Holographic Fermi surfaces from top-down constructions

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1. Introduction

Fermi surfaces at finite chemical potential $\mu$ are straightforward at weak coupling:

- System’s response to adding a fermion is measured by Green’s function:

  \[
  G_R(t, x) = i\theta(t)\langle\{\psi^\dagger(t, x), \psi(0, 0)\}\rangle_{\text{Fermi sea}}
  \]

- Find a pole in Fourier space:

  \[
  G_R(\omega, k) = \frac{Z}{\epsilon(k) - \mu - \omega - i\Gamma + \ldots}
  \approx \frac{Z}{v_F(k - k_F) - \omega - i\Gamma}
  \]

- $\Gamma \sim \omega^2 + \text{(finite temperature)}$ in Landau theory: long-lived quasi-particles.

- I plotted $\text{Im} \frac{1}{(k-1) - \omega - \frac{i}{10}(\omega^2 + 0.02)}$ to convey the idea.
Fermi surfaces arise in $\mathcal{N} = 4$ super-Yang-Mills and ABJM theory, exhibiting several different behaviors. Here are typical examples from $AdS_5 \times S^5$:

A significant caveat: states showing Fermi surfaces often have bosonic instabilities.
Here are typical examples from $AdS_4 \times S^7$.

The key question: Knowing the dual operator in $\mathcal{N} = 4$ SYM or ABJM theory, how can we anticipate whether there will be a Fermi surface, and/or estimate $k_F$?
1.1. An overview of holographic fermions

- Study finite density theories in $\mathbb{R}^{2,1}$ at finite density by analyzing a dual black hole in $AdS_4$ with electric charge behind the horizon.

$$S = \int d^4x \left[ R - \frac{1}{4} F_{\mu\nu}^2 + \frac{6}{L^2} \right]$$

$A_\mu$ gauges in the bulk a $U(1)$ symmetry of the boundary theory.

- Dual field theory (from coincident M2-branes) has fermionic operators $O_\chi \sim \text{tr} \lambda X$ where $\lambda$ is a fermion and $X$ a boson. Trace is over color.

- Use AdS/CFT to compute $\langle O_\chi O_\chi^\dagger \rangle$.

Electric field comes from charge behind horizon.

AdS/CFT prescription is to insert a supergravity fermion from the boundary with $O_\chi$ and see how much of it falls into the black hole.
• Boundary theory is a strongly interacting CFT involving both fermions and bosons, so it’s not \textit{a priori} clear there should be fermi surfaces at all.
• But supergravity picture makes it fairly clear that there should be if fermion charge is large enough.

Gravitational attraction and electrostatic repulsion compete to determine behavior of test particles.

Normalizable fermions at $\omega = 0$ and $k = k_F \neq 0$ stay above the horizon and below the boundary.
1.2. A generic answer from AdS/CFT for the fermionic Green’s function


\[ G(\omega, k) = \left\langle O_\chi(\omega, \vec{k}) O_\chi^\dagger(-\omega, -\vec{k}) \right\rangle \approx \frac{h_1}{(k - k_F) - \frac{1}{v_F}\omega - h_2 e^{i\gamma} \omega^{2\nu_F}} \]

when \( k \approx k_F \) and \( \omega \approx 0 \).

- A singularity in \( G(\omega, k) \) at \( \omega = 0 \) and finite \( k = k_F \) defines the presence of a Fermi surface.

- \( v_F \) is Fermi velocity.

- Assuming \( \nu_F > 1/2 \), low-energy dispersion relation is \( \omega \approx v_F(k - k_F) \).

- If \( \nu_F > 1/2 \) or if \( e^{i\gamma} \) is nearly real, quasi-particles’ width is much smaller than their energy.

- Can easily obtain \( \nu_F \leq 1/2 \), i.e. far from perturbative Landau regime.
1.3. The simplest supergravity calculation

[Faulkner et al ’09; Hartman-Hartnoll 1003.1918; DeWolfe-Gubser-Rosen 1112.3036]

- Simplest charged black hole background is extremal RNAdS$_4$:

\[ ds^2 = \frac{r^2}{L^2} (f dt^2 - dx^2) - \frac{L^2}{r^2} \frac{dr^2}{f} \]

\[ A_\mu dx^\mu = \mu \left( 1 - \frac{r_H}{r} \right) \]

\[ f = 1 - 4 \left( \frac{r_H}{r} \right)^3 + 3 \left( \frac{r_H}{r} \right)^4 \]  

(1)

- Simplest fermion to consider obeys massless charged Dirac equation:

\[ \gamma^\mu (\nabla_\mu - iqA_\mu) \chi = 0. \]

(2)

Fermi surfaces in boundary theory correspond to fermion normal modes in the bulk.

- Supergravity gives relations \( q = \frac{1}{\sqrt{2}L} \) and \( \mu = \frac{\sqrt{6}r_H}{L} \). Generally we’ll choose \( L = 1 \). If also \( r_H = 1 \), then one finds a normal mode at

\[ \omega = 0 \quad k = k_F \equiv 0.9185. \]  

(3)
1.4. Problems with holographic fermions and their solutions

- Previous calculations mainly focus on ad hoc lagrangians, e.g. [Liu-McGreevy-Vegh ’09, Cubrovic-Zaanen-Schalm ’09].
  - Instead, let’s work with fermions of maximal gauged supergravity in $D = 4$ and $D = 5$: reductions / truncations of M-theory on $S^7$ and type IIB on $S^5$.

- AdS-Reissner-Nordstrom black holes have non-zero entropy at $T = 0$, which is hard to understand in field theory.
  - Work with classical variants of RN $AdS_4$ which can be embedded in M-theory or type IIB and have no zero-point entropy.

- Field theory interpretation, e.g. in ABJM theory or $\mathcal{N} = 4$ super-Yang-Mills theory, has been obscure.
  - Formulate “boson rule” and “fermion rule” which capture results of many supergravity calculations in terms of field theory quantities.

- Supergravity calculations are hard work!
  - Find some strong collaborators.
2. Supergravity backgrounds and spinning branes

Charged black holes in $AdS_5$ come from spinning D3-branes.
Charged black holes in $AdS_4$ come from spinning M2-branes.

$D = 4, \mathcal{N} = 8$ supergravity [de Wit and Nicolai, 1982] has $SO(8)$ gauge symmetry associated with the $S^7$ directions coming from $y^1 \ldots y^8$.

- For a semi-pedagogical refresher, see [de Wit, hep-th/0212245].
- Field content is: graviton $g_{\mu\nu}$, 8 gravitini $\psi^i_\mu$, 28 gauge fields $A^{ij}_\mu$, 56 Majorana spinors $\chi^{ijk}$, and 70 real scalars $\phi^{ijkl}$.
- Eight-valued indices $i, j, \ldots$ characterize either the internal symmetry group $SU(8)$ or the gauge group $SO(8)$ (in a spinorial rep wrt $S^7$).
It’s useful to pass to an $SO(8)$ triality frame more simply related to $S^7$:

$$
\begin{pmatrix}
A_{\mu}^{12} \\
A_{\mu}^{34} \\
A_{\mu}^{56} \\
A_{\mu}^{78}
\end{pmatrix}
= \frac{1}{\sqrt{8}}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
A_{\mu}^a \\
A_{\mu}^b \\
A_{\mu}^c \\
A_{\mu}^d
\end{pmatrix}.
$$

(4)

Now $A_{\mu}^a$ encodes spin in the $y^1$-$y^2$ plane, $A_{\mu}^b$ encodes spin in the $y^3$-$y^4$ plane, etc.

With $A_{\mu}^a \neq A_{\mu}^b \neq A_{\mu}^c \neq A_{\mu}^d$, one must turn on three of the 70 scalars to find consistent solutions. Relevant part of $D = 4, \mathcal{N} = 8$ action is

$$
\mathcal{L} = R - \frac{1}{2} (\partial \phi)^2 + \frac{2}{L^2} (\cosh \phi_1 + \cosh \phi_2 + \cosh \phi_3) - \frac{1}{4} \sum_{i=a,b,c,d} e^{-\lambda_i} (F_{\mu\nu}^i)^2
$$

(5)

where

$$
\begin{pmatrix}
\lambda_a \\
\lambda_b \\
\lambda_c \\
\lambda_d
\end{pmatrix}
= \begin{pmatrix}
-1 & -1 & -1 & -1 \\
-1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix}.
$$

(6)

These scalars parametrize oblateness / prolateness of the $S^7$. 

Holographic Fermi surfaces
The general charged 4-d black brane solution we want to consider is

\[ ds_4^2 = e^{2A(r)} \left[ -h(r) dt^2 + d\vec{x}^2 \right] + \frac{e^{2B(r)}}{h(r)} dr^2 \]

\[ A^i = \Phi_i(r) dt \quad \phi_A = \phi_A(r) \]

where

\[ A = -B = \log \frac{r}{L} + \frac{1}{4} \sum_i \log H_i \]

\[ h = 1 - \frac{r}{r_H} \prod_i \frac{r_H + Q_a}{r + Q_a} \]

\[ \lambda_i = -2 \log H_i + \frac{1}{2} \sum_j \log H_j \]

\[ \Phi_i = \frac{1}{L} \sqrt{\frac{Q_i}{r_H}} \sqrt{\prod_j (r_H + Q_j)} \left( 1 - \frac{r_H + Q_i}{r + Q_i} \right), \]

and one can show

\[ s = \frac{1}{4GL^2} \sqrt{\prod_j (r_H + Q_j)} \]

with \( s \to 0 \) as \( r_H \to 0 \) provided at least one of the \( Q_j = 0 \).
There are several qualitatively different behaviors for these charged black branes, and we aim to explore all of them, especially the cases with $s \to 0$.

- $1Q-4d$, $2Q-4d$, $3Q-4d$ are the main cases we’ll consider; $4Q-4d$ was the simplest RN $AdS_4$ case, already discussed.

- $r_H \to 0$ limit is singular for $1Q-4d$, $2Q-4d$, $3Q-4d$.

- To make sure that supergravity is applicable, we’ll turn on small non-zero $r_H$.

- Order of limits gets subtle: For example, $2Q-4d$ is a $r_H \to 0$ limit with $\mu_a = \mu_b = 0$, not the same as a $\mu_a = \mu_b \to 0$ limit with $T = 0$. 
3. Fermion equations of motion

\( D = 4, \mathcal{N} = 8 \) supergravity lagrangian is schematically

\[
\mathcal{L} = \mathcal{L}_b + \frac{1}{2} \overline{\chi} D \chi \chi + \overline{\psi}_\mu O_{\text{mix}} \chi + \frac{1}{2} \overline{\psi}_\mu D_{\text{Rarita–Schwinger}} \psi_\mu + \mathcal{O}(\text{fermion}^4) \tag{9}
\]

Our main task is to decouple the quadratic fermion action and solve resulting linear equations to get two-point functions \( \langle \mathcal{O}_\chi \mathcal{O}_\chi^\dagger \rangle \).

- Some of the 56 fermions \( \chi_{ijk} \) can mix with the 8 gravitini \( \psi^i_\mu \), giving them a mass (super-Higgs). We don’t want these.

- Because bosonic background has no charged fields under \( U(1)^4 \), we know that \( \chi_{ijk} \) can’t couple with \( \psi^i_\mu \) if it has an \( SO(8) \) weight not in the 8. There are 32 such \( \chi_{ijk} \), and dual operators are schematically \( \text{tr} \lambda Z \).

- Of the 24 remaining \( \chi_{ijk} \), there are 16 which don’t couple to the \( \psi^i_\mu \), and 8 that do, but we haven’t worked out which are which. So ignore them all and focus on the special 32.

- Similar results are available from [Gubser-DeWolfe-Rosen ’13] in the case of \( D = 5, \mathcal{N} = 8 \) supergravity; fields of interest are dual to operators \( \text{tr} \lambda Z \).
In 4-dim: The fermion equations of motion we want to study take the form

\[
\left[ i\gamma^\mu \nabla_\mu + \gamma^\mu A^j_\mu Q_j + \sigma^{\mu\nu} F^j_{\mu\nu} P_j + M \right] \vec{\chi} = 0. \tag{10}
\]

\(\vec{\chi}\) is a 32-component vector, and the matrices \(Q_j, P_j,\) and \(M\) all commute (!).

Simultaneous eigenvectors satisfy

\[
\left[ i\gamma^\mu \nabla_\mu + \frac{1}{4} \sum_j \left( q_j \gamma^\mu A^j_\mu + \frac{i}{2} p_j e^{-\lambda_j/2} \sigma^{\mu\nu} F^j_{\mu\nu} + m_j e^{\lambda_j/2} \right) \right] \chi = 0. \tag{11}
\]

\(\nabla_\mu\) includes spin connection but not gauge connections. \(\gamma^\mu\) includes spin connection but not gauge connections. Gauge couplings and Pauli couplings. Spatially variable mass term, \(m \to 0\) at \(\partial AdS_4\).

and we can tabulate the parameters \((q_j, p_j, m_j)\).

Dual operators follow from values of \(q_j\): E.g. \(q_j = (3, 1, 1, -1)\) corresponds to \(\text{tr} \lambda Z\) where

\[
[\lambda]_{SO(8)} = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right) \quad [Z]_{SO(8)} = (1, 0, 0, 0) \quad \text{i.e.} \quad Z = X_1 + iX_2 \tag{12}
\]

We’ll denote \(Z_j = X_{2j-1} + iX_{2j}\) for \(j = 1, 2, 3, 4\).
In 5-dim: Gauge group is $SO(6) \supset U(1)^3$, but we restricted to the case

$$A^a_\mu \equiv A^{12}_\mu = a_\mu \quad A^b_\mu \equiv A^{34}_\mu = A^c_\mu \equiv A^{56}_\mu = A_\mu$$

$$\mu_a = \mu_1 \quad \mu_b = \mu_c = \mu_2/\sqrt{2}$$

(13)

Only one scalar in supergravity is active, $\phi$ in the 20' of $SO(6)$; it is dual to $O_\phi = \text{tr}(2|Z_1|^2 - |Z_2|^2 - |Z_3|^2)$, where $Z_j = X_{2j-1} + iX_{2j}$.

24 of the 48 fermions $\chi_{abc}$ are dual to $\text{tr} \lambda Z$ and obey equations of the form

$$\left[ i \gamma^\mu \nabla_\mu + 2q_1 \gamma^\mu a_\mu + 2q_2 \gamma^\mu A_\mu + ip_1 e^{-2\phi/\sqrt{6}} \gamma^{\mu \nu} f_{\mu \nu} + ip_2 e^{\phi/\sqrt{6}} \gamma^{\mu \nu} F_{\mu \nu} - 2(m_1 e^{-\phi/\sqrt{6}} + m_2 e^{2\phi/\sqrt{6}}) \right] \chi = 0$$

(14)
4. Green’s functions and Fermi surfaces

Setting $\chi(t, \vec{x}, r) = \frac{1}{\sqrt{-\det g_{mn}}} e^{-i\omega t + ikx^1} \psi(r)$ where $m, n = t, 1, 2$, we find:

- Infalling solution at the horizon is $\psi \propto (r - R_H)^{-\frac{i\omega}{4\pi T}}$.
- Asymptotic forms at large $r$ are related to retarded Green’s function:

$$
\psi_{\alpha+} = A_\alpha r^{m-d/2} + B_\alpha r^{-m-1-d/2} \quad \psi_{\alpha-} = C_\alpha r^{m-1-d/2} + D_\alpha r^{-m-d/2}
$$

$$
G_R(\omega, \vec{k}) = -i \int \frac{d^3k}{(2\pi)^3} e^{i\omega t - i\vec{k} \cdot \vec{x}} \theta(t) \langle [\mathcal{O}_\chi(t, x), \mathcal{O}_\chi^{\dagger}(0, 0)] \rangle = \frac{D_\alpha}{A_\alpha}.
$$

- $\alpha = 1, 2$ refers to spinor index of boundary operator. $G_R = G_{\alpha\beta}$ is diagonal, so we can consider one value of $\alpha$ at a time.
- $A_\alpha = 0$ makes fermion wave-function normalizable at boundary.
- Dissipationless modes are possible at $\omega = 0$: Fermion normal mode if also $A_\alpha = 0$. Thus a Fermi surface ($G_R = \infty$) corresponds to a normal mode.
- As far as we can tell, no analytic results are available; all results for $G_R$ were obtained by numerically solving (a close equivalent of) the Dirac equation.
4.1. Examples

Thanks to a relation $G_{11}(\omega, k) = G_{22}(\omega, -k)$, we can get all information from $G_{22}$. Cases examined in 5-d were the following:

<table>
<thead>
<tr>
<th>#</th>
<th>Dual operator</th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>1Q-5d</th>
<th>2Q-5d</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\lambda_1 Z_1$</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{3}{4}$</td>
<td>$\frac{3}{2}$</td>
<td>$1$</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\text{Y}^A$</td>
<td>$\text{N}$</td>
</tr>
<tr>
<td>2</td>
<td>$\lambda_2 Z_1$</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{3}{4}$</td>
<td>$\frac{3}{2}$</td>
<td>$-1$</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\text{Y}^A$</td>
<td>$\text{N}$</td>
</tr>
<tr>
<td>3</td>
<td>$\overline{\lambda}_3 Z_1, \overline{\lambda}_4 Z_1$</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{3}{4}$</td>
<td>$\frac{3}{2}$</td>
<td>$0$</td>
<td>$-\frac{1}{4}$</td>
<td>$0$</td>
<td>$\text{Y}^A$</td>
<td>$\text{N}$</td>
</tr>
<tr>
<td>4</td>
<td>$\lambda_1 Z_2, \lambda_1 Z_3$</td>
<td>$\frac{1}{2}$</td>
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<td>$\frac{1}{2}$</td>
<td>$2$</td>
<td>$\frac{1}{4}$</td>
<td>$0$</td>
<td>$\text{N}^B$</td>
<td>$\text{Y}^G$</td>
</tr>
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<td>$\overline{\lambda}_2 Z_2, \overline{\lambda}_2 Z_3$</td>
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<td>$-\frac{1}{2}$</td>
<td>$2$</td>
<td>$-\frac{1}{4}$</td>
<td>$0$</td>
<td>$\text{N}$</td>
<td>$\text{Y}^G$</td>
</tr>
<tr>
<td>6</td>
<td>$\lambda_3 Z_2, \lambda_4 Z_3$</td>
<td>$\frac{1}{2}$</td>
<td>$-\frac{1}{4}$</td>
<td>$-\frac{1}{2}$</td>
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<td>$\text{N}$</td>
<td>$\text{Y}^H$</td>
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<td>7</td>
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<td>$\text{N}^B$</td>
<td>$\text{Y}^H$</td>
</tr>
</tbody>
</table>

“Boson Rule:” You get a Fermi surface for $\text{tr} \lambda Z$ iff $Z$ has an expectation value.

- 1Q-5d has $\langle \text{tr}(2|Z_1|^2 - |Z_2|^2 - |Z_3|^2) \rangle > 0$, so $\langle \text{tr} |Z_1|^2 \rangle > 0$.
- 2Q-5d has $\langle \text{tr}(2|Z_1|^2 - |Z_2|^2 - |Z_3|^2) \rangle < 0$, so $\langle \text{tr} |Z_2|^2 \rangle > 0$, $\langle \text{tr} |Z_3|^2 \rangle > 0$. 
4-d cases are a bit more intricate:

<table>
<thead>
<tr>
<th>#</th>
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<th>$q_b$</th>
<th>$q_c$</th>
<th>$q_d$</th>
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<th>$m_b$</th>
<th>$m_c$</th>
<th>$m_d$</th>
<th>1Q-4d</th>
<th>2Q-4d</th>
<th>3Q-4d</th>
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</thead>
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<td>1</td>
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<td>1</td>
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<td>1</td>
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<td>-1</td>
<td>3</td>
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<td>N</td>
<td>Y</td>
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<tr>
<td>13</td>
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<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-3</td>
<td>1</td>
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<td>1</td>
<td>Y</td>
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<td>N</td>
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<tr>
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<tr>
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<td>-3</td>
<td>N</td>
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</tbody>
</table>

But boson rule works in every case: non-zero bosons are $Z_1$ for 1Q-4d; $Z_3$ and $Z_4$ for 2Q-4d; and $Z_2$, $Z_3$, $Z_4$ for 3Q-4d.

Suggested interpretation: The singularity in $\langle \mathcal{O}_\chi \mathcal{O}^{\dagger}_\chi \rangle$ is due to a Fermi surface of a colored fermion, co-existing with a scalar condensate which (at large $N$) leaves the $U(1)$ symmetry unbroken.
Easiest for me to think about the case of $\mathcal{N} = 4$ SYM in $d = 4$. Large $N$ allows $U(1)$ to remain unbroken even with non-zero scalar condensate:

A common worry is that scalar condensate can run away along flat directions. But perhaps this is not relevant at large $N$. Here’s why:

- Only a subleading fraction of directions satisfy $[X^I, X^J] = 0$.
- Cases considered are finitely far from SUSY limit, so it’s probably more representative to think of non-commuting directions.
- In non-commuting directions, condensate is limited by $V \sim g^2 \text{tr}[X^I, X^J]^2$. 
So—plausibly—the singularity at $k = k_F$, with residue $\sim N^2$ in $AdS_5$ calculations, owes to diagrams in $\mathcal{N} = 4$ SYM roughly like this:

This account contrasts strongly with the proposal the Fermi surfaces are best understood in terms of color singlet fermions \textit{in the gauge theory} [Huijse-Sachdev ’11], and if colored fermions have Fermi surfaces, they are hidden from supergravity calculations.
A closer look at examples shows that $k_F$ is often significantly smaller than the natural scale

$$\mu_* = \sqrt{T^2 + \mu_1^2 + \mu_2^2} \quad (5\text{-}d) \quad \mu_* = \sqrt{T^2 + \sum_j \mu_j^2} \quad (4\text{-}d). \quad (16)$$
There are two unrelated reasons for this:

1. $\mu_1 \ll \mu_*$ for the 1Q-5d (Case A), so we naturally have small Fermi surfaces.

2. Case G involves the gaugino $\lambda_1^{(1/2,1/2,1/2)}$, which carries charge under $U(1)$ of the 2Q-BH background, whereas Case H involves the gaugino $\lambda_3^{(-1/2,1/2,-1/2)}$, which is neutral under this $U(1)$.

Viewing #1 as trivial, we suggest the following

```
“Fermion Rule:” The value of $k_F$ is suppressed, though it may not vanish, when $\lambda$ is neutral under the $U(1)$ charge of the black hole.
```
A detailed look at 4-d cases provide supports the boson rule and gives some additional evidence in favor of the fermion rule.

- Chemical potential $\mu_a$ is small for case A.
- $k_F$ is rather larger for case F (charged $\lambda$) than for case G (neutral $\lambda$).
5. A tentative Luttinger count

Consider just the 2Q-5d case: An $AdS_5$ black hole with two equal charges, $A^{34}_\mu = A^{56}_\mu$ with $A^{12}_\mu = 0$.

- A lovely sharp Fermi surface arises for $\lambda^{+++}_1$ and $\bar{\lambda}^{+++}_2$ (more precisely for $\text{tr} \lambda Z$ operators involving these fermions): $k_F/\mu_* \approx 0.812$.

- A count of the total charge carried by the fermions is given, according to Luttinger, by

$$\rho_{\text{fermions}} = N_{\text{fermions}} q_{\text{fermions}} \int_{k<k_F} \frac{d^3k}{(2\pi)^3}. $$  \hspace{1cm} (17)

We know $q_{\text{fermions}} = 1$ (from $SO(6)$ group theory), and it’s reasonable to suppose $N_{\text{fermions}} = N^2$.

- On the other hand, we know the total $\rho_{\text{total}} = \frac{Q_2}{\pi r_H^3}$.

- After a bit of calculation, find $\frac{\rho_{\text{fermion}}}{\rho_{\text{total}}} \approx 0.972$.

OK since it’s not decisively greater than one—but a bit strange since I’d have thought more charge would be carried by bosons.
6. Summary

- Field theory understanding of holographic Fermi surfaces is probably easier without extremal entropy complicating the story.

- Holographic Fermi surfaces appear or don’t appear in correlators of $O_\chi = \text{tr} \lambda Z$ precisely if $Z$ has an expectation value.

- Probably what’s going on is that we’re seeing a Fermi surface of the color-charged fermions $\lambda$, not some composite color-singlet created by $O_\chi$.

- Neutral fermions (wrt black hole’s chemical potential) have smaller Fermi surfaces, though their $k_F$ may not be exactly 0.

- Better understanding of field theory is very desirable. Also, we need some examples without boson instabilities.
Revisiting cases with non-zero entropy

What about 5d cases with all three charges non-zero? Hard because $s$ remains finite as $T \to 0$. A few examples are helpful:

Fermion is $\text{tr} \lambda_1 Z_2 = \text{tr} \lambda^{+++} Z^{010}$.
There “should” be a Fermi surface everywhere.
Oscillatory region is where BF bound is violated in $AdS_2$ region.

Fermion is $\text{tr} \bar{\lambda}_3 Z_1 = \text{tr} \lambda^{+-} Z^{100}$.
There “should” be a Fermi surface everywhere.

$\mu_R = \frac{\mu_a}{\sqrt{2} \mu_b}$, with $\mu_b = \mu_c$.

$\mu_R = \frac{1}{\sqrt{2}}$ is the equal-charge black hole.
A Fermi surface vanishing into an oscillatory region probably indicates only finite but small width developing at zero temperature; c.f. [Liu-McGreevy-Vegh ’09]

Contrast with fermions such as $\text{tr} \lambda_4 Z_1 = \text{tr} \lambda^{--} Z^{100}$, where there are no Fermi surfaces. Makes sense because overall charge of $\lambda_4$ is negative—so no Fermi sea wants to form.