Generalised Mirror Symmetry and the Weyl Anomaly

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Abstract

- Classically, Weyl invariance

$$S(g,\phi) = S(g',\phi')$$

under

$$g'_{\mu\nu}(x) = \Omega(x)^2 g_{\mu\nu}(x) \quad \phi' = \Omega(x)^\alpha \phi$$

implies

$$g^{\mu\nu} T_{\mu\nu} = 0$$

- But in the quantum theory

$$g^{\mu\nu} < T_{\mu\nu} > \neq 0$$

Capper and Duff 1973
Over the period 1973-2012 this Weyl anomaly has found a variety of applications in quantum gravity, black hole physics, inflationary cosmology, string theory and statistical mechanics.

CFTs

- Weyl anomalies appear in their most pristine form when CFTs are coupled to an external gravitational field. In this case
$$g^{\mu\nu}\langle T_{\mu\nu}\rangle = \frac{1}{(4\pi)^2}(cF - aG)$$
where F is the square of the Weyl tensor:
$$F = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2,$$
G is proportional to the Euler density:
$$G = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2,$$
- Note no R^2 term.
- We ignore $\Box R$ terms whose coefficient can be adjusted to any value by adding the finite counterterm
$$\int d^4x\sqrt{g}R^2.$$

Central charges c and a

- In the CFT a and c are the central charges given in terms of the field content by

$$\bar{a} \equiv 720a = 2N_0 + 11N_{1/2} + 124N_1$$

$$\bar{c} \equiv 720c = 6N_0 + 18N_{1/2} + 72N_1$$

where N_s are the number of fields of spin s.

- In the notation of Duff 1977

$$(4\pi)^2 b = c \quad (4\pi)^2 b' = -a$$

Total versus anomalous trace

- For fields that are classically non-conformal, such as gravity itself, the total trace T of the quantum stress tensor

$$T \equiv g_{\mu\nu} \frac{\delta W}{\delta g_{\mu\nu}}$$

involves a "naive" term in addition to the Weyl anomaly.

$$T = T_N + T_A \qquad (1)$$

- The anomaly is given by the De Witt b_4 coefficient in the asymptotic exansion of the heat kernel

$$T_A = b_4 = \frac{1}{(4\pi)^2}(cF - aG + eR^2)$$

c and a are gauge-dependent for spins $3/2$ and 2, though $c - a$ is not

Euler number

- When $F - G$ vanishes, anomaly reduces to

$$T_A = A\frac{1}{32\pi^2}R^{*\mu\nu\rho\sigma}R^{*}{}_{\mu\nu\rho\sigma}$$

where

$$360A = \bar{c} - \bar{a}$$

so that in Euclidean signature

$$\int d^4x\sqrt{g}g^{\mu\nu}T_{\mu\nu} = A\chi(M^4)$$

where $\chi(M^4)$ is the Euler number of spacetime.

Arbitrary spin

- Calculate b_4 for arbitrary (n, m) reps of Lorentz group, then physical anomaly given by

$$A = A(n,m) + A(n-1,m-1) - 2A(n-1/2,m-1/2)$$

so in total

$$A_{total} = 4N_0 + 7N_{1/2} - 52N_1 - 233N_{3/2} + 848N_2$$

where N_s are the number of fields of spin s.

- The b_4 coefficient for chiral reps (1/2,0) (1,0) etc also involve R*R unless we add (0,1/2) (0,1) etc

Christensen and Duff 1978

p-forms and inequivalent anomalies

- Inequivalence:

$$A_2 - A_0 = 1$$

$$A_3 = -2$$

$$A_4 = 3$$

  Duff and van Nieuwenhuizen 1980

- Confirmed by string calculations Antoniadis, Gava and Narain 1992
- Can arrange $A = 0$ for $\mathcal{N} \geq 3$ Nicolai and Townsend 1980
- But, according to Grisaru et al 1980 and Siegel 1980 total stress tensors are equivalent.

Generalized mirror symmetry and trace anomalies
M-theory on X^7

- We consider compactification of $(\mathcal{N} = 1, D = 11)$ supergravity on a 7-manifold X^7 with betti numbers $(b_0, b_1, b_2, b_3, b_3, b_2, b_1, b_0)$ and define a generalized mirror symmetry

$$(b_0, b_1, b_2, b_3) \to (b_0, b_1, b_2 - \rho/2, b_3 + \rho/2)$$

under which

$$\rho(X^7) \equiv 7b_0 - 5b_1 + 3b_2 - b_3$$

changes sign

$$\rho \to -\rho$$

Duff and Ferrara 2010

- The massless sectors of these compactifications have

$$f = 4(b_0 + b_1 + b_2 + b_3)$$

degrees of freedom

Anomalies

	Field	f	ΔA	$360A$	$360A'$	X^7
g_{MN}	$g_{\mu\nu}$	2	-3	848	-232	b_0
	$\mathcal{A}_\mu$	2	0	-52	-52	b_1
	$\mathcal{A}$	1	0	4	4	$-b_1 + b_3$
ψ_M	ψ_μ	2	1	-233	127	$b_0 + b_1$
	χ	2	0	7	7	$b_2 + b_3$
A_{MNP}	$A_{\mu\nu\rho}$	0	2	-720	0	b_0
	$A_{\mu\nu}$	1	-1	364	4	b_1
	A_μ	2	0	-52	-52	b_2
	A	1	0	4	4	b_3
total ΔA			0			
total A				$-\rho/24$		
total A'					$-\rho/24$	

Vanish without a trace!

- Remarkably, we find that the anomalous trace depends on ρ
$$A = -\frac{1}{24}\rho(X^7)$$
So the anomaly flips sign under generalized mirror symmetry and vanishes for generalized self-mirror theories. For $X^{(8-\mathcal{N})} \times T^{(\mathcal{N}-1)}$ with $\mathcal{N} \geq 3$ the anomaly vanishes identically but not the Townsend-Nicolai way!
- Equally remarkable is that we get the same answer for the total trace using the numbers of Grisaru et al 1980.

Type IIA

- In the case of $(\mathcal{N}=1, D=11)$ on $X^6 \times S^1$, or equivalently (Type IIA, D=10) on X^6,

$$A = -\frac{1}{24}\chi(X^6)$$

and so in Euclidean signature

$$\int d^4x\sqrt{g}g_{\mu\nu} < T^{\mu\nu} >= -\frac{1}{24}\chi(M^4)\chi(X^6) = -\frac{1}{24}\chi(M^{10})$$

where $\chi(M^4)$ is the Euler number of spacetime.

Four curious supergravities

- Of particular interest are the four cases

$$(b_0, b_1, b_2, b_3) = (1, \mathcal{N} - 1, 3\mathcal{N} - 3, 4\mathcal{N} + 3)$$

  with $\mathcal{N} = 1, 2, 4, 8$, namely the four “curious” supergravities, which enjoy some remarkable properties.
  $\mathcal{N} = 1$, 7 WZ multiplets, $f = 32$,
  $\mathcal{N} = 2$, 3 vector multiplets, 4 hypermultiplets, $f = 64$,
  $\mathcal{N} = 4$, 6 vector mutiplets, $f = 128$,
  $\mathcal{N} = 8$, $f = 256$.
- Reduction of Supergravities for Membranes in $D = 4, 5, 7, 11$

Fano plane

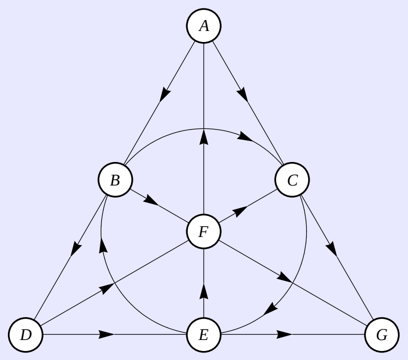

Figure: The Fano plane has seven points and seven lines (the circle counts as a line) with three points on every line and three lines through every point. The truncation from 7 lines to 3 to 1 to 0 corresponds to the truncation from N=8 to N=4 to N=2 to N=1.

O, H, C, R theories

Field	$360A$	**O**	**H**	**C**	**R**
$g_{\mu\nu}$	848	1	1	1	1
B_μ	-52	7	6	0	0
S	4	28	16	10	7
ψ_μ	-233	8	4	2	1
χ	7	56	28	14	7
$A_{\mu\nu\rho}$	-720	1	1	1	1
$A_{\mu\nu}$	364	7	3	1	0
A_μ	-52	21	6	4	0
A	4	35	19	11	7
		$A = 0$	$A = 0$	$A = 0$	$A = 0$

Table: Vanishing anomaly in **O**, **H**, **C** **R** theories.

Acknowledgements

- Congratulations Paul
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