$D^6\,\mathcal{R}^4$

Pierre Vanhove

Eurostring DAMTP
March 26th, 2015
1001.2535 1004.0163 1111.2983 1404.2192
Michael B. Green, Stephen D. Miller, Jorge Russo



Low-energy string effective action

The low-energy limit of type II superstring on tori T^d is given by

$$\mathcal{L}^D = \frac{1}{\ell_D^{D-2}}\Big(\mathcal{R} + \ell_D^6 \mathcal{E}_{(0,0)}(\varphi)\mathcal{R}^4 + \ell_D^{10}\mathcal{E}_{(1,0)}(\varphi)D^4\mathcal{R}^4 + \ell_D^{12}\mathcal{E}_{(0,1)}(\varphi)D^6\mathcal{R}^4 + \cdots\Big)$$

- ℓ_D is the Planck length in D dimensions
- $\mathcal{E}_{(p,q)}(\varphi)$ with $\varphi \in G_D(\mathbb{R})/K_D(\mathbb{R})$ the G_D the duality group
- From the analytic piece of the four-graviton amplitude S-matrix

$$\mathcal{S}_4^{an,D} = \sum_{p,q} \mathcal{E}^D_{(p,q)}(\varphi)\,\ell_D^{2p+3q}\,\partial^{2p+3q}\mathcal{R}^4$$

- They are constrained by supersymmetry and the discrete duality symmetries of the theory

String theory and Quantum field theory

These string theory interactions are strongly constraints by maximal supersymmetries and duality symmetries

We can ask about what can we learn about

- black-hole contributions from non-perturbative sector
- the ultraviolet behaviour of the low-energy limit : $\mathcal{N} = 8$ supergravity

String theory and Quantum field theory

The non-perturbative modular/automorphic symmetry are fundamental symmetries

There are five elementary arithmetical operations: addition, subtraction, multiplication, division, and … modular forms.

Constraint from Supersymmetry

- On-shell supersymmetry invariance of the effective action imposes

$$\mathcal{L}^D = \frac{1}{\ell_D^{D-2}}\Big(\mathcal{R} + \ell_D^6 \mathcal{E}_{(0,0)}(\varphi)\mathcal{R}^4 + \ell_D^{10}\mathcal{E}_{(1,0)}(\varphi)D^4\mathcal{R}^4 + \ell_D^{12}\mathcal{E}_{(0,1)}(\varphi)D^6\mathcal{R}^4 + \cdots\Big)$$

$$\left(\frac{\delta_{(0,-1)}}{\ell_D{}^6} + \sum_{2p+3q\geqslant 0} \ell_D{}^{2(2p+3q)}\delta_{(p,q)}\right)\left(\frac{\mathcal{L}_{(0,-1)}}{\ell_D{}^6} + \sum_{2p+3q\geqslant 0} \ell_D{}^{2(2p+3q)}\mathcal{L}_{(p,q)}\right) = 0$$

Constraint from Supersymmetry

- On-shell supersymmetry invariance of the effective action imposes

$$\mathcal{L}^D = \frac{1}{\ell_D^{D-2}}\Big(\mathcal{R} + \ell_D^6 \mathcal{E}_{(0,0)}(\varphi)\mathcal{R}^4 + \ell_D^{10}\mathcal{E}_{(1,0)}(\varphi)D^4\mathcal{R}^4 + \ell_D^{12}\mathcal{E}_{(0,1)}(\varphi)D^6\mathcal{R}^4 + \cdots\Big)$$

$$\begin{aligned}
\delta_{(0,-1)}\mathcal{L}_{(0,-1)} &= 0 && \text{Einstein-Hilbert action}\\
\delta_{(0,-1)}\mathcal{L}_{(0,0)} + \delta_{(0,0)}\mathcal{L}_{(0,-1)} &= 0 && \mathcal{R}^4 \text{ term}\\
\delta_{(0,-1)}\mathcal{L}_{(1,0)} + \delta_{(1,0)}S_{(0,-1)} &= 0 && D^4\mathcal{R}^4 \text{ term}\\
\delta_{(0,-1)}\mathcal{L}_{(0,1)} + \delta_{(0,0)}\mathcal{L}_{(0,0)} + \delta_{(0,1)}\mathcal{L}_{(0,-1)} &= 0 && D^6\mathcal{R}^4 \text{ term}
\end{aligned}$$

Constraint from Supersymmetry

- In $D = 10$ for type IIB $\varphi = \Omega \in SL(2,\mathbb{R})/SO(2)$
- It was shown in [Green, Sethi;Sinha] that this implies

$$
\begin{aligned}
(\Delta^{(10)} - \frac{3}{4})\, \mathcal{E}^{(10)}_{(0,0)} &= 0; \qquad \mathcal{R}^4 \text{ interaction}\\
(\Delta^{(10)} - \frac{5}{4})\, \mathcal{E}^{(10)}_{(1,0)} &= 0; \qquad D^4\mathcal{R}^4 \text{ interaction}
\end{aligned}
$$

$\Delta^{(10)} = \Omega_2^2\,(\partial^2_{\Omega_1} + \partial^2_{\Omega_2})$ is the invariant Laplacian for $SO(2)\backslash SL(2,\mathbb{R})$

Constraint from Supersymmetry

From the properties of 2-loop amplitude in $D = 11$ was derived that the $D^6\mathcal{R}^4$ satisfies [Green, Vanhove]

$$(\Delta^{(10)} - 12)\, \mathcal{E}^{(10)}_{(0,1)} = -(\mathcal{E}^{(10)}_{(0,0)})^2$$

This conjecture that has received various confirmations over the years

- Matching the perturbative string expansion up to and including genus 3 ([see Mafra's talk])
- Fitting expectations from supersymmetry (non-renormalisation theorems)

Dimensional reduction

Consider a one-loop QFT amplitude on a spacetime $\mathcal{M}^{1,D-1} \times S^1(r_d)$

$$\mathcal{A}_4^{1-\text{loop}} = \frac{1}{r_d} \sum_{n\in\mathbb{Z}} \int \frac{d^D \ell}{\prod_{i=1}^4 ((\ell + p_1 + \cdots p_i)^2 + \frac{n^2}{r_d^2})}$$

- For $r_d \to \infty$ we have a massless amplitude in $D+1$ dimensions
- For finite values of r_d the thresholds are shifted to $p_i \cdot p_j \sim n^2/r_d^2$

Dimensional reduction

The analytic structure of the massless thresholds in higher dimensions by an infinite resumation of the Kaluza-Klein mode

$$\frac{1}{r_d}\sum_{n\geqslant 0}\left(\frac{n^2}{r_d^2}-s\right)^{1-\frac{d}{2}} = \frac{1}{r_d}\Big((-s)^{1-\frac{d}{2}}+\sum_{k\geqslant 0} c(k)\,\zeta(2k+d-2)\,\frac{(r_d^2 s)^k}{r_d^{2-d}}\Big)$$

For $d=2$ we have $\log$ thresholds

$$\frac{1}{r_d}\sum_{n\geqslant 0}\log\left(\frac{n^2}{r_d^2}-s\right) = \frac{1}{r_d}\Big(\log(-sr_d^2)+\sum_{k\geqslant 1}\zeta(2k)\,\frac{1}{k}\,(r_d^2 s)^k\Big)$$

This has important consequences for the consistency of the contributions to the string effective action

Dimensional reduction

- The higher derivative couplings to the type II string effective action are not really independent of each others [Green, Russo, Vanhove]
- They are connected when performing a decompactification $10 = D + d$
  $T^d = \lim_{r_d/\ell_D \to \infty} T^{d-1} \times S^1(r_d)$

$$\left(\frac{\ell_D}{\ell_{D+1}}\right)^{8-D} \mathcal{E}^{(D)}_{(0,0)} \to \frac{r_d}{\ell_{D+1}} \mathcal{E}^{(D+1)}_{(0,0)} + \left(\frac{r_d}{\ell_{D+1}}\right)^{8-D}$$

Dimensional reduction

- The higher derivative couplings to the type II string effective action are not really independent of each others [Green, Russo, Vanhove]

- They are connected when performing a decompactification $10 = D + d$ $T^d = \lim_{r_d/\ell_D\to\infty} T^{d-1} \times S^1(r_d)$

$$\left(\frac{\ell_D}{\ell_{D+1}}\right)^{12-D} \mathcal{E}^{(D)}_{(1,0)} \to \frac{r_d}{\ell_{D+1}}\mathcal{E}^{(D+1)}_{(1,0)} + \left(\frac{r_d}{\ell_{D+1}}\right)^{6-D} \mathcal{E}^{(D+1)}_{(0,0)} + \left(\frac{r_d}{\ell_{D+1}}\right)^{12-D}$$

Dimensional reduction

- The higher derivative couplings to the type II string effective action are not really independent of each others [Green, Russo, Vanhove]
- They are connected when performing a decompactification $10 = D + d$ $T^d = \lim_{r_d/\ell_D \to \infty} T^{d-1} \times S^1(r_d)$

$$\left(\frac{\ell_D}{\ell_{D+1}}\right)^{14-D} \mathcal{E}^{(D)}_{(0,1)} \to \frac{r_d}{\ell_{D+1}} \mathcal{E}^{(D+1)}_{(0,1)} + \left(\frac{r_d}{\ell_{D+1}}\right)^{4-D} \mathcal{E}^{(D+1)}_{(1,0)} + \left(\frac{r_d}{\ell_{D+1}}\right)^{8-D} \mathcal{E}^{(D+1)}_{(0,0)} + \left(\frac{r_d}{\ell_{D+1}}\right)^{14-D} + \left(\frac{r_d}{\ell_{D+1}}\right)^{15-2D}$$

Dimensional reduction

The decompactification of the Laplacian on G_D/K_D in D dimensions to the one in $D+1$ dimensions (for $D \geqslant 3$)

$$\Delta^{(D)} \to \Delta^{(D+1)} + \frac{D-2}{2(D-1)}(\partial_{\log r_{10-D}})^2 + \frac{D^2-3D-58}{2(D-1)}\,\partial_{\log r_{10-D}}$$

This implies that for $D \geqslant 3$

$$\lambda^{(D)}_{(p,q)} - \lambda^{(D+1)}_{(p,q)} = \frac{2p+3(q+1)}{(D-1)(D-2)}\,(D^2-3D-53+4p+6q)$$

Differential equations and critical dimensions

From the $D=10$ result one derives [Green, Russo, Vanhove]

$$\left(\Delta^{(D)}-\frac{3(11-D)(D-8)}{D-2}\right)\mathcal{E}^{(D)}_{(0,0)} = \delta^{(0,0)}_D$$
$$\left(\Delta^{(D)}-\frac{5(12-D)(D-7)}{D-2}\right)\mathcal{E}^{(D)}_{(1,0)} = \delta^{(1,0)}_D$$
$$\left(\Delta^{(D)}-\frac{6(14-D)(D-6)}{D-2}\right)\mathcal{E}^{(D)}_{(0,1)} = -(\mathcal{E}^{(D)}_{(0,0)})^2+\delta^{(0,1)}_D$$

The eigenvalues vanish in the critical dimensions for UV divergence $D^{2L}\mathcal{R}^4$ interactions

$$D_c=\begin{cases}8 & \text{for } L=0\\ 4+6/L & \text{for } 2\leqslant L\leqslant 4\end{cases}$$

Differential equations and supersymmetry

- The δ contributions are needed for regularizing the 'quasi'-zero mode contribution
- Eigenvalues derived from properties of the $\mathcal{N} = 8$ in $D = 4$ S-matrix by [Elvang et al.]
- threshold conditions from supersymmetry [Bossard, Verschinin]
- Supersymmetry hierarchy [Basu, Sethi]

Counter-term to UV divergences

Vanishing eigenvalue : the coupling is a quasi-zero mode with behaviour

$$\mathcal{E}^{(D_c)}_{(p,q)} = c_L \log(g_D) + \cdots$$

Changing frame $g_D^{-2} = g_s^{-2} v^{10-D}$

$$\ell_s^2 g_{\mu\nu}^{\text{string}} = g_D^{-\frac{2}{D-2}} \ell_D^2 \, g_{\mu\nu}^{\text{Einstein}}$$

$$\mathcal{A}_L^{D_c}(\ell_s^2 k_i \cdot k_j) = \mathcal{A}_L^{D_c}(\ell_D^2 k_i \cdot k_j) + c_L \log(g_D) \, \partial^{2\beta_L} \mathcal{R}^4$$

Counter-term to UV divergences

Vanishing eigenvalue : the coupling is a quasi-zero mode with behaviour

$$\mathcal{E}^{(D_c)}_{(p,q)} = c_L \, \log(g_D) + \cdots$$

Changing frame $g_D^{-2} = g_s^{-2} v^{10-D}$

$$\ell_s^2 g_{\mu\nu}^{\text{string}} = g_D^{-\frac{2}{D-2}} \, \ell_D^2 \, g_{\mu\nu}^{\text{Einstein}}$$

$$\mathcal{A}_L^{D_c}(\ell_s^2 k_i \cdot k_j) = \mathcal{A}_L^{D_c}(\ell_D^2 k_i \cdot k_j) + c_L \log(g_D) \, \partial^{2\beta_L} \mathcal{R}^4$$

This coefficient is identified to the UV counter-term [Green, Russo, Vanhove]

$$\delta\mathcal{L} = \frac{1}{\ell_D^{D-8}} \, c_L \log(g_D) \, (\ell_D \partial)^{2L} \, \mathcal{R}^4$$

Counter-term to UV divergences

A derivation of the value of the c_L needs to use the discrete symmetry of the theory

$$\mathcal{E}^{(D)}_{(p,q)}(\gamma\cdot\vec{\varphi}) = \mathcal{E}^{(D)}_{(p,q)}(\vec{\varphi}); \qquad \gamma \in E_{d+1}(\mathbb{Z})$$

Counter-term to UV divergences

A derivation of the value of the c_L needs to use the discrete symmetry of the theory

$$\mathcal{E}^{(D)}_{(p,q)}(\gamma\cdot\vec{\varphi}) = \mathcal{E}^{(D)}_{(p,q)}(\vec{\varphi}); \qquad \gamma\in E_{d+1}(\mathbb{Z})$$

Perfect match of the UV divergence at $1 \leqslant L \leqslant 3$ by [Bern et al.]:

- $\mathcal{R}^4$at $(D=8, L=1)$
- $\partial^4\mathcal{R}^4$ at $(D=7, L=2)$
- $\partial^6\mathcal{R}^4$ at $(D=6, L=3)$ & $(D=8, L=2)$

[Green, Schwarz, Brink; Green, Russo, Vanhove; D'Hoker, Green,Pioline, Russo; Pioline]

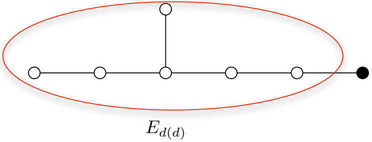
Duality symmetries

[Hull, Townsend]

D	$E_{11-D(11-D)}(\mathbb{R})$	K_D	$E_{11-D(11-D)}(\mathbb{Z})$
10A	$\mathbb{R}^+$	1	1
10B	$Sl(2,\mathbb{R})$	$SO(2)$	$Sl(2,\mathbb{Z})$
9	$Sl(2,\mathbb{R})\times\mathbb{R}^+$	$SO(2)$	$Sl(2,\mathbb{Z})$
8	$Sl(3,\mathbb{R})\times Sl(2,\mathbb{R})$	$SO(3)\times SO(2)$	$Sl(3,\mathbb{Z})\times Sl(2,\mathbb{Z})$
7	$Sl(5,\mathbb{R})$	$SO(5)$	$Sl(5,\mathbb{Z})$
6	$Spin(5,5,\mathbb{R})$	$(Spin(5)\times Spin(5))/\mathbb{Z}_2$	$Spin(5,5,\mathbb{Z})$
5	$E_{6(6)}(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_{6(6)}(\mathbb{Z})$
4	$E_{7(7)}(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_{7(7)}(\mathbb{Z})$
3	$E_{8(8)}(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_{8(8)}(\mathbb{Z})$

Embedding of the duality group

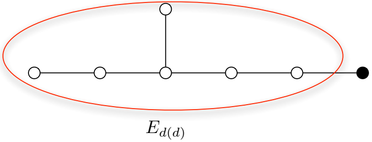
- The duality group in $D+1$ dimensions E_d is embedded into the Levi component of the standard maximal parabolic subgroup of duality group E_{d+1} associated with the last node $P_{\alpha_{d+1}} = L_{\alpha_{d+1}} U$ where $L_{\alpha_{d+1}} = GL(1) \cdot E_d$



- The $GL(1)$ parameter $r^2 = r_{d+1}/\ell_D$ is the radius of circle compactification of the theory from dimensions $D+1$ to D

Embedding of the duality group

- The duality group in $D+1$ dimensions E_d is embedded into the Levi component of the standard maximal parabolic subgroup of duality group E_{d+1} associated with the last node $P_{\alpha_{d+1}} = L_{\alpha_{d+1}} U$ where $L_{\alpha_{d+1}} = GL(1) \cdot E_d$



- The zero Fourier modes with respect to the angular variables $\theta \in U_{\alpha_{d+1}}$ gives the constant term that must satisfy the hierarchy we described

Instanton contributions

The non-perturbative contributions arise from the non-zero Fourier modes

- The Fourier coefficients the angular variables $\theta \in U_{\alpha_{d+1}}$

$$\mathcal{F}^{(D)}_{(p,q)}[Q,\varphi] := \int_{[0,1]^m} d\theta\, \mathcal{E}^{(D)}_{(p,q)}\, e^{2i\pi Q\cdot\theta}$$

- In the decompactification limit $r_d \gg \ell_{D+1}$

$$\mathcal{F}^{(D)}_{(p,q)}[Q,\varphi] \sim \sum_{Q\in\mathbb{Z}^n} d_Q f_Q(\varphi)\, e^{-2\pi r_d\, m_Q}$$

Instanton contributions

- In the decompactification limit $r_d \gg \ell_{D+1}$

$$\mathcal{F}^{(D)}_{(p,q)}[Q,\varphi] \sim \sum_{Q\in\mathbb{Z}^n} d_Q f_Q(\varphi)\, e^{-2\pi r_d\, m_Q}$$

- m_Q is the mass of the BPS particle state in dimension $D+1$ that lead to an instanton once wrapped on the (euclidean) circle of radius r_d
- d_Q is a number theoretic function counting the instanton configurations
- $f_Q(\varphi)$, due to the quantum fluctuations, is a function of the moduli invariant under the symmetry group E_d in dimension $D+1$
- The Fourier transform induces a condition on the discrete charges Q in the charge lattice which lies in *discrete orbits* inside $E_{d+1}(\mathbb{Z})$

Supersymmetric orbits

- The relevant continuous BPS orbits have been classified by [Ferrara, Maldacena; Pope, Lu, Stelle]
- One important orbits is the $\frac{1}{2}$-BPS orbits in dimension D

$$\mathcal{O}_{\frac{1}{2}-BPS} = \frac{E_{d+1}}{E_d \ltimes \mathbb{R}^{n_d}}$$

- Where n_d is the number of BPS charges in dimension $D-1$

E_{d+1}	$M_{\alpha_{d+1}}$	V_{α_d}
E_8	E_7	$q^i : \mathbf{56}$, $q : \mathbf{1}$
E_7	E_6	$q^i : \mathbf{27}$
E_6	$SO(5,5)$	$S_\alpha : \mathbf{16}$
$SO(5,5)$	$SL(5)$	$v_{[ij]} : \mathbf{10}$
$SL(5)$	$SL(3) \times SL(2)$	$v_{ia} : \mathbf{3} \times \mathbf{2}$
$SL(3) \times SL(2)$	$SL(2) \times \mathbb{R}^+$	$vv_a : \mathbf{2}$

Supersymmetric orbits

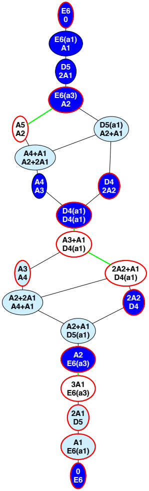
$M_{\alpha_{d+1}} = E_d$	BPS	BPS condition	Orbit	Dim.
$SL(2)$	$\frac{1}{2}$	-	1	0
$SL(2)\times\mathbb{R}^+$	$\frac{1}{2}$	$v\,v_\alpha = 0$	$\frac{\mathbb{R}^*\times SL(2,\mathbb{R})}{SL(2,\mathbb{R})}$	1
	$\frac{1}{4}$	$v\,v_\alpha \neq 0$	$\mathbb{R}^*\times\frac{SL(2,\mathbb{R})}{SO(2,\mathbb{R})}$	3
$SL(3)\times SL(2)$	$\frac{1}{2}$	$\epsilon^{ab}\,v_{i\,a}v_{j\,b} = 0$	$\frac{SL(3,\mathbb{R})\times SL(2,\mathbb{R})}{(\mathbb{R}^+\times SL(2,\mathbb{R}))\ltimes\mathbb{R}^3}$	5
	$\frac{1}{4}$	$\epsilon^{ab}\,v_{i\,a}v_{j\,b} \neq 0$	$\frac{SL(3,\mathbb{R})\times SL(2,\mathbb{R})}{SL(2,\mathbb{R})\ltimes\mathbb{R}^2}$	6
$SL(5)$	$\frac{1}{2}$	$\epsilon^{ijklm}\,v_{ij}\,v_{kl} = 0$	$\frac{SL(5,\mathbb{R})}{(SL(3,\mathbb{R})\times SL(2,\mathbb{R}))\ltimes\mathbb{R}^6}$	7
	$\frac{1}{4}$	$\epsilon^{ijklm}\,v_{ij}\,v_{kl} \neq 0$	$\frac{SL(5,\mathbb{R})}{O(2,3)\ltimes\mathbb{R}^4}$	10
$SO(5,5)$	$\frac{1}{2}$	$(S\Gamma^m S)=0$	$\frac{SO(5,5,\mathbb{R})}{SL(5,\mathbb{R})\ltimes\mathbb{R}^{10}}$	11
	$\frac{1}{4}$	$(S\Gamma^m S)\neq 0$	$\frac{SO(5,5,\mathbb{R})}{O(3,4)\ltimes\mathbb{R}^8}$	16

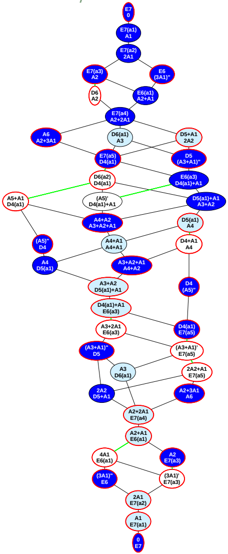
Supersymmetric orbits

$M_{\alpha_{d+1}} = E_d$	BPS	BPS condition	Orbit	Dim.
E_6	$\frac{1}{2}$	$I_3 = \frac{\partial I_3}{\partial q^i} = 0$, and $\frac{\partial^2 I_3}{\partial q^i \partial q^j} \neq 0$.	$\frac{E_6}{O(5,5)\ltimes\mathbb{R}^{16}}$	17
	$\frac{1}{4}$	$I_3 = 0,\ \frac{\partial I_3}{\partial q^i} \neq 0$	$\frac{E_6}{O(4,5)\ltimes\mathbb{R}^{16}}$	26
	$\frac{1}{8}$	$I_3 \neq 0$	$\mathbb{R}^* \times \frac{E_6}{F_{4(4)}}$	27
E_7	$\frac{1}{2}$	$I_4 = \frac{\partial I_4}{\partial q^i} = \frac{\partial^2 I_4}{\partial q^i \partial q^j}\Big\vert_{Adj_{E_7}} = 0$, and $\frac{\partial^3 I_4}{\partial q^i \partial q^j \partial q^k} \neq 0$.	$\frac{E_7}{E_6\ltimes\mathbb{R}^{27}}$	28
	$\frac{1}{4}$	$I_4 = \frac{\partial I_4}{\partial q^i} = 0$, and $\frac{\partial^2 I_4}{\partial q^i \partial q^j}\Big\vert_{Adj_{E_7}} \neq 0$.	$\frac{E_7}{(O(5,6)\ltimes\mathbb{R}^{32})\times\mathbb{R}}$	45
	$\frac{1}{8}$	$I_4 = 0,\ \frac{\partial I_4}{\partial q^i} \neq 0$	$\frac{E_{7(7)}}{F_{4(4)}\ltimes\mathbb{R}^{26}}$	55
	$\frac{1}{8}$	$I_4 > 0$	$\mathbb{R}^+ \times \frac{E_7}{E_{6(2)}}$	56

The Hasse diagram for E_6 and E_7 [The Atlas group]

The closure diagrams of unipotent orbits for E_6 and E_7





The solutions

We need to construct the invariant automorphic function $\mathcal{E}_{(p,q)}$ with Fourier mode only supported on these orbits

$$\mathcal{E}^{(D)}_{(p,q)}(\gamma\cdot\vec{\varphi}) = \mathcal{E}^{(D)}_{(p,q)}(\vec{\varphi}); \qquad \gamma \in E_{d+1}(\mathbb{Z})$$

This is an a priori non trivial and difficult mathematical problem

Comments on the impact of Dynkin's work on current research in representation theory

David A. Vogan, Jr.
Department of Mathematics,
Massachusetts Institute of Technology

A central problem in the representations of reductive Lie groups is constructing unitary representations attached to the nilpotent coadjoint orbits.

In a related direction, Arthur's conjectures (still unproved) relate homomorphisms of $SL(2)$ to residues of Eisenstein series. Colette Moeglin has done great work in the direction of proving that the residues predicted by Arthur (with Dynkin's tables) actually exist. The residues give rise to interesting unitary automorphic representations that are difficult to construct in any other way. Her first paper on this subject is "Orbites unipotentes

Induced Eisenstein series

With information from perturbative string theory one can determine the solution for the $\mathcal{R}^4$ and $D^4\,\mathcal{R}^4$

$$\begin{aligned}
\mathcal{E}^{(D)}_{(0,0)} &= 2\zeta(3)\, E^{E_{d+1}}_{\alpha_1;\frac{3}{2}}; \qquad \text{for } 3 \leqslant d = 11 - D \leqslant 7 \text{ and } d = 0\\
\mathcal{E}^{(D)}_{(1,0)} &= \zeta(5)\, E^{E_{d+1}}_{\alpha_1;\frac{5}{2}}; \qquad \text{for } 5 \leqslant d = 11 - D \leqslant 7 \text{ and } d = 0
\end{aligned}$$

Induced Eisenstein series

- Boundary conditions from string/M-theory allow to determine the *unique* solution [Green, Russo, Miller, Vanhove]

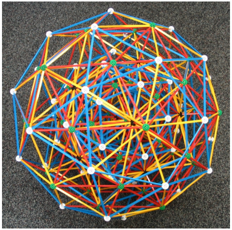
$E_{d+1}(\mathbb{Z})$	$\mathcal{E}^{D}_{(0,0)}$	$\mathcal{E}^{D}_{(1,0)}$
$E_{8(8)}(\mathbb{Z})$	$2\zeta(3)\mathbf{E}^{E_8}_{[1\,0^7];\frac{3}{2}}$	$\zeta(5)\,\mathbf{E}^{E_8}_{[1\,0^7];\frac{5}{2}}$
$E_{7(7)}(\mathbb{Z})$	$2\zeta(3)\mathbf{E}^{E_7}_{[1\,0^6];\frac{3}{2}}$	$\zeta(5)\,\mathbf{E}^{E_7}_{[1\,0^6];\frac{5}{2}}$
$E_{6(6)}(\mathbb{Z})$	$2\zeta(3)\mathbf{E}^{E_6}_{[1\,0^5];\frac{3}{2}}$	$\zeta(5)\,\mathbf{E}^{E_6}_{[1\,0^5];\frac{5}{2}}$
$SO(5,5,\mathbb{Z})$	$2\zeta(3)\mathbf{E}^{SO(5,5)}_{[10000];\frac{3}{2}}$	$\zeta(5)\,\widehat{\mathbf{E}}^{SO(5,5)}_{[10000];\frac{5}{2}} + \frac{8\zeta(6)}{45}\widehat{\mathbf{E}}^{SO(5,5)}_{[00001];3}$
$SL(5,\mathbb{Z})$	$2\zeta(3)\mathbf{E}^{SL(5)}_{[1000];\frac{3}{2}}$	$\zeta(5)\,\widehat{\mathbf{E}}^{SL(5)}_{[1000];\frac{5}{2}} + \frac{6\zeta(5)}{\pi^3}\widehat{\mathbf{E}}^{SL(5)}_{[0010];\frac{5}{2}}$
$SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})$	$2\zeta(3)\widehat{\mathbf{E}}^{SL(3)}_{[10];\frac{3}{2}} + 2\widehat{\mathbf{E}}_1(U)$	$\zeta(5)\,\mathbf{E}^{SL(3)}_{[10];\frac{5}{2}} - 8\zeta(4)^{SL(3)}_{[10];-\frac{1}{2}}\,\mathbf{E}_2(U)$
$SL(2,\mathbb{Z})$	$2\zeta(3)\mathbf{E}_{\frac{3}{2}}(\Omega)\,\nu_1^{-\frac{3}{7}} + 4\zeta(2)\,\nu_1^{\frac{4}{7}}$	$\frac{\zeta(5)\mathbf{E}_{\frac{5}{2}}}{\nu_1^{\frac{5}{7}}} + \frac{4\zeta(2)\zeta(3)}{15}\,\nu_1^{\frac{9}{7}}\mathbf{E}_{\frac{3}{2}} + \frac{4\zeta(2)\zeta(3}{15\nu_1^{\frac{12}{7}}}$
$SL(2,\mathbb{Z})$	$2\zeta(3)\mathbf{E}_{\frac{3}{2}}(\Omega)$	$\zeta(5)\,\mathbf{E}_{\frac{5}{2}}(\Omega)$

Induced Eisenstein series

The simplicity of the solution is visible on the perturbative contribution giving the expected non-renormalisation theorems

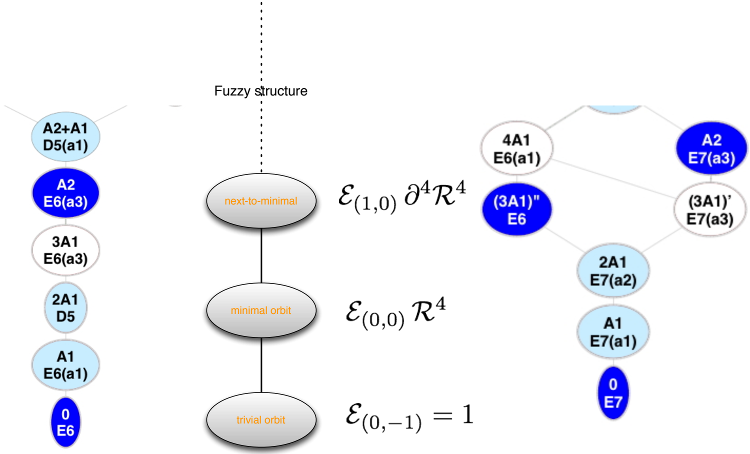
$$\mathcal{E}^D_{(0,0)}\Big|_{\text{pert}} = g_D^{-2\frac{8-D}{D-2}} \left(\frac{a_{\text{tree}}}{g_D^2} + I_{1-\text{loop}}\right)$$

$$\mathcal{E}^D_{(1,0)}\Big|_{\text{pert}} = g_D^{-4\frac{7-D}{D-2}} \left(\frac{a_{\text{tree}}}{g_D^4} + \frac{1}{g_D^2} I_{1-\text{loop}} + I_{2-\text{loop}}\right)$$



We could have as many terms as elements of the Weyl group $|W(E_7)| = 2903040$; $|W(E_8)| = 696729600$ but the answer picked by string theory is much simpler

The closure diagram of an Eisenstein series



[Bossard, Verschinin] assign the $D^6\,\mathcal{R}^4$ term to the node $A2\;E7(a3)$ of the E_7 Hasse diagram

In $D=10$ dimensions type IIB superstring theory we have to solve the differential equation

$$(\Omega_2^2\,(\partial_{\Omega_1}^2+\partial_{\Omega_2}^2)-12)\,\mathcal{E}^{(10)}_{(0,1)}(\Omega)=-(\mathcal{E}^{(10)}_{(0,0)}(\Omega))^2$$

$$\mathcal{E}^{(10)}_{(0,0)}(\Omega)=2\zeta(3)E_{\frac{3}{2}}(\Omega)=\sum_{(m,n)\neq(0,0)}\frac{\Omega_2^{\frac{3}{2}}}{|m\Omega+n|^{\frac{3}{2}}}$$

The $D^6\,\mathcal{R}^4$ term

Let $\mathcal{E}^{(10)}_{(0,1)}(\Omega) = \sum_{n\in\mathbb{Z}} f_n(\Omega_2)\, e^{2i\pi n\Omega_1}$ a modular function

Tree level constraint $\lim_{\Omega_2\to\infty} \mathcal{E}^{(10)}_{(0,1)}(\Omega) = O(\Omega_2^3)$

Less obvious are the constraint on the individual Fourier modes

Theorem (Green-Miller-Vanhove)

If $f(x+iy)$ is an $SL(2,\mathbb{Z})$-invariant function on the upper half plane satisfying the large-y growth condition $f(x+iy) = O(y^s)$ for some $s>1$, then each Fourier mode of f satisfies the bound $\widehat{f}_n(y) = O(y^{1-s})$ for small y.

The $D^6 \mathcal{R}^4$ term

The solution takes the form

$$\begin{aligned} \widehat{f}_n(y) &= \delta_{n,0}\left(\frac{2\zeta(3)^2}{3}y^3 + \frac{4\zeta(2)\zeta(3)}{3}y + \frac{4\zeta(4)}{y}\right) \\ &+ \alpha_n\, y^{\frac{1}{2}}\, K_{\frac{7}{2}}(2\pi|n|y) \\ &+ \sum_{\substack{n_1+n_2=n\\(n_1,n_2)\neq(0,0)}} \sum_{i,j=0,1} M^{ij}_{n_1,n_2}(\pi|n|y)\, K_i(2\pi|n_1|y)\, K_j(2\pi|n_2|y) \end{aligned}$$

M_{ij} are Laurent polynomials of degree at most 2

The constant term

$$\widehat{f}_0(y) \;=\; \frac{2\,\zeta(3)^2}{3}\,y^3 + \frac{4\,\zeta(2)\,\zeta(3)}{3}\,y + \frac{4\,\zeta(4)}{y} + \frac{4\,\zeta(6)}{27\,y^3} + \sum_{n\neq 0}\widehat{f}^{P}_{n,-n}(y)$$

The perturbative contributions match the tree-level, one-loop, two-loop and three-loop term from the 4-graviton scattering

[Green, Vanhove; Green, Vanhove, Russo; D'Hoker, Green, Pioline, Russo; Gómez, Mafra]

The constant term

$$\widehat{f}_0(y) \;=\; \frac{2\,\zeta(3)^2}{3}\,y^3 + \frac{4\,\zeta(2)\,\zeta(3)}{3}\,y + \frac{4\,\zeta(4)}{y} + \frac{4\,\zeta(6)}{27\,y^3} + \sum_{n\neq 0}\widehat{f}^{P}_{n,-n}(y)$$

at large y we have a behaviour characteristic of D-instanton/anti-D-instanton pairs

$$\widehat{f}^{P}_{n_1,-n_1}(y) \;\simeq -\,e^{-4\pi|n_1|y}\left(\frac{\sigma_2(|n_1|)^2}{|n_1|^5\,y^2} + O(y^{-3})\right).$$

Measure is the square of the $\frac{1}{2}$-BPS measure found by [Green, Gutperle]

The $D^6\,\mathcal{R}^4$ term

An alternative form reads

$$\mathcal{E}^{(10)}_{(0,1)}(\Omega) = \frac{2\zeta(3)^2}{3} E_{\frac{3}{2}}(\Omega) + \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Phi(\gamma\Omega)$$

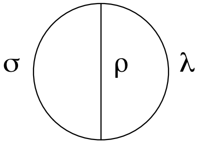
$$\Phi(x+iy) = 4\zeta(3)\int_{\mathbb{R}} \left(\sum_{n\in\mathbb{Z}} \sigma_{-2}(|n|)\, e^{2i\pi n(x+u)}\right) h\left(\frac{x}{y}\right) du$$

where $h(x)$ is the unique smooth even real function with $h(x) \sim_{x\to\pm\infty} 1/(6|x|^3)$ solving

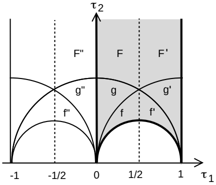
$$\left(\frac{d}{dx}(1+x^2)\frac{d}{dx} - 12\right) h(x) = -\frac{1}{(1+x^2)^{\frac{3}{2}}}$$

Conjectures at higher order in derivative

An integral representation for the $D^6\mathcal{R}^4$ term was derived from the 2-loop amplitude in $D = 11$ supergravity [Green, Vanhove]



$$\mathcal{E}^{(10)}_{(0,1)}(\Omega) = \int_{\Gamma_0(2)} \frac{d^2\tau}{\tau_2^2}\,\Gamma_{(2,2)}(\tau,\Omega)\,A_{(0,1)}(\tau)$$



$$A_{(0,1)} = \frac{|\tau|^2 - \tau_1 + 1}{\tau_2} + 5\,\frac{(\tau_1^2 - \tau_1)(|\tau|^2 - \tau_1)}{\tau_2^3}$$

$$(\Delta - 12)A_{(0,1)} = 0$$

The inhomogeneous term comes from the boundary of the domain of integration

Conjectures at higher order in derivative

At higher derivative order we found in $D = 9$ dimensions a similar but more complicated pattern [Green, Russo, Vanhove]

$$\mathcal{E}^{(10)}_{(p,q)}(\Omega) = \sum_r e^r_{(p,q)}(\Omega)$$

where

$$(\Delta - \lambda_r)\, e^r_{(p,q)}(\Omega) = \sum_{s_1,s_2} c_{s_1,s_2}\, E_{s_1} E_{s_2} \qquad s_1, s_2 \in \frac{1}{2} + \mathbb{Z}$$

Kronecker-Eisenstein series

For integer weight such $s \in \mathbb{N}$ a system of equation is solved the function [Green, D'Hoker, Vanhove]

$$C_{a_1,a_2,a_3}(\tau) = \sum_{\substack{p_1,p_2,p_3 \in \mathbb{Z}\tau+\mathbb{Z} \\ p_1+p_2+p_3=0}} \prod_{i=1}^{3} \frac{\tau_2^{a_i}}{|p_i|^{2a_i}} \qquad \tau \in \mathfrak{h}$$

For instance

$$(\Delta - 12)(C_{2,2,2} - 6C_{1,2,3}) = 36(E_3^2 - 3E_6)$$

$$(\Delta - 12)(10C_{1,1,4} + 4C_{1,2,3} + C_{2,2,2}) = -4(E_3^2 + 15E_2E_4 - 68E_6)$$

appearing as a natural basis for the expansion of the 1-loop 4-graviton amplitude in type II string at order $\alpha'^6 D^{12} \mathcal{R}^4$.

This will be discussed in D'Hoker's talk.

Conclusion and outlook

- Supersymmetry and duality symmetries of string theory forces us to consider new type of automorphic function
- These function satisfy inhomogeneous differential equations and present striking instanton/anti-instanton contributions in their zero mode sector
- It is remarkable that the automorphic program matches so well the amplitude computations when comparison is available
- From the spectrum of eigenvalues in $D = 10$ one get deduce the eigenvalues in $D = 4$ and test for possible UV divergences