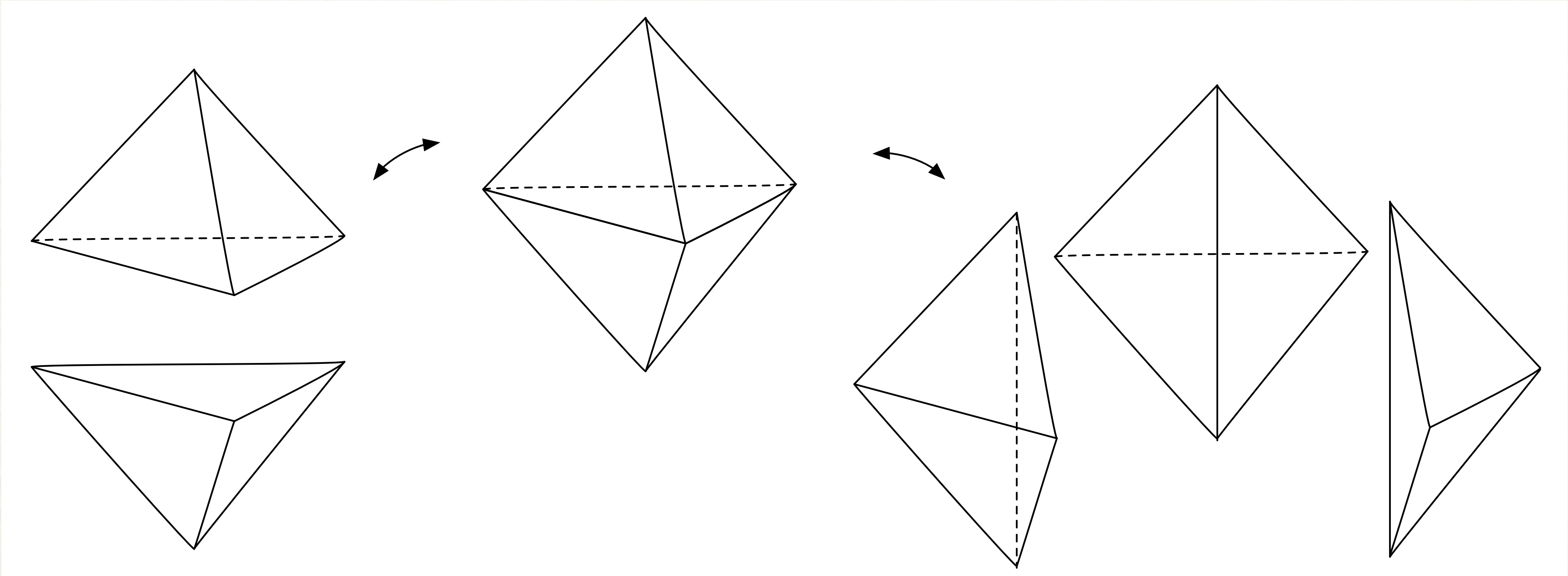
The Amplituhedron

(Arkani-Hamed, JT 2013)

Volume of polyhedron

(Hodges 2009)

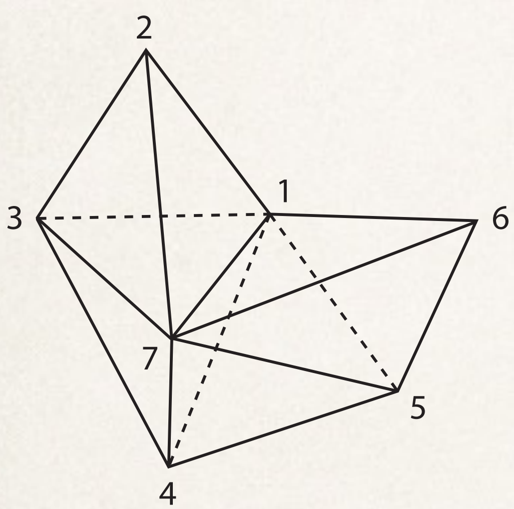
- New kinematical variables — momentum twistors
  $Z \in \mathbb{C}^3$
- Tree-level process: $gg \to 5g$
- Comparison of two calculations of recursion relations



(Picture by Stavros Garoufalidis)

Evidence for a new structure

Volume of polyhedron



$gg \to gg \ldots g$

at tree-level

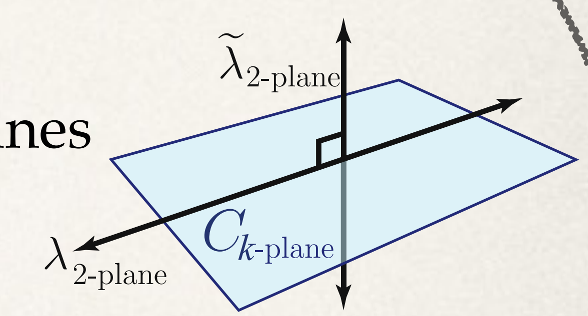
$$\int_{\widetilde{P}_n} \frac{D^4\mathcal{W}}{(\mathcal{Z}_0 \cdot \mathcal{W})^5}$$

Amplitude = volume

(Arkani-Hamed, Bourjaily, Cachazo, Hodges, JT 2010)

Grassmannian

Configurations of k-planes in n dimensions



$$\oint_{C\subset\Gamma_\sigma} \frac{d^{k\times n}C}{\mathrm{vol}(GL(k))} \frac{\delta^{k\times 4}(C\cdot\widetilde{\eta})}{(1\cdots k)\cdots(n\cdots k{-}1)} \delta^{k\times 2}(C\cdot\widetilde{\lambda})\,\delta^{2\times(n-k)}(\lambda\cdot C^{\perp})$$

All-loop order information

(Arkani-Hamed, Cachazo, Cheung, Kaplan 2009)

"Conjecture"

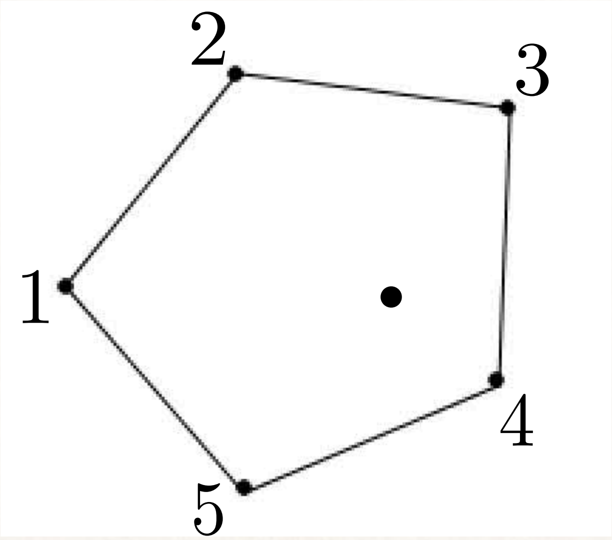
Amplitudes are volumes
of *some regions* in *some space*

Strategy

- Simple intuitive geometric ideas: use equations
- Generalization:
  - More complicated geometry
  - Higher dimensions
- Same equations persist

Road to Amplituhedron

<u>Start:</u>
Point inside a convex polygon



Road to Amplituhedron

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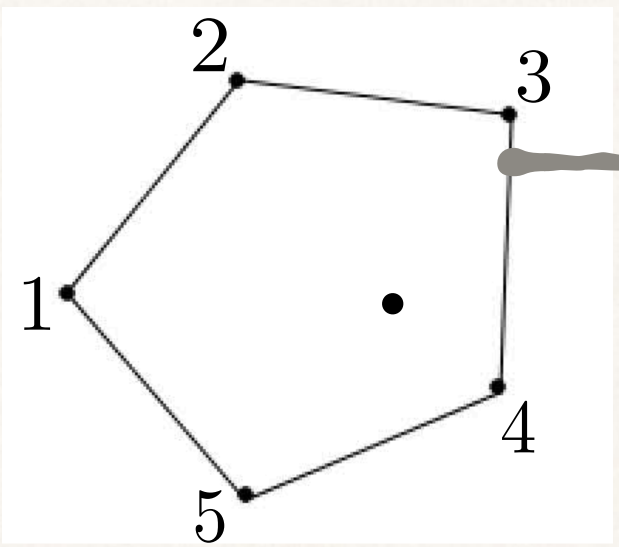
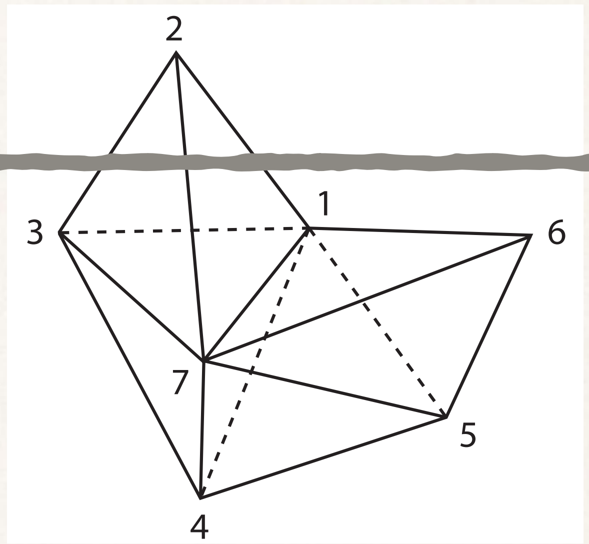
Point inside a convex polygon





Road to Amplituhedron

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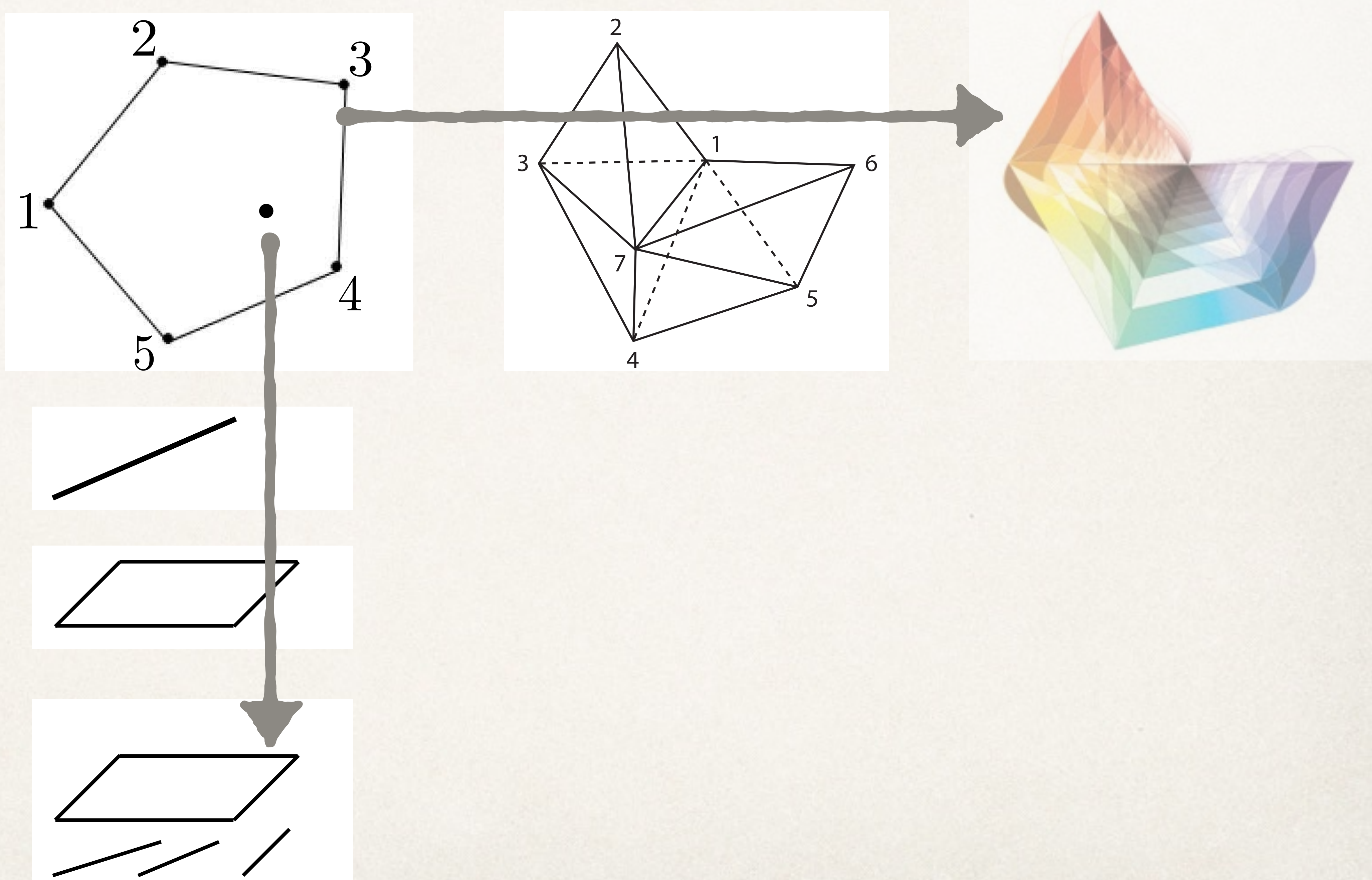
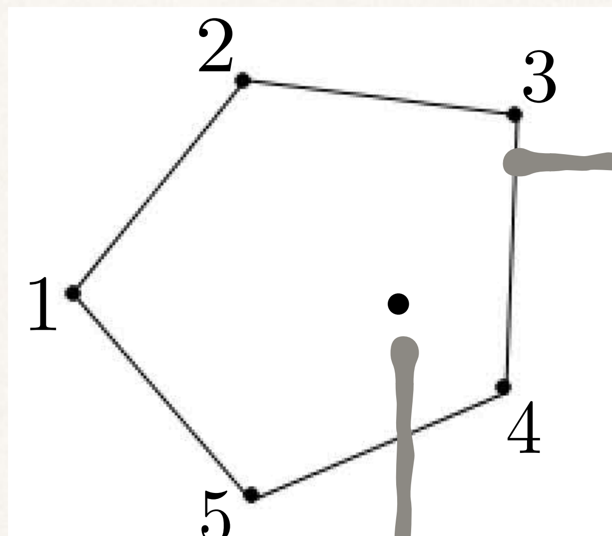
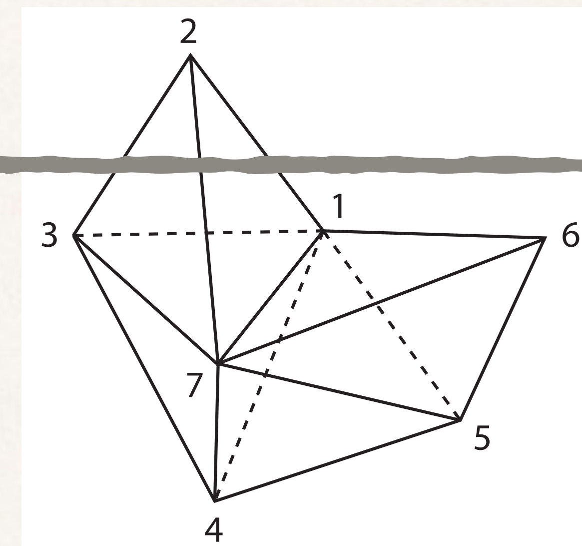
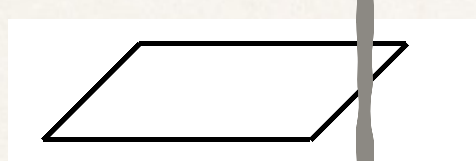


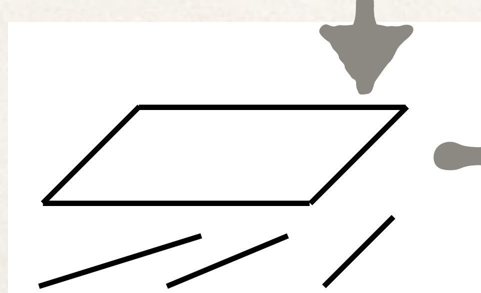
Road to Amplituhedron

<u>Start:</u>
Point inside a convex polygon









<u>Amplituhedron $\mathcal{A}_{n,k,\ell}$</u>

A k-dim plane and ℓ lines inside a $(k+4)$-dim convex space defined by n vertices

The Amplituhedron

- Volume of $\mathcal{A}_{n,k,\ell}$:

Amplitudes in maximally supersymmetric Yang-Mills theory

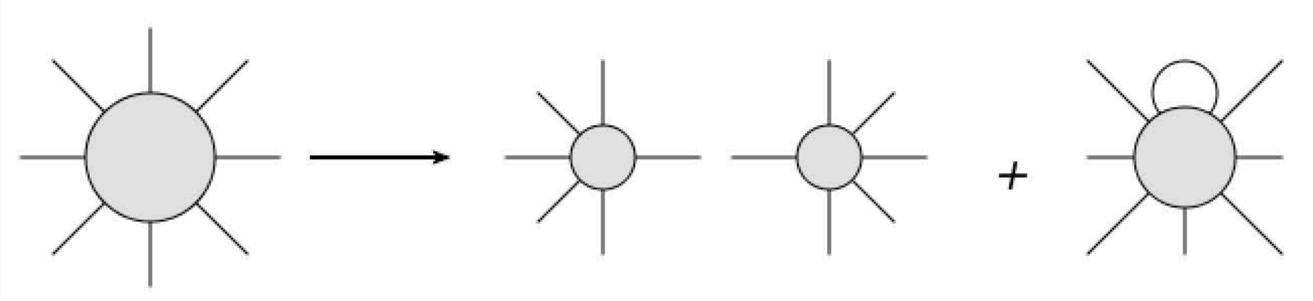
$\ell = 0$: Amplitudes of gluons in QCD

n
number of particles

k
helicity information

ℓ
number of loops

- Consistency check: Locality and Unitarity



- Explicit checks against reference theoretical data

Volume of the space

- Differential form with logarithmic singularities
- Simple examples: $$\Omega \sim \frac{dx}{x} \quad \text{near} \quad x = 0$$

$$x > 0 : \qquad \text{Vol} = \frac{dx}{x}$$

$$y > 0,\, x > 0 : \qquad \text{Vol} = \frac{dx}{x}\frac{dy}{y}$$

$$y > x > 0 : \qquad \text{Vol} = \frac{dx}{x}\frac{dy}{y - x}$$

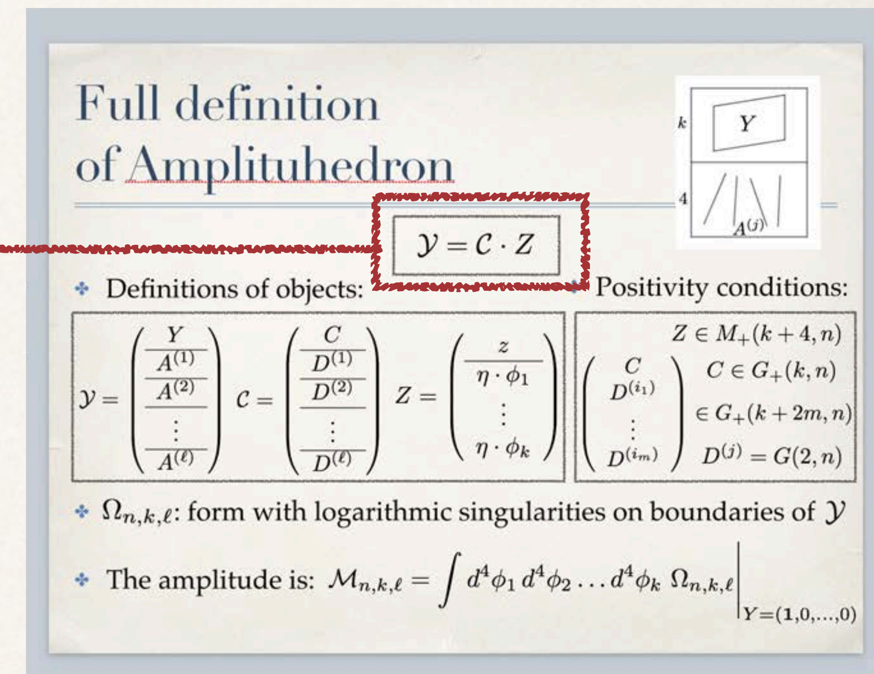
The Amplituhedron

- In the definition of Amplituhedron

$$\mathcal{Y} = \mathcal{C} \cdot Z$$

Amplituhedron

Positive matrices: Minors are positive $\begin{vmatrix} * & * \\ * & * \end{vmatrix} > 0$



- <u>Positivity</u>: crucial property of geometry
  - Locality, unitarity, even planarity derived from it
  - Hidden symmetry of this theory (Yangian) manifest

Inequalities

- Amplituhedron variables z_i

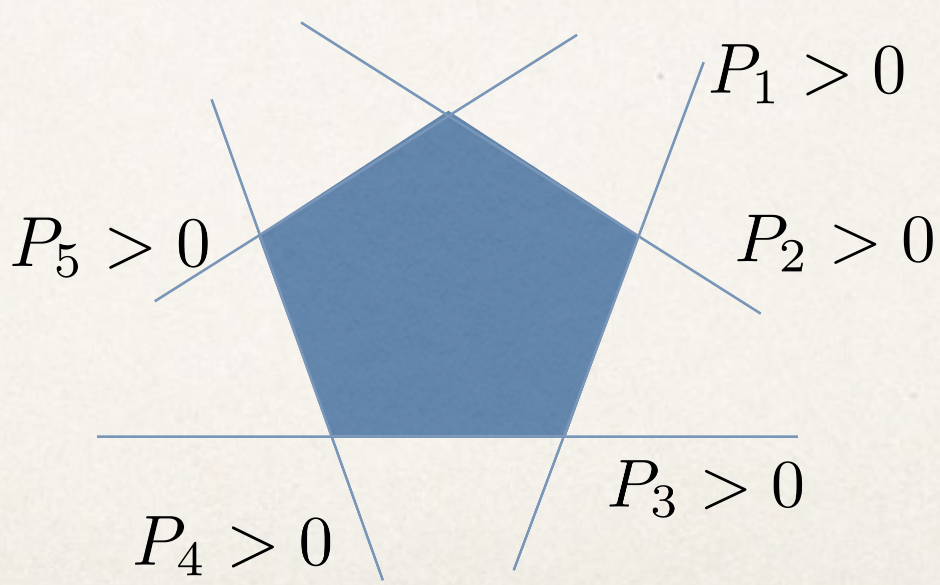
$$(p_i, \epsilon_j, \ell_k) \to (x_i, \tilde{\eta}_j, y_k) \to (Z_i, \eta_j, Z^{(k)}_{AB}) \to z_i$$

- The definition of Amplituhedron: inequalities

$$P_j(z_i) \geq 0$$

- Boundaries of the space

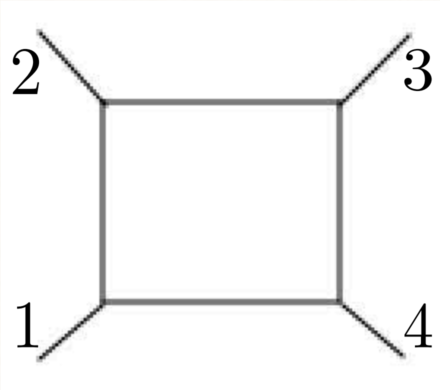
$$P_j(z_i) = 0$$



Legal and illegal boundaries

- Singularities and cuts of the amplitude: localize z_i
- Inequalities hold $P_j(z_i) \geq 0$ $\ell_k \in \mathbb{C} \leftrightarrow z_i > 0$
  - Point inside the Amplituhedron space
  - Physical cut or singularity of the amplitude
- One or more inequalities violated $P_j(z_i) < 0$
  - Point outside the Amplituhedron space
  - Unphysical cut or singularity of the amplitude

Example 1: One-loop amplitude

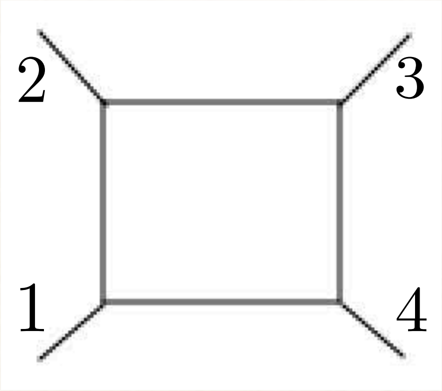


- Consider 4pt one-loop amplitude
- Inequalities: $z_1, z_2, z_3, z_4 \geq 0$
- Boundaries of the space: $z_1, z_2, z_3, z_4 = (0, \infty)$
- Differential form

$$\Omega = \frac{dz_1}{z_1}\frac{dz_2}{z_2}\frac{dz_3}{z_3}\frac{dz_4}{z_4}$$

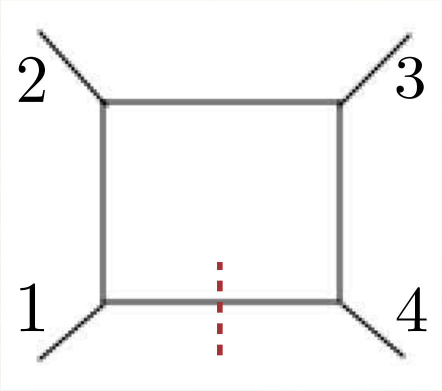
Example 1: One-loop amplitude

- Cuts of the amplitude



Example 1: One-loop amplitude

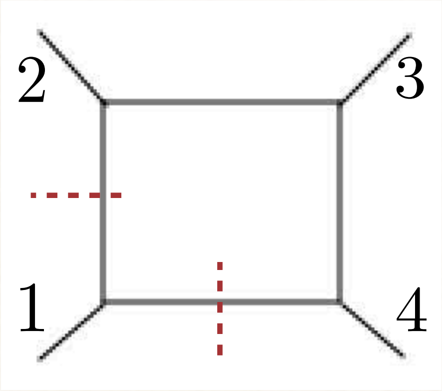
- Cuts of the amplitude



$$z_1 = 0$$

Example 1: One-loop amplitude

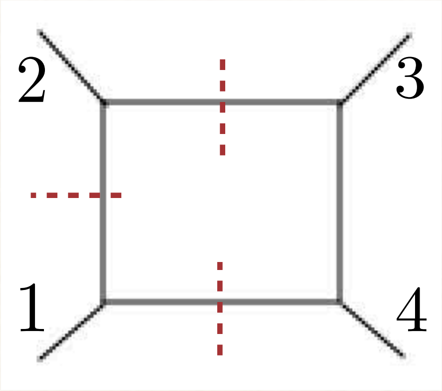
- Cuts of the amplitude



$$z_1 = 0$$
$$z_2 = 0$$

Example 1: One-loop amplitude

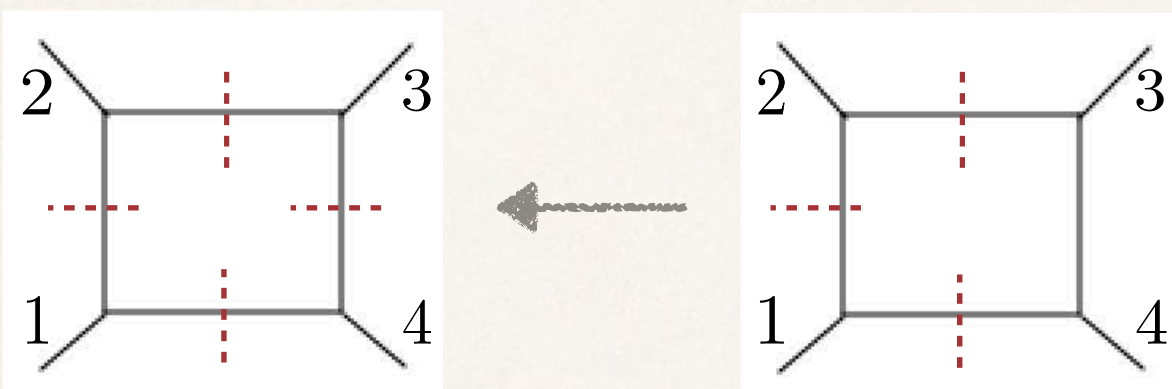
- Cuts of the amplitude



$$z_1 = 0$$
$$z_2 = 0$$
$$z_3 = 0$$

Example 1: One-loop amplitude

- Cuts of the amplitude



$z_4 = 0$

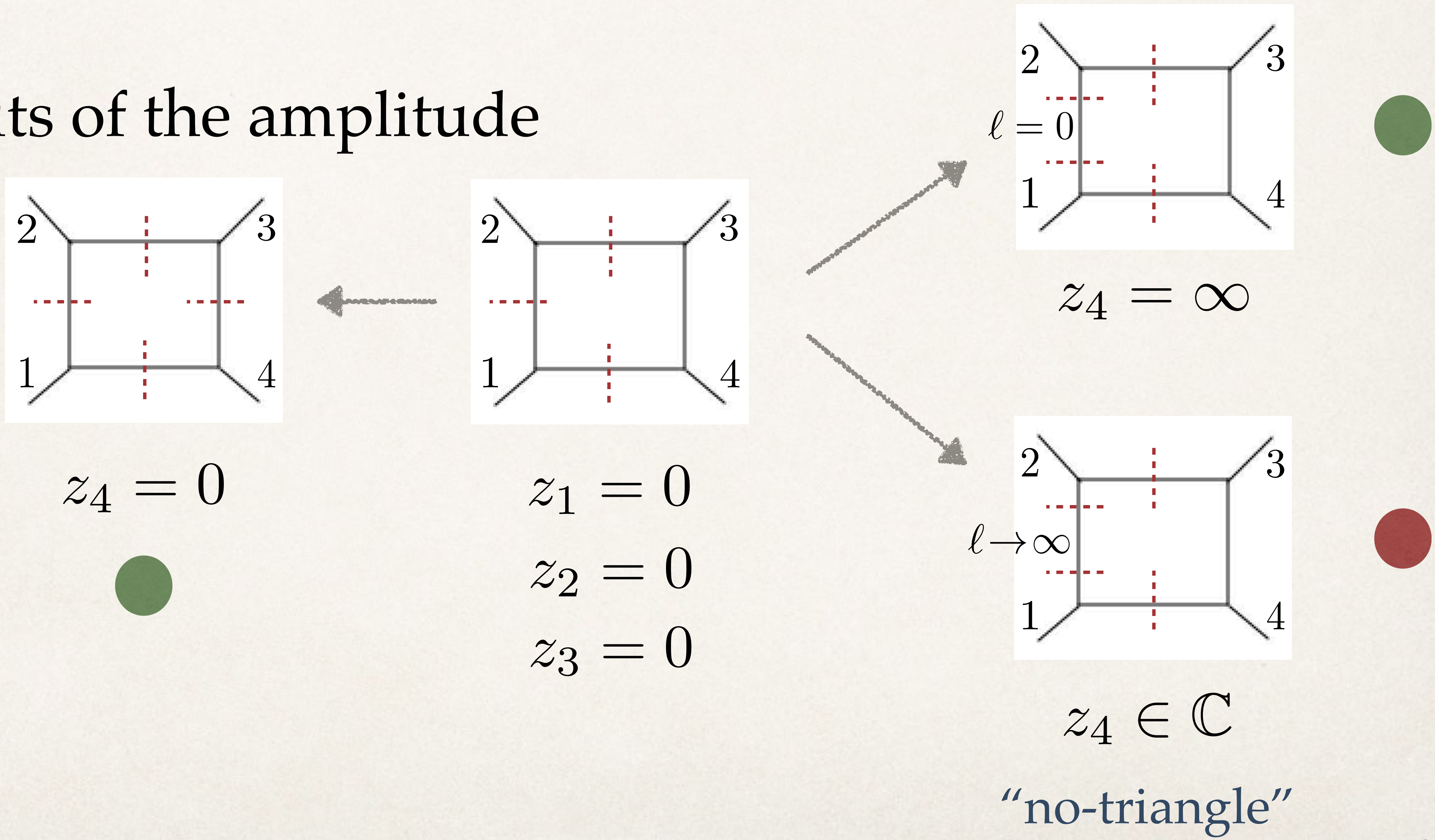
$z_1 = 0$

$z_2 = 0$

$z_3 = 0$

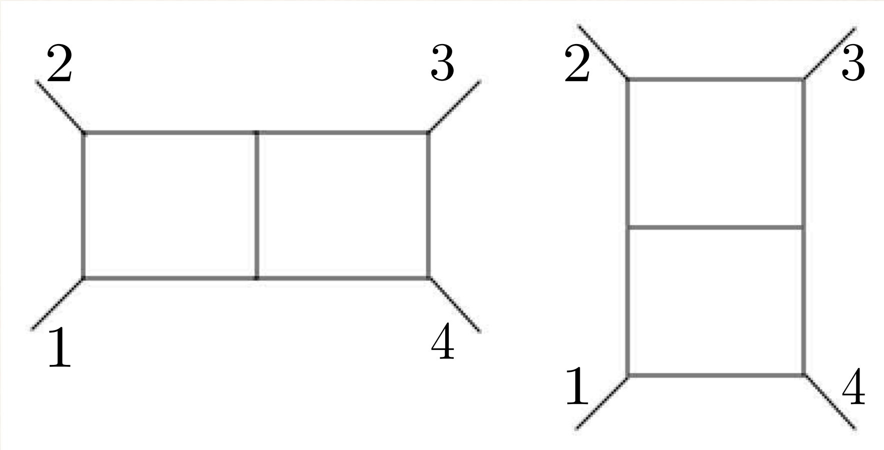
Example 1: One-loop amplitude

- Cuts of the amplitude



Example 2: Two-loop amplitude

- Consider 4pt two-loop amplitude

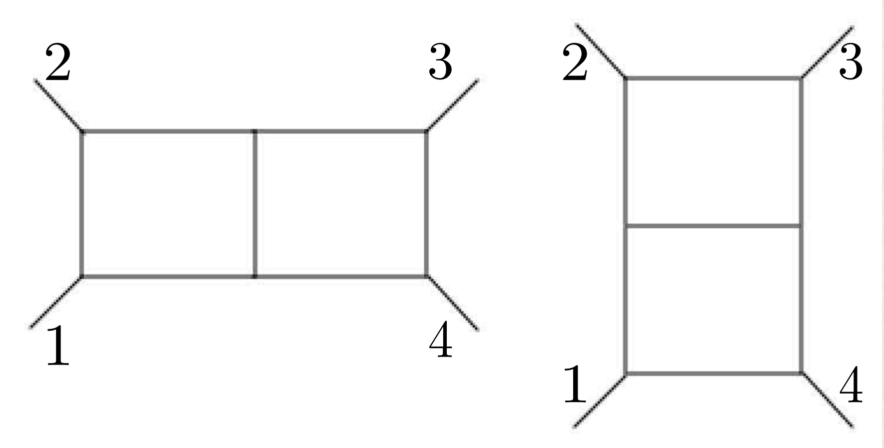


- Inequalities: $$z_1, z_2, z_3, z_4 \geq 0$$ $$z_5, z_6, z_7, z_8 \geq 0$$

$$(z_1 - z_5)(z_6 - z_2) + (z_3 - z_7)(z_8 - z_4) \geq 0$$

Example 2: Two-loop amplitude

- Consider 4pt two-loop amplitude

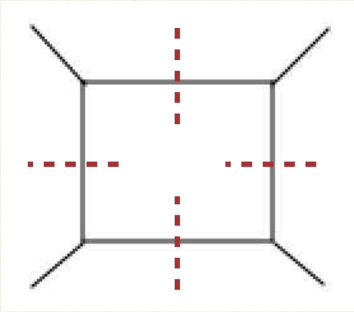


- Inequalities: $z_1, z_2, z_3, z_4 \geq 0$

$$z_5, z_6, z_7, z_8 \geq 0$$

$$(z_1 - z_5)(z_6 - z_2) + (z_3 - z_7)(z_8 - z_4) \geq 0$$

- Check: one-loop cut



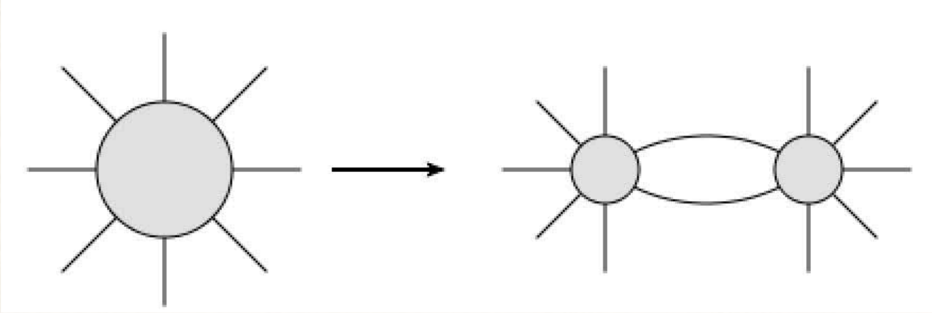
$$z_1 = 0$$
$$z_2 = 0$$
$$z_3 = 0$$
$$z_4 = 0$$

$$-z_5 z_6 - z_7 z_8 \geq 0$$

Ω vanishes on this cut

Example 3: Unitarity cut

- Standard formulation



$$\text{Cut}\, M_{n,\ell} = \sum_{\ell_1+\ell_2=\ell-1} M_{n_1,\ell_1} M_{n_2,\ell_2}$$

- Set of inequalities split into two sets

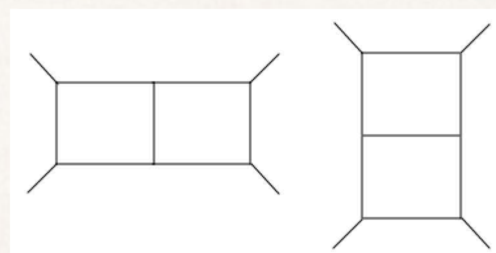
$$P_j(z_i) \geq 0 \quad i = 1, \ldots, m \quad \xrightarrow{z_1 = z_2 = 0} \quad \begin{array}{ll} P_j^{(1)}(z_i) \geq 0 & i = 3, \ldots, k \\ P_j^{(2)}(z_i) \geq 0 & i = k+1, \ldots, m \end{array}$$

where k is a free parameter

Physics vs geometry

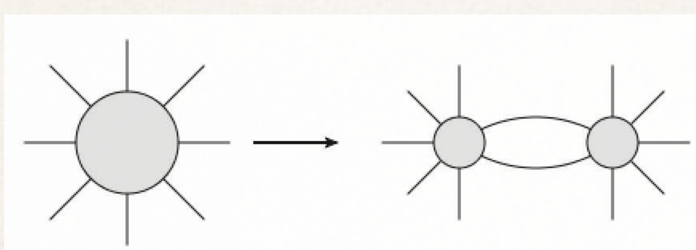
Standard methods

- Planar diagrams



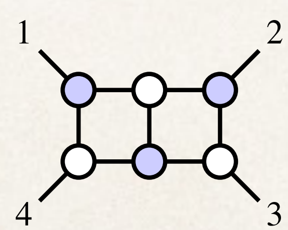
Locality + Planarity

- Match physical cuts/singularities



Unitarity

$$\text{Cut}(I) =$$



Construction not known in general

Amplituhedron

- Inequalities

$$P_j(z_i) \geq 0$$

- Logarithmic form

$$\Omega \sim \frac{dx}{x}$$

<u>Complete set known</u>

Matching zeroes

(Arkani-Hamed, Hodges, JT 2014)

- Inequalities $P_j(z_i) \geq 0$ $\rightarrow$ Unique form Ω
- Smaller set of information fixes the form

up to an overall constant

$$\Omega = \frac{N(z_i)}{D(z_i)}$$

Physical poles

Fixed by zeroes $N(z_i) = 0$

- Points outside the Amplituhedron
- Points inside with multiple poles

- Checked explicitly for several examples

Summary of the planar part

- Amplituhedron $P_j(z_i) \geq 0$
  - Points on boundaries — physical poles
  - Points outside — unphysical poles

Homogeneous problem

- Differential form Ω

Standard:	Conjecture:
Physical, logarithmic poles $\sim \frac{dx}{x}$	Yes
No poles outside Amplituhedron	Yes
Residues on the inside: reduced inequalities	Redundant

Generalizations

- Next step:
  - Stay planar, go beyond $\mathcal{N} = 4$ SYM
  - Go to non-planar $\mathcal{N} = 4$ SYM

Generalizations

- Next step:
  - Stay planar, go beyond $\mathcal{N} = 4$ SYM
  - Go to non-planar $\mathcal{N} = 4$ SYM

In this talk

Non-planar amplitudes

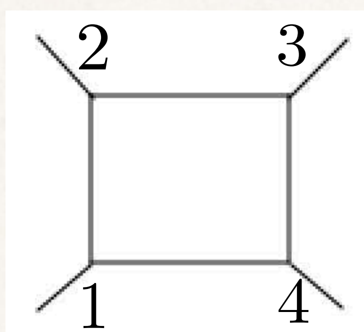
(Arkani-Hamed, Bourjaily, Cachazo, JT 2014)

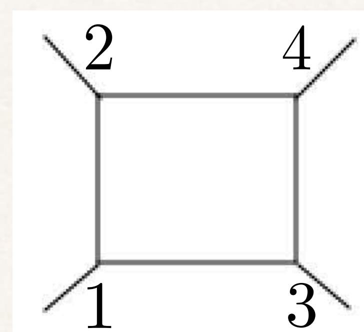
(Bern, Herrmann, Litsey, Stankowicz, JT 2014 + in progress)

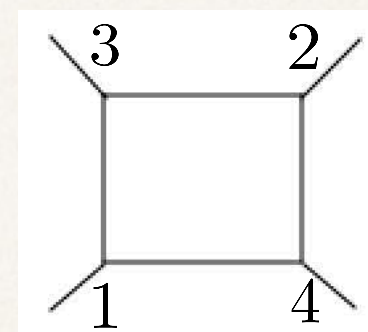
No global variables

- Absence of global variables

What is ℓ ?







- We can not guess/test inequalities immediately

- Check implications for the amplitude form Ω

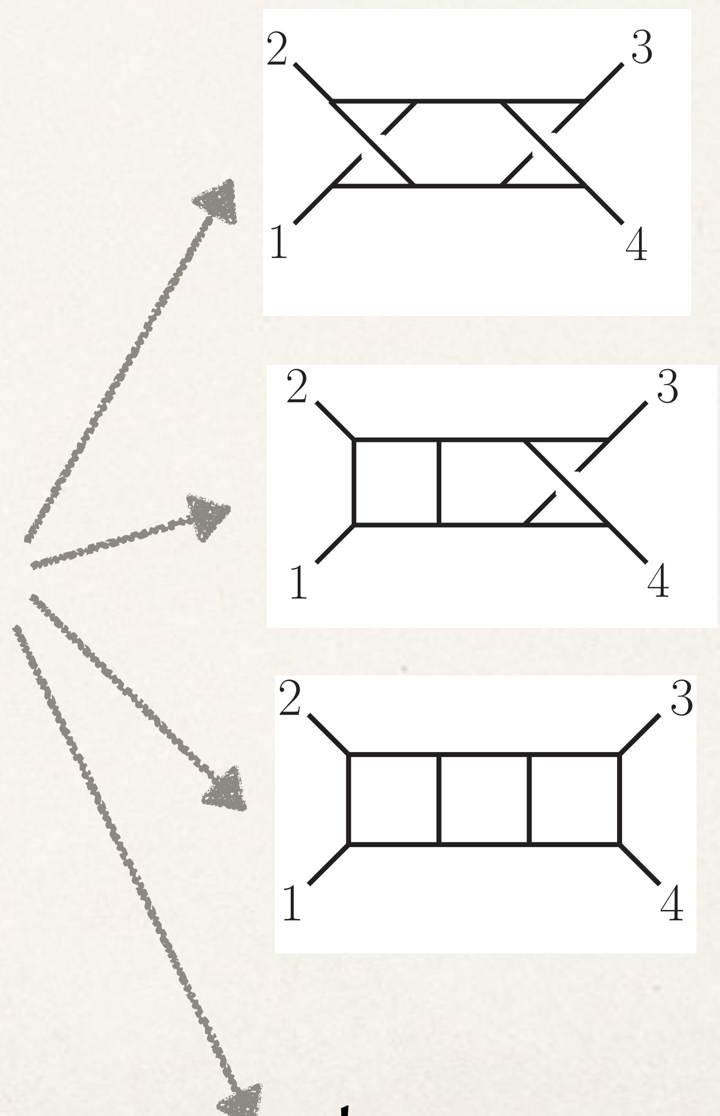
Non-planar form

- Use of standard momenta k_i, ℓ_k
- No single form, sum of diagrams

$$\Omega = \sum_{\sigma, j} C_j \cdot \Omega_j(k_i, \ell_k)$$

C_j color factor

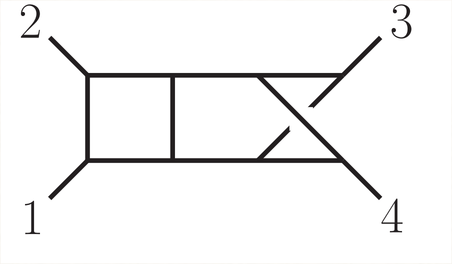
- Each has its own variables



etc

Constraints

- Inspired by the planar sector we conjecture:
  - Logarithmic singularities $\Omega \sim \frac{dx}{x}$
  - No poles at $\ell \to \infty$
- Stronger condition: each diagram individually



$$I_j(k_i, \ell_k) = \frac{N_j(k_i, \ell_k)}{P_1^2 P_2^2 \dots P_m^2}$$

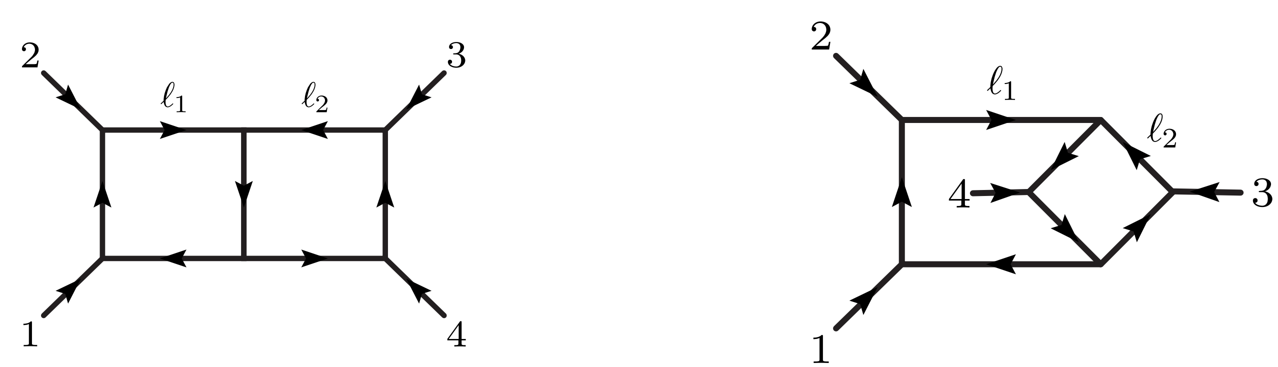
- Find the basis and expand the amplitude

Evidence 1: Two-loop amplitude

- Expansion of the 4pt two-loop amplitude

(Bern, Rozowsky, Yan 1997)

Two basis integrals



$$N_1 = (k_1 + k_2)^2 \qquad N_2 = (k_1 + k_2)^2$$

Double Poles

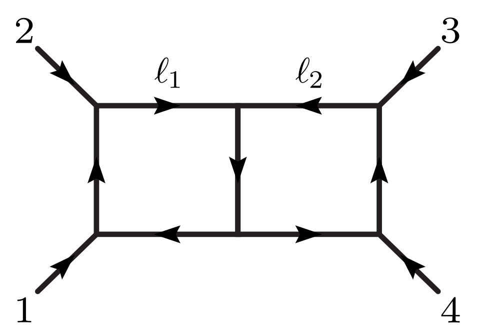
Poles at infinity

Evidence 1: Two-loop amplitude

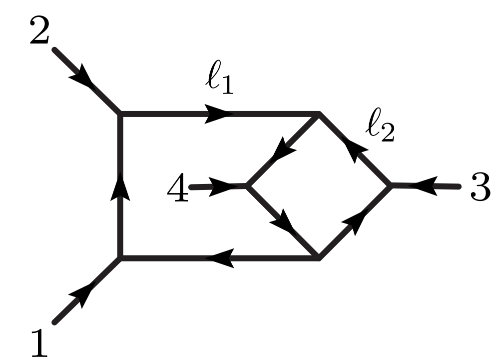
- Expansion of the 4pt two-loop amplitude

(Bern, Rozowsky, Yan 1997)

Two basis integrals



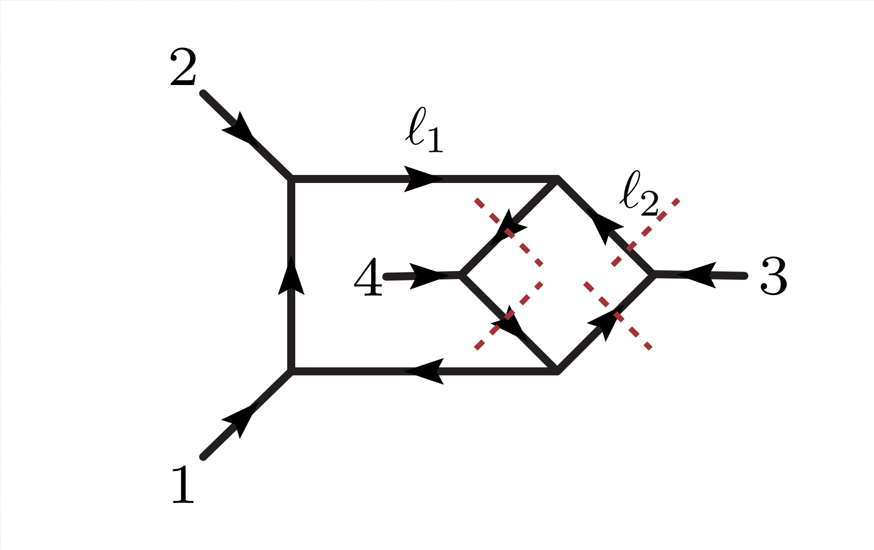
$$N_1 = (k_1 + k_2)^2$$



$$N_2 = (k_1 + k_2)^2$$

Double Poles	NO	YES
Poles at infinity	NO	YES

Evidence 1: Two-loop amplitude

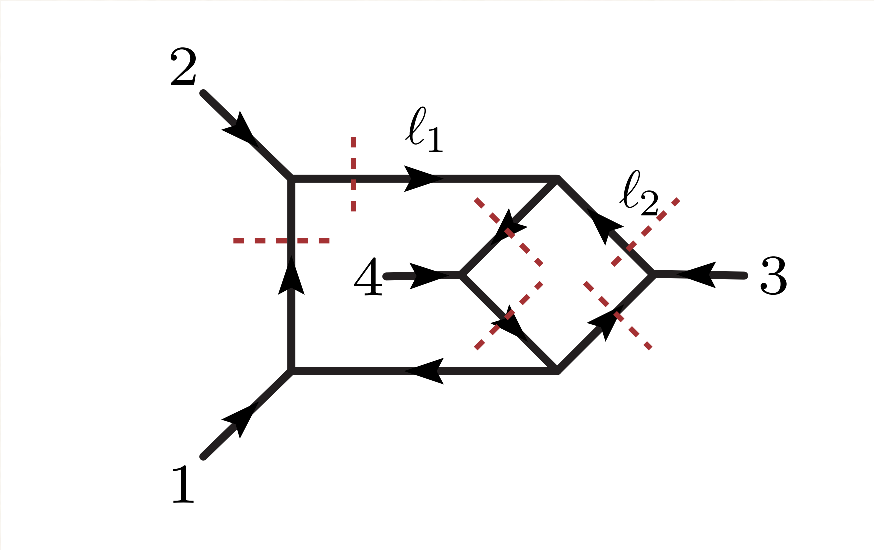


$$dI = \frac{d^4\ell_1\, d^4\ell_2\ (p_1+p_2)^2}{\ell_1^2(\ell_1-k_2)^2(\ell_1-k_1-k_2)^2\ell_2^2(\ell_2-k_3)^2(\ell_1+\ell_2)^2(\ell_1+\ell_2+k_4)^2}$$

Perform cuts $\quad \ell_2^2 = (\ell_2-k_3)^2 = (\ell_1+\ell_2)^2 = (\ell_1+\ell_2+k_4)^2 = 0$

Localize ℓ_2 completely

Evidence 1: Two-loop amplitude

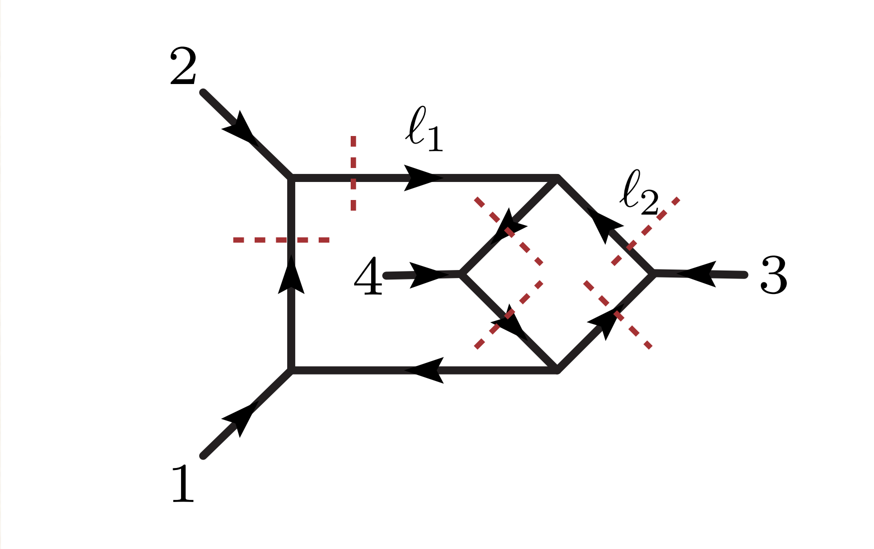


$$\mathrm{Cut}_1\, dI = \frac{d^4\ell_1}{\ell_1^2(\ell_1-k_2)^2(\ell_1-k_1-k_2)^2[(\ell_1+k_3)^2(\ell_1+k_4)^2-\ell_1^2(\ell_1+k_3+k_4)^2]}$$

Localize $\ell_1 = \alpha k_2$ by cutting $\ell_1^2 = (\ell_1 - k_2)^2 = 0$

and the Jacobian

Evidence 1: Two-loop amplitude

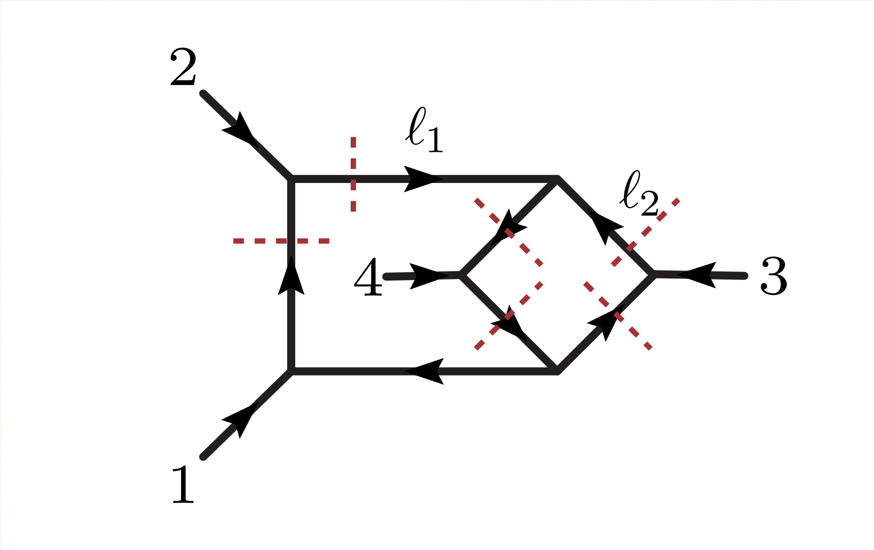


$$\text{Cut}_{1,2}\, dI = \frac{d\alpha}{(\alpha+1)\alpha^2 tu}$$

Double pole for $\alpha = 0$

- There is also pole at infinity
- We want to find a numerator which cancels all that

Evidence 1: Two-loop amplitude



$$\mathrm{Cut}_{1,2}\, dI = \frac{d\alpha}{(\alpha+1)\alpha^2 tu}$$

Double pole for $\alpha = 0$

New numerator

$$N = (\ell_1 + k_3)^2 + (\ell_1 + k_4)^2$$

Cancels double pole

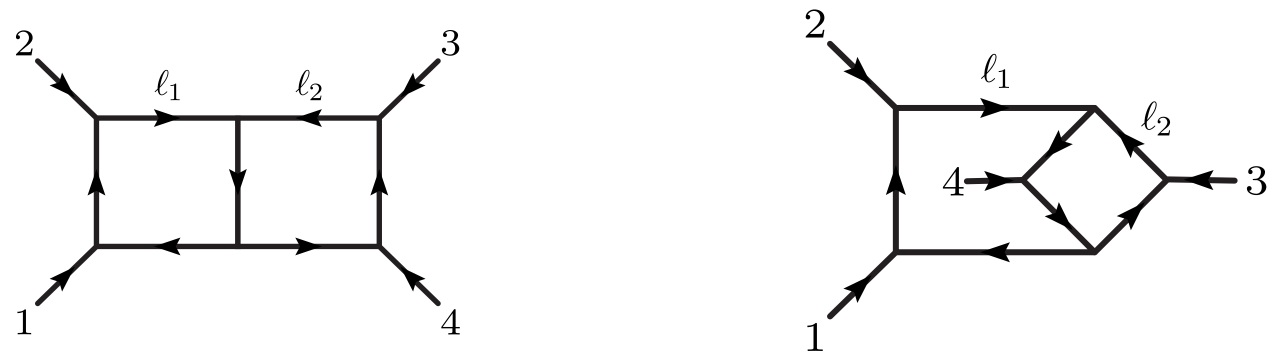
$$N \to \alpha s$$

Evidence 1: Two-loop amplitude

- New expansion of the 4pt two-loop amplitude

(Arkani-Hamed, Bourjaily, Cachazo, JT, 2014)

Two basis integrals



$N_1 = (k_1 + k_2)^2$ $\qquad$ $N_2 = (\ell_1 + k_3)^2 + (\ell_1 + k_4)^2$

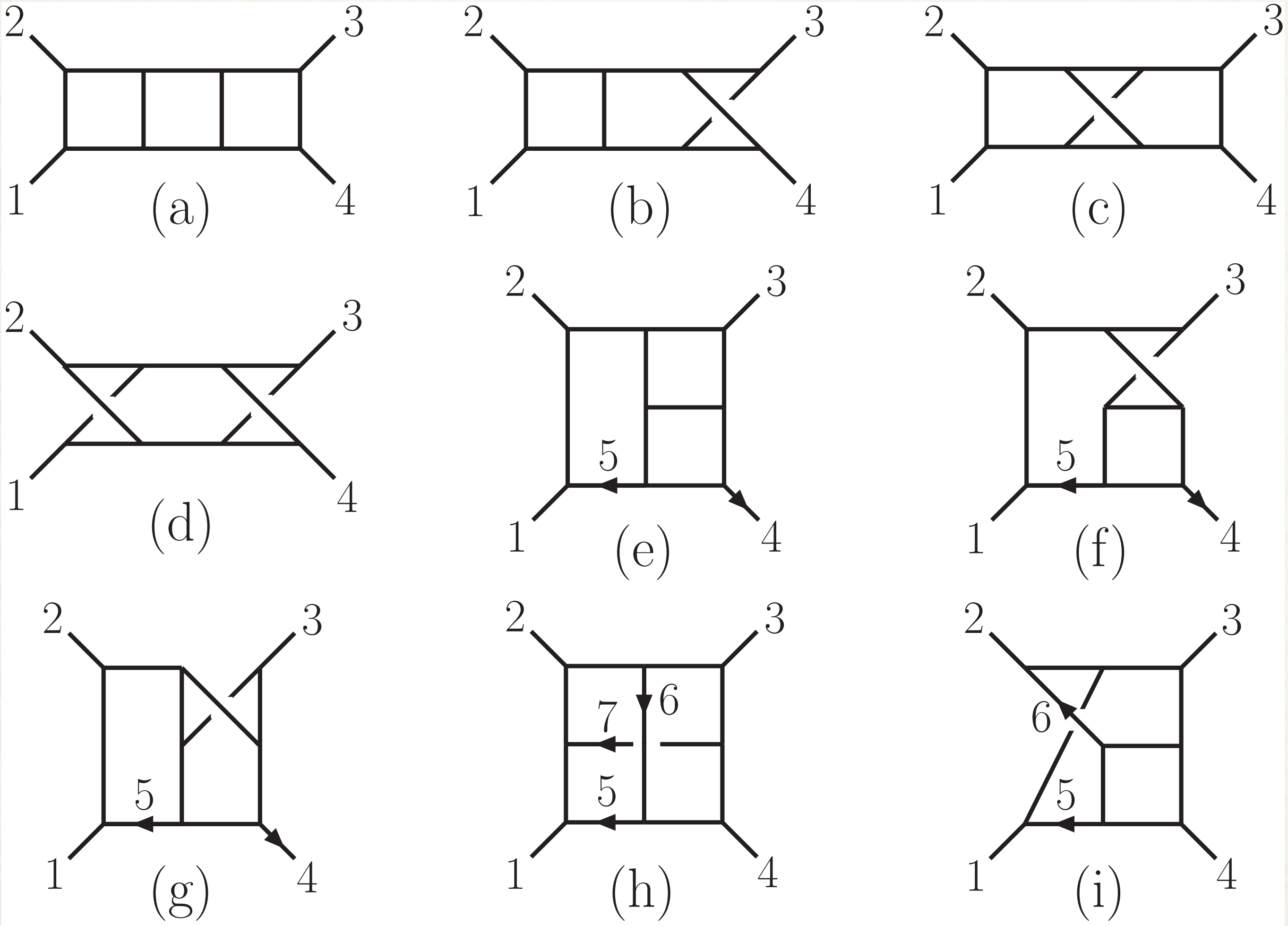
Double Poles	NO	NO
Poles at infinity	NO	NO

<u>Expand amplitude in the basis:</u> YES

Evidence 2: Three-loop amplitude

- Basis for three-loop four point amplitude

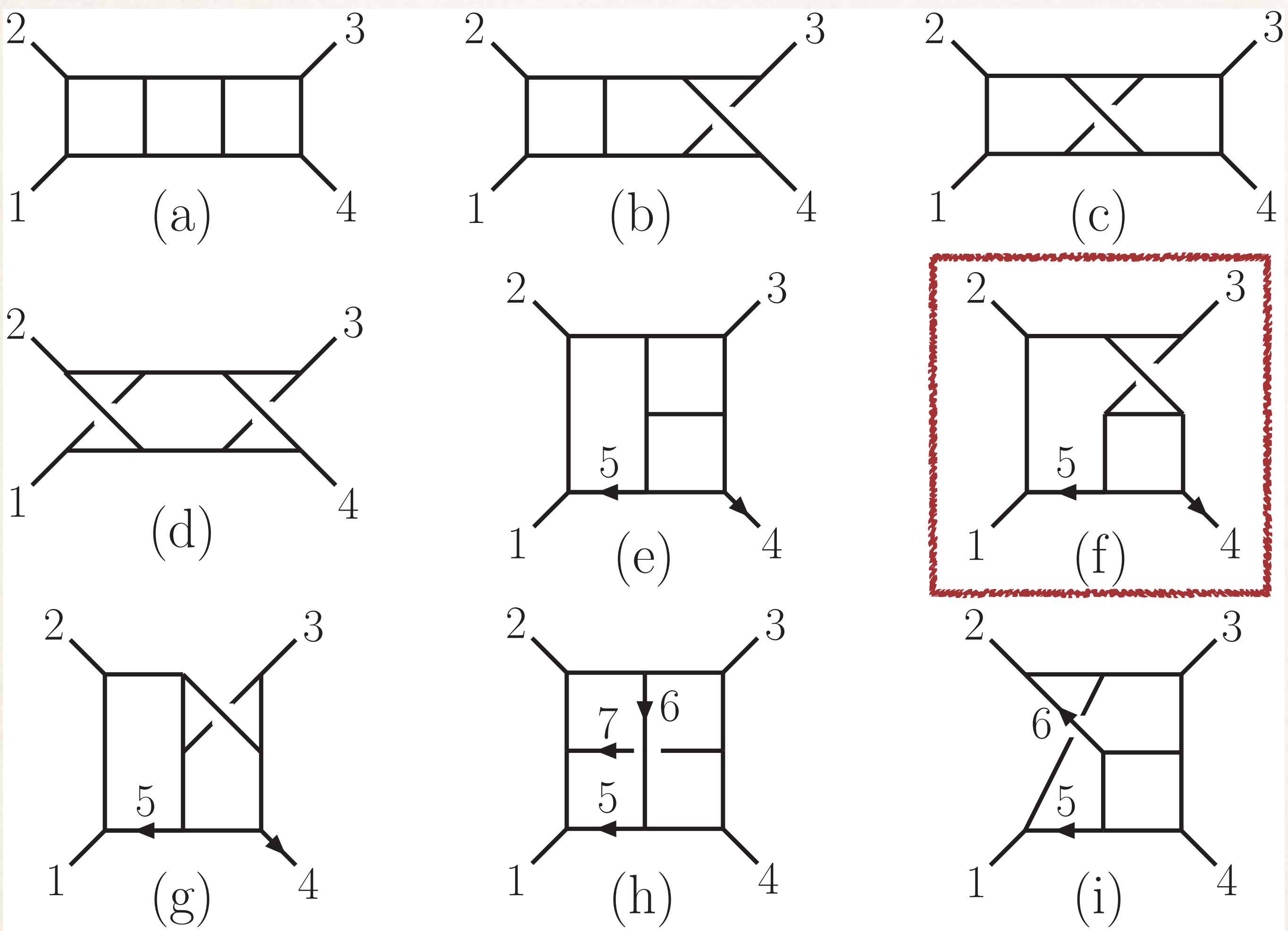
(Bern, Carrasco, Dixon, Johansson, Kosower 2007)



Numerator	Double pole	Pole at infinity
Original	YES	YES
BCJ	YES	YES

Evidence 2: Three-loop amplitude

- Basis for three-loop four point amplitude

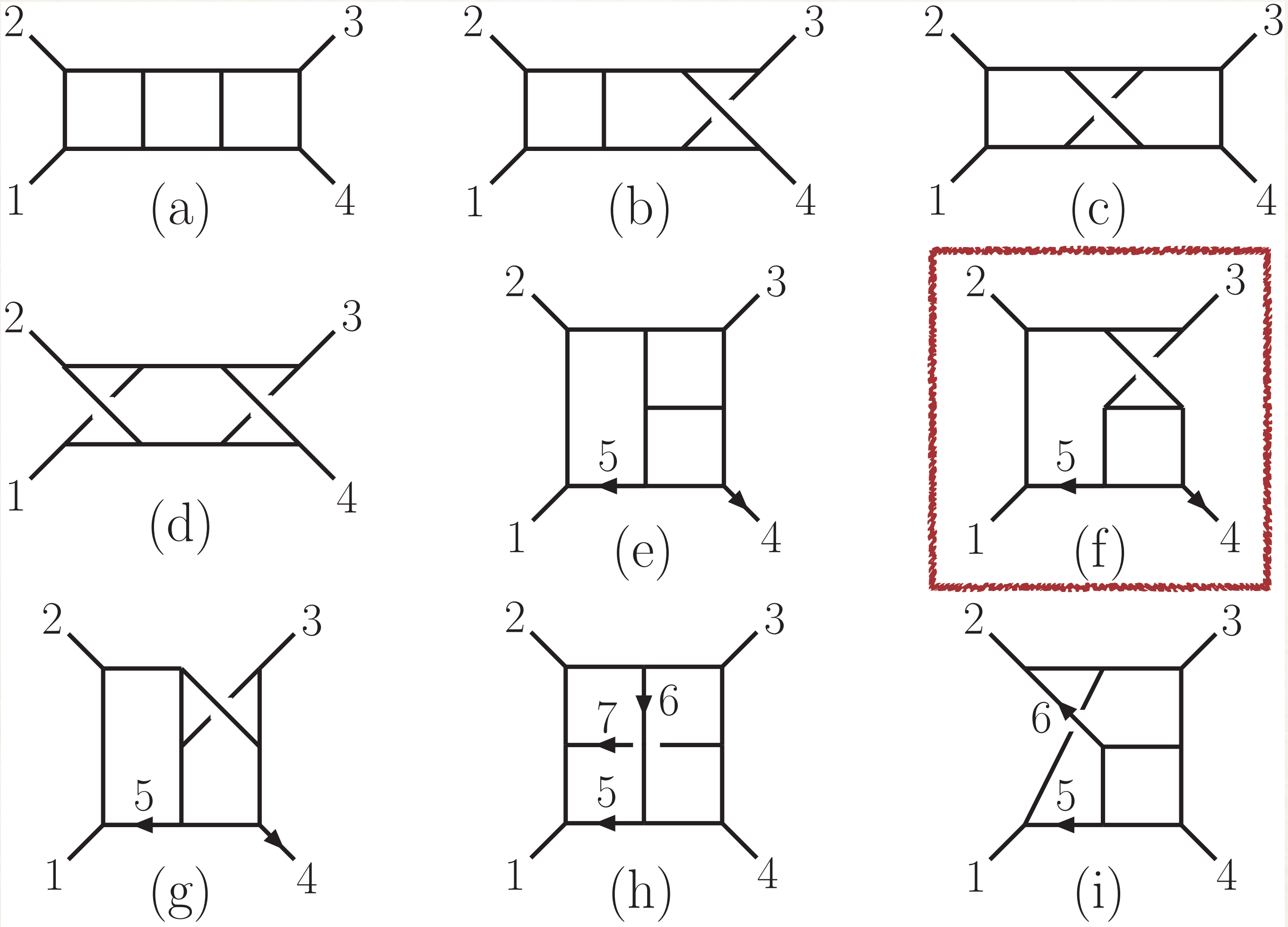


Old numerator

$$N = (\ell_5 + k_4)^2 (k_1 + k_2)^2$$

Evidence 2: Three-loop amplitude

- Basis for three-loop four point amplitude



Old numerator

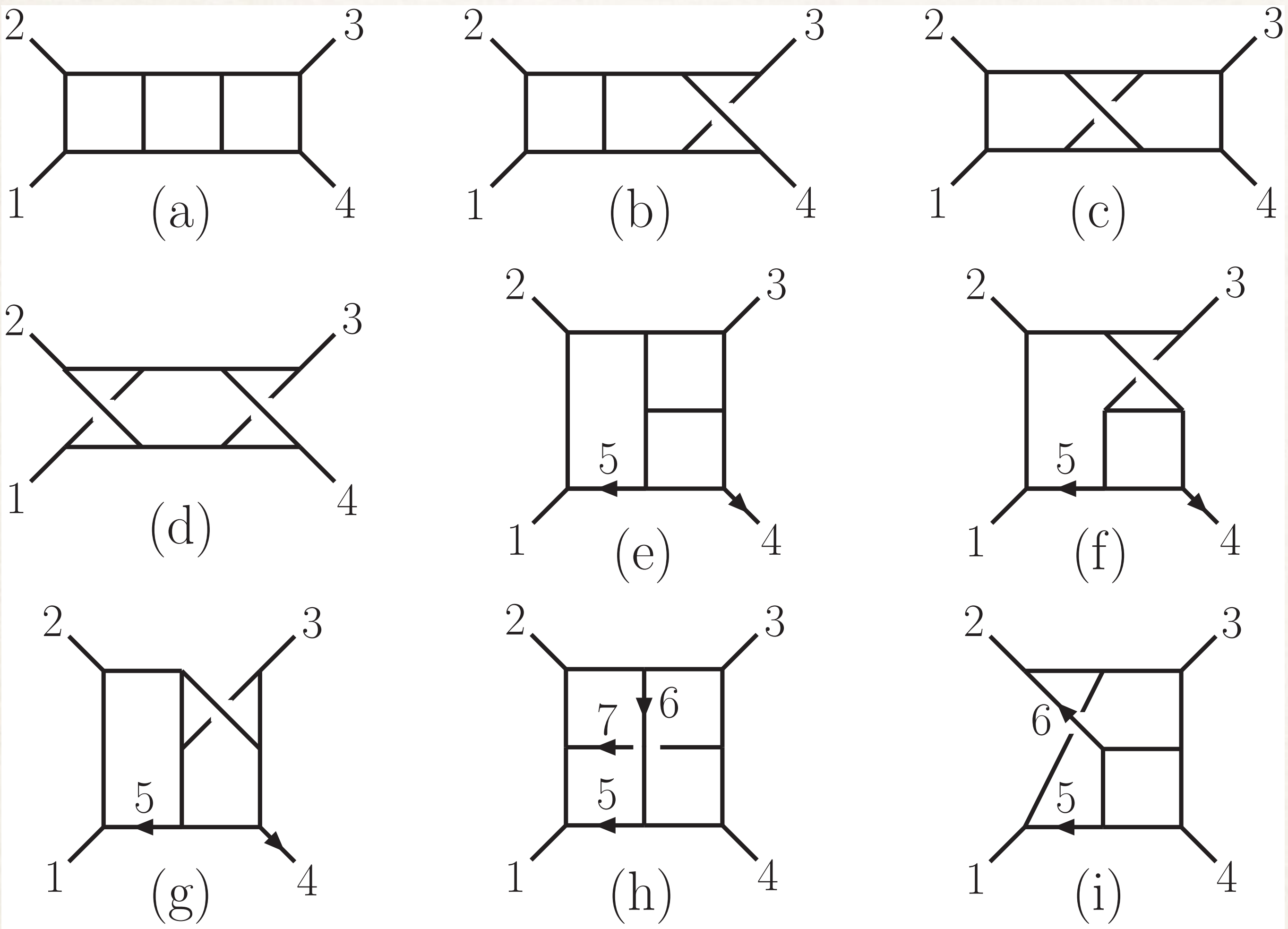
$$N = (\ell_5 + k_4)^2 \boxed{(k_1 + k_2)^2}$$

New numerator

$$N = (\ell_5 + k_4)^2 \boxed{[(\ell_5 + k_3)^2 + (\ell_5 + k_4)^2]}$$

Evidence 2: Three-loop amplitude

- Basis for three-loop four point amplitude



(Bern, Herrmann, Litsey, Stankowicz, JT 2014)

Numerator	Double pole	Pole at infinity
Original	YES	YES
BCJ	YES	YES
New	NO	NO

Expansion of the amplitude: YES

Fixing coefficients

- Standard approach:
  - Unitarity cut
  - Maximal cut
  - Leading singularity

  Non-zero RHS

  $$\mathrm{Cut}(I) = \ldots$$

- Proposal:

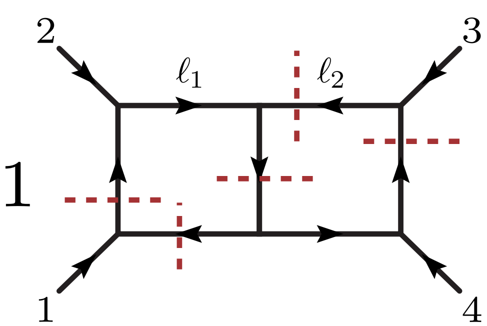
  Illegal cuts $\mathrm{Cut}(I) = 0$ fix uniquely result!

  (up to an overall constant)

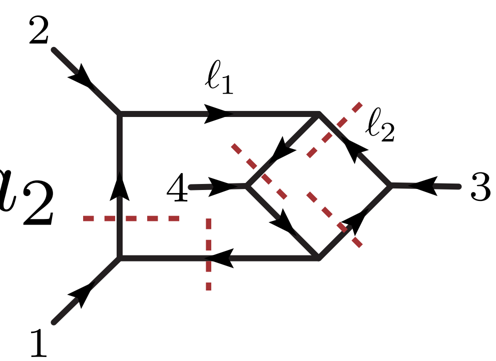
Explicit check

- Two-loop amplitude

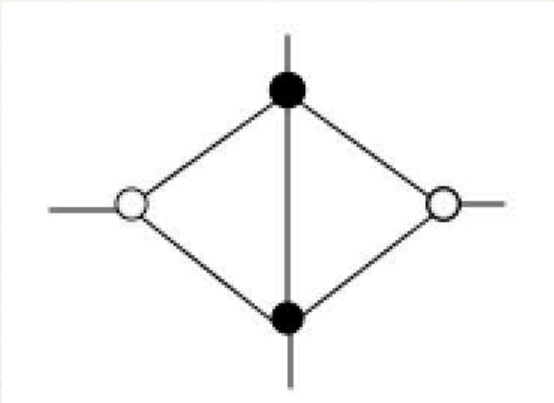
$$M_2 = \sum_{\sigma} \quad a_1$$



$$+ \quad a_2$$



Illegal 5-cut



$k = 1$

Fixes relative coefficient

$$a_1 = a_2$$

Non-planar summary

- Absence of global variables: no inequalities yet
- Test of implications:
    - Logarithmic form
    - No poles at infinity
    - Diagrams + Zeroes fix the answer

<u>Homogeneous conditions</u>

Outlook

- Final conjecture:
  - Logarithmic singularities
  - Fixed by zeroes

Amplitudes in $\mathcal{N} = 4$ SYM are fixed by homogeneous conditions

- Future directions:
  - Search for global variables
  - Inequalities and geometric interpretation
  - Exploring $\mathcal{N} = 8$ SUGRA and $\mathcal{N} < 4$ SYM

Thank you for your attention