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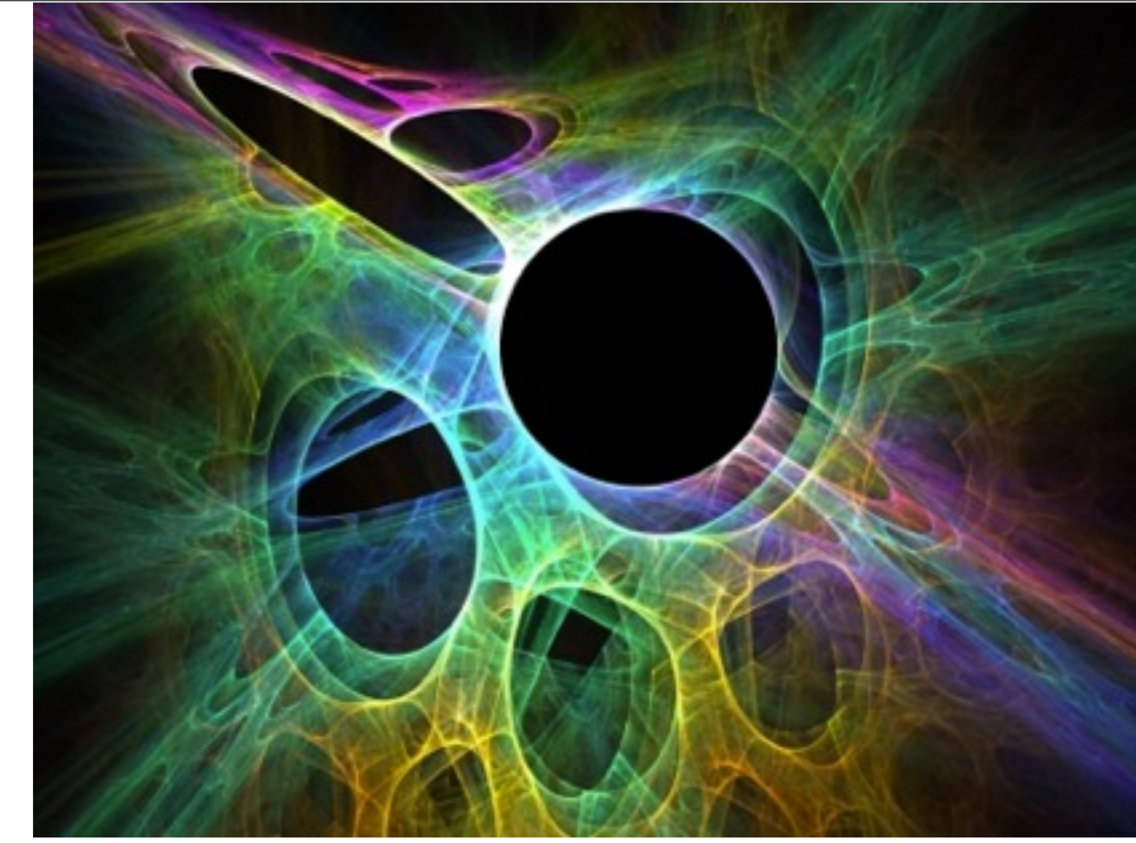

2nd - 3rd July, 2012

Branes, Supergravity and M-Theory

A conference to celebrate the 60th birthday of Paul K Townsend

Hamiltonian Formulation of Open (Super)strings

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LM work with Paul K. Townsend - 10 - 08, 08, 11; 11 - 06, 11

work in progress with Alisdair Routh and Paul K. Townsend - 12.

Introduction

- Hamiltonian Formulation - Not Pursued in String Theory
- $3D$ (Super)particles, Closed (Super)strings Hamiltonian Formulation Developed LM and Paul K. Townsend 10,11.
- Work in $3D$ even if many results valid in any dimension
  $D=3$ Exception Closed (Super)string in Light Cone Gauge
  Dangerous Commutators $\left[\mathcal{J}^{-i},\mathcal{J}^{-j}\right]$ zero by default
  The States of the $3D$ Closed (Super)string are Anyons!
- Hamiltonian formulation for (Super)strings with Boundaries leads bc's which must be obeyed and the conditions on the local parameters of the gauge symmetries
- Gauges appropriate for quantization:
  LC - Neumann bc's, Arvis - Dirichlet bc's

Hamiltonian form of closed string

LC-gauge & absence of Lorentz anomalies in $3D$

$D=3, \mathcal{N}=2$ **Superstring**

$D=3, \mathcal{N}=2$ **Summary of Results**

Hamiltonian Formulation - Open (Super)string

Arvis & LC Gauges

Massive $D=3, \mathcal{N}=2$ Superparticle -Hamiltonian Form

Conclusions

Hamiltonian form of closed string

- Nambu-Goto action for the closed string of tension T in Hamiltonian form

$$S[\mathbb{X},\mathbb{P};\ell,u] = \int d\tau \oint \frac{d\sigma}{2\pi}\left\{\dot{\mathbb{X}}^\mu \mathbb{P}_\mu - \frac{1}{2}\ell\left[\mathbb{P}^2 + (T\mathbb{X}')^2\right] - u\,\mathbb{X}'^\mu \mathbb{P}_\mu\right\}$$

where ℓ and u 'lapse' and 'shift' spirit of ADM

- Noether Poincaré generators $\mathcal{P}_\mu = \oint \frac{d\sigma}{2\pi}\mathbb{P}_\mu\,,\;\; \mathcal{J}^\mu = \oint \frac{d\sigma}{2\pi}\left[\mathbb{X}\wedge\mathbb{P}\right]^\mu$

The action invariant under diff transformations

$$\delta\mathbb{X} = \alpha\mathbb{P} + \beta\mathbb{X}'\,, \qquad \ell = \dot{\alpha} + u'\alpha - u\alpha' + (\ell'\beta - \ell\beta')$$

$$\delta\mathbb{P} = T^2(\alpha\mathbb{X}')' + (\beta\mathbb{P})' \qquad u = \dot{\beta} + u'\beta - u\beta' + T^2\left(\alpha\ell' - \ell\alpha'\right)$$

- Upon elimination of auxiliary $\mathbb{P}$

$$S[\mathbb{X};\ell,u] = -\frac{1}{2}T\int d\tau \int_0^{2\pi}\frac{d\sigma}{2\pi}\sqrt{-\det g}\, g^{ij}\partial_i\mathbb{X}\cdot\partial_j\mathbb{X}$$

World sheet metric conformally equivalent to

$$ds^2 = g_{ij}d\xi^i d\xi^j = (u^2 - T^2\ell^2)d\tau^2 + 2u\, d\tau d\sigma + d\sigma^2$$

LC-gauge & absence of Lorentz anomalies in $3D$

- LC-gauge components : $(X^+, X^-, X), \ (P^+, P^-, P)$

$X^+ = \tau, P_- = p_-(\tau),$ residual gauge invariance $p_-(\tau)$

We separate the Average over String: $a(\tau) = \oint \frac{d\sigma}{2\pi} A(\sigma, \tau)$

$\bar{A} = A - a \qquad A^\pm = 1/\sqrt{2}\left(A^1 \pm A^0\right)$

$$L = \dot{x}p + \dot{x}^- p_- + \frac{1}{2\pi}\oint d\sigma\, \dot{\bar{X}}\bar{P} + p_+ - \frac{1}{2\pi}\oint d\sigma\, uX'P$$
$$+\frac{p_-}{2\pi}\oint d\sigma \left\{\bar{X}^- u' - \ell\left(P_+ + \frac{1}{2p_-}\left[P^2 + (TX')^2\right]\right)\right\}$$

Eliminate Lagrange multipliers: $\ell \Rightarrow \quad P_+ = -\frac{1}{2p_-}\left[P^2 + (TX')^2\right]$

$\bar{X}^- \Rightarrow u' = 0, \quad u \Rightarrow \left(\bar{X}^-\right)' = -\frac{1}{p_-}\bar{X}'P\,,\ p_+ \Rightarrow \ell = \frac{1}{p_-}$

- $$L = \left\{ \dot{x} p + \dot{x}^- p_- + \frac{1}{2\pi} \oint d\sigma\, \dot{\bar{X}} \bar{P} \right\} - H - \frac{u_0}{2\pi} \oint d\sigma\, \bar{X}' \bar{P}$$

$$H = -p_+ = \frac{1}{2p_-} \oint \frac{d\sigma}{2\pi} \left[P^2 + (TX')^2 \right],\ u_0 :$$ Level Matching Constraint

- Poincaré Generators in LC:

$$\mathcal{P} = p\,, \qquad \mathcal{P}_- = p_- \qquad \mathcal{P}_+ = -H\,, \quad \mathcal{J} = x^- p_- + \tau H\,, \qquad \mathcal{J}^+ = \tau p - x p_-\,,$$

$$\mathcal{J}^- = -x^- p - xH + \Lambda/p_-\,, \qquad \Lambda = \frac{p_-}{2\pi} \oint d\sigma \left[\bar{X} \bar{P}_+ - \bar{X}^- \bar{P} \right].$$

- Casimirs: $\mathcal{P}^2 = -\mathcal{M}^2 = \frac{1}{2\pi} \oint d\sigma \left[\bar{P}^2 + (T\bar{X}')^2 \right], \qquad \mathcal{P} \cdot \mathcal{J} = \Lambda\,.$
- Quantization most transparent in normal modes:

$$\bar{X} = \frac{i}{\sqrt{2T}} \sum_{n=1}^{\infty} \frac{1}{n} \left[e^{in\sigma} \left(\alpha_n - \tilde{\alpha}_n^* \right) - e^{-in\sigma} \left(\alpha_n^* - \tilde{\alpha}_n \right) \right]$$

$$\bar{P} = \sqrt{\frac{T}{2}} \sum_{n=1}^{\infty} \left[e^{in\sigma} \left(\alpha_n + \tilde{\alpha}_n^* \right) + e^{-in\sigma} \left(\alpha_n^* + \tilde{\alpha}_n \right) \right]$$

- To compute Poincaré generators we need:

$$\bar{X}^{-} = -\frac{1}{p_-}\left\{p\bar{X} + \sum_{n=1}^{\infty}\frac{i}{n}[e^{in\sigma}(\beta_n - \tilde{\beta}_n^*) - e^{-in\sigma}(\beta_n^* - \tilde{\beta}_n)]\right\}$$

$$\bar{P}_{+} = -\frac{1}{p_-}\left\{p\bar{P} + T\sum_{n=1}^{\infty}[e^{in\sigma}(\beta_n + \tilde{\beta}_n^*) + e^{-in\sigma}(\beta_n^* + \tilde{\beta}_n)]\right\}.$$

These last expressions are composite!

$$\beta_n = \frac{1}{2}\sum_{m=1}^{n-1}\alpha_m\alpha_{n-m} + \sum_{m>n}\alpha_m\alpha^*_{m-n},$$

and correspondingly for the star and tilde ones.

$$L = \left\{\dot{x}p + \dot{x}^{-}p_{-} + i\sum_{n=1}^{\infty}\frac{1}{n}\left[\dot{\alpha}_n\alpha_n^* + \dot{\tilde{\alpha}}_n\tilde{\alpha}_n^*\right]\right\} - H + u_0\sum_{n=1}^{\infty}\left[\alpha_n^*\alpha - \tilde{\alpha}_n^*\tilde{\alpha}_n\right]$$

- $$H = -\mathcal{P}_{+} = \frac{1}{2p_-}\left(p^2 + \mathcal{M}^2\right), \quad \mathcal{M}^2 = 2T\sum_{n=1}^{\infty}\left[\alpha_n^*\alpha_n + \tilde{\alpha}_n^*\tilde{\alpha}_n\right]$$

$$\Lambda = \sqrt{2T}\sum_{n=1}^{\infty}\frac{i}{n}\left[(\alpha_n^*\beta_n - \beta_n^*\alpha_n) + (\tilde{\alpha}_n^*\tilde{\beta}_n - \tilde{\beta}_n^*\tilde{\alpha}_n)\right]$$

- Quantization: $\left[\alpha_n, \alpha_n^\dagger\right] = n\,, \left[\tilde{\alpha}_n, \tilde{\alpha}_n^\dagger\right] = n$ otherwise commuting
- Ordering Ambiguities: only in $\mathcal{M}^2 = 2T\left(N + \tilde{N} - a\right)$

where $N = \sum_{n=1}^{\infty} \alpha_n^\dagger \alpha_n, \qquad \tilde{N} = \sum_{n=1}^{\infty} \tilde{\alpha}_n^\dagger \tilde{\alpha}_n$

- States: $|n_1, n_2, \cdots, \tilde{n}_1, \tilde{n}_2, \cdots >= (\alpha_1^\dagger)^{n_1} \cdots (\tilde{\alpha}_1^\dagger)^{\tilde{n}_1} \cdots |p, p_- >$
- Level Matching Condition: $(N - \tilde{N})|n_1, n_2, \cdots, \tilde{n}_1, \tilde{n}_2, \cdots > = 0$
- Poincaré Algebra Closes!
- Ground state: $\mathcal{M}^2 = 2T(-a), \qquad \Lambda = 0$
- Level 1: $\mathcal{M}^2 = 2T(2-a), \qquad \Lambda = 0$
- Level 2: $\mathcal{M}^2 = 2T(4-a), \qquad \Lambda$ - non diagonal

diagonalize and divide by the mass of the states $s_2 = \left(0, 0, \frac{\pm 3}{\sqrt{4-a}}\right)$

Level 3: similar $s_3 = \left(0, 0, 0, \pm\sqrt{\frac{179}{12(6-a)}}, \pm\sqrt{\frac{179}{12(6-a)}}, \pm\sqrt{\frac{179}{3(6-a)}}\right)$

- For $D = 3$ irreps of Poincaré group induced from irreps of the universal cover $\overline{SO(2)} \cong \mathbb{R}$. All irreps are one-dimensional and characterized by the helicity β, which can be **any real number**

Binegar - 82.

- Manifest Lorentz covariance requires fields in irreps of the relevant cover of the Lorentz group $SO(1,2)$. **Irrational helicity requires infinite dimensional irrep of the universal cover** $\overline{SO}(1,2)$.

Jackiw & Nair, '91

Because massive string states contain anyons this new $3D$ string theory **should not be accessible through standard covariant quantization techniques, as irrational spin anyons seem to require infinite dimensional component fields.**

$D=3, \mathcal{N}=2$ Superstring

- $D=3, \mathcal{N}=2$ action for the closed string of tension T in semi-Hamiltonian form
- $D=3, \mathcal{N}=2$ *SUSY* $\delta_\epsilon \mathbb{X}^\mu = i\bar{\Theta}_a \Gamma^\mu \epsilon_a \,, \delta_\epsilon \Theta_a = \epsilon_a \,, \{\Theta_a; a=1,.\,\mathcal{N}\}$
- Cartan Forms $\quad \Pi^\mu = d\mathbb{X}^\mu + i\bar{\Theta}_a \Gamma^\mu d\Theta_a \,, \Pi_a = d\Theta_a$
- Green-Schwarz action for the closed string of tension T in semi Hamiltonian form

$$S[\mathbb{X},\mathbb{P},\Theta_a;\ell,u] = \int d\tau \oint \frac{d\sigma}{2\pi} \left\{ \Pi^\mu_\tau \mathbb{P}_\mu - \frac{1}{2}\ell\left[\mathbb{P}^2 + (T\Pi_\sigma)^2\right] - u\,\Pi^\mu_\sigma \mathbb{P}_\mu \right.$$

pullback $\left. + iT\left[\left(\dot{\mathbb{X}}^\mu + \frac{i}{2}\bar{\Theta}_a\Gamma^\mu\dot{\Theta}_a\right)(\bar{\Theta}_1\Gamma_\mu\Theta_1' - \bar{\Theta}_2\Gamma_\mu\Theta_2') - \Big((\,\dot{}\,) \Leftrightarrow (\,{}'\,)\Big)\right]\right\}$

where $h_3^{\mathcal{N}=2} = dh_2^{\mathcal{N}=2}, \; h_2^{\mathcal{N}=2} = -\left(d\mathbb{X}^\mu + \frac{i}{2}\bar{\Theta}_a\Gamma^\mu d\Theta_a\right)(\bar{\Theta}_1\Gamma_\mu d\Theta_1 - \bar{\Theta}_2\Gamma_\mu d\Theta_2)$

The action invariant under modified diff transformations

$$\delta\mathbb{X} = \alpha\left[\mathbb{P} - i\ell^{-1}\bar{\Theta}_a\Gamma\left(\dot{\Theta}_a - u\Theta_a'\right)\right] + \beta\mathbb{X}'\,,\quad \delta\Theta_a = \alpha\ell^{-1}\left(\dot{\Theta}_a - u\Theta_a'\right) + \beta\Theta_a'\,,$$

$$\delta\mathbb{P} = \left(T^2\alpha\Pi_\sigma + \beta\mathbb{P}\right)' + 2i\alpha\ell^{-1}T\left(\bar{\Theta}_1'\Gamma\dot{\Theta}_1 - \bar{\Theta}_2'\Gamma\dot{\Theta}_2\right)$$

$$\delta_\kappa\mathbb{X}^\mu = -i\bar{\Theta}_a\Gamma^\mu\delta_\kappa\Theta_a\,,\qquad \delta_\kappa\Theta_1 = \Gamma_\mu\left(\mathbb{P}^\mu - T\Pi_\sigma^\mu\right)\kappa_1 \qquad \delta_\kappa\Theta_2 = \Gamma_\mu\left(\mathbb{P}^\mu + T\Pi_\sigma^\mu\right)\kappa_2$$

κ - symmetry $\delta_\kappa\mathbb{P}_\mu = 2iT\left(\bar{\Theta}_1'\Gamma_\mu\delta_\kappa\Theta_1 - \bar{\Theta}_2'\Gamma_\mu\delta_\kappa\Theta_2\right) \quad \delta_\kappa u = -T\left(\delta_{\kappa_1}\ell - \delta_{\kappa_2}\ell\right)$

$$\delta_\kappa\ell = -4i\bar{\kappa}_1\left[\dot{\Theta}_1 + (\ell T - u)\,\Theta_1'\right] - 4i\bar{\kappa}_2\left[\dot{\Theta}_2 + (-\ell T - u)\,\Theta_2'\right]$$

- Noether Poincaré generators

$$\mathscr{P}_\mu = \oint\frac{d\sigma}{2\pi}\left\{\mathbb{P}_\mu + iT\left[\bar{\Theta}_1\Gamma_\mu\Theta_1' - \bar{\Theta}_2\Gamma_\mu\Theta_2'\right]\right\} \quad \mathscr{J}^\mu \doteq \oint\frac{d\sigma}{2\pi}\{[\mathbb{X}\wedge(\mathbb{P} + iT(\bar{\Theta}_1\Gamma\Theta_1' - \bar{\Theta}_2\Gamma\Theta_2'))]^\mu$$

$$+\frac{i}{2}\bar{\Theta}_1\Theta_1(\mathbb{P} - T\mathbb{X}')^\mu + \frac{i}{2}\bar{\Theta}_2\Theta_2(\mathbb{P} + T\mathbb{X}')^\mu + (T/2)(\bar{\Theta}_2\Gamma^\mu\Theta_2'\bar{\Theta}_1\Theta_1 - \bar{\Theta}_1\Gamma^\mu\Theta_1'\bar{\Theta}_2\Theta_2)\}$$

- *SUSY*

$$\mathscr{Q}_1^\alpha = \sqrt{2}\oint\frac{d\sigma}{2\pi}\{(\mathbb{P}^\mu - T\Pi_\sigma)(\Gamma^\mu\Theta_1)^\alpha - 2iT\left(\bar{\Theta}_1\Theta_1\right)\Theta_1'\}$$

$$\mathscr{Q}_2^\alpha = \sqrt{2}\oint\frac{d\sigma}{2\pi}\{(\mathbb{P}^\mu + T\Pi_\sigma)(\Gamma^\mu\Theta_2)^\alpha + 2iT\left(\bar{\Theta}_2\Theta_2\right)\Theta_2'\}$$

$D = 3, \mathcal{N} = 2$ Summary of Results

Follow talk Padua: Paul K. Townsend - 11.

- Quantize in the light-cone gauge $\Longrightarrow$ bosonic annihilation operators $(\alpha_n, \tilde{\alpha}_n)$, and fermionic annihilation operators $(\xi_n, \tilde{\xi}_n)$.
- The following 'odd' operator plays a crucial role:

$$\Xi \ \propto \ \sum_{n=0}^{\infty} \left[\left(\alpha_n \xi_n^\dagger + \alpha_n^\dagger \xi_n \right) + \left(\tilde{\alpha}_n \tilde{\xi}_n^\dagger + \tilde{\alpha}_n^\dagger \tilde{\xi}_n \right) \right]$$

Ξ squares to the even mass-squared operator M^2(using level-matching constraint), so it determines the spectrum.

- Ξ commutes with super-Casimir $\mathcal{P} \cdot \mathcal{J} + \frac{i}{4} \bar{\mathcal{Q}} \mathcal{Q}$, $\Rightarrow$ spectrum is super-Poincaré invariant $\Longrightarrow$ No Super-Poincaré Anomalies

- Spectrum
- 2 fermionic zero modes $\Rightarrow$ 4 massless ground states at level $N = 0$: 2 bosons and 2 fermions.
- All other states are massive. At level $N = 1$ we get 4 copies of the scalar supermultiplet with helicities $(-1/2,\, 0,\, 0,\, 1/2)$
- At level $N = 2$ we get 8 copies of scalar supermultiplet plus 4 copies of spin-2 supermultiplet $(1,\, 3/2,\, 3/2,\, 2)$ and its parity conjugate $(-2,\, -3/2,\, -3/2,\, -1)$
- At level $N = 3$ we get another 8 copies of the scalar super-multiplet. Remaining 28 + 28 supermultiplets all have irrational helicities $\Rightarrow$ Lorentz group is the infinite cover of $SO(1,2)$

Hamiltonian Formulation - Open (Super)string

work in progress with Alisdair Routh and Paul K. Townsend - 12.

- Same Action as Closed String but it has Ends!

General Variation: a Bulk term zero on shell, and a Boundary term:

$$\delta S_{\mathrm{on-shell}} = -\frac{1}{2\pi}\int d\tau\,\left[\left(T^2\ell\mathbb{X}' + u\mathbb{P}\right)\cdot\delta\mathbb{X}\right]_0^{2\pi}$$

which must vanish due to the bc's. For fixed endpoints: $\delta\mathbb{X} = 0$

We do not fix $\mathbb{X}^0$, so we must have:

$$\left[T^2\ell\mathbb{X}_0' + u\mathbb{P}_0\right]_0^{2\pi} = 0$$

Its α variation is: $\left[\dot{\alpha}\mathbb{X}_0' + \alpha\left(u\mathbb{X}_0' + \ell\mathbb{P}_0\right)'\right]_0^{2\pi} = 0$

Then we demand that ℓ , $\mathbb{P}^0$, and α are free variables, parameters

$$\mathbb{X}_0'|_{\mathrm{ends}} = 0\,,\ \left(\ell\mathbb{P}_0\right)'|_{\mathrm{ends}} = 0,\ u|_{\mathrm{ends}} = 0\,.$$

then: $\mathbb{X}_0(\tau,\sigma) = \sum_{n=0}^{\infty}\cos\left(n\sigma/2\right)X_0^{(n)}$, $u(\tau,\sigma) = \sum_{n=1}^{\infty}\sin\left(n\sigma/2\right)u_{(n)}(\tau)$

It follows that $\mathbb{X}_0'''|_{\text{ends}} = 0$, and $u''|_{\text{ends}} = 0$. There remains to check the β gauge invariance of all the bc's and the α gauge invariance of the new set of bc's. Then with the assumption that u' , $\mathbb{X}_0''$ and α are also free (unrestricted!) variables, parameters, we end up with the following system of gauge invariant bc's under the restricted diff transformations:

$$\mathbb{X}_0'|_{\text{ends}} = 0\,, \mathbb{P}_0'|_{\text{ends}} = 0,\, u|_{\text{ends}} = 0\,, \ell'|_{\text{ends}} = 0\,,\ \beta|_{\text{ends}} = 0\,,$$

$$\alpha'|_{\text{ends}} = 0\,.$$

So far we discussed only the implications of the fact that $\mathbb{X}^0$ is not fixed, taking this into account in the variation of the action

$$\delta S_{\text{on-shell}} = -\frac{T^2}{2\pi}\int d\tau\,\left[\ell\vec{\mathbb{X}}'\cdot\delta\vec{\mathbb{X}}\right]_0^{2\pi}\,.$$

With $*$ denoting a generic coordinate we are going to consider:

$$\text{Free end: } \mathbb{X}'_*|_{\text{end}} = 0 \text{ ~ Neumann,}$$
$$\text{Fixed end: } \mathbb{X}_*|_{\text{end}} = R_* \text{ ~ Dirichlet.}$$

Demand Now Gauge Invariance of the Constraints end up with:

$$\text{Free end: } \mathbb{X}'_*|_{\text{end}} = 0 \text{ ~ Neumann, } \mathbb{P}'_*|_{\text{end}} = 0 \ ,$$
$$\text{Fixed end: } \mathbb{X}_*|_{\text{end}} = R_* \text{ ~ Dirichlet, } \mathbb{P}_*|_{\text{end}} = 0$$

- Hamiltonian Formulation permits Systems of Gauge Invariant bc's.
- $D=3, \mathcal{N}=2$ Open Superstring action will have a corresponding boundary term in $\delta S_{\text{on-shell}}$. I will present two systems of gauge invariant (modified diff α and β transformation, κ transformation)

Common bc's

$$\mathbb{X}_0'|_{\text{ends}} = 0\,, \mathbb{P}_0'|_{\text{ends}} = 0,\ u|_{\text{ends}} = 0\,, \ell'|_{\text{ends}} = 0\,,\ \ \beta|_{\text{ends}} = 0\,,$$

$$\alpha'|_{\text{ends}} = 0\,.$$

Free ends:

$$\vec{\mathbb{X}}'|_{\text{ends}} = 0,\ \ \vec{\mathbb{P}}'|_{\text{ends}} = 0,$$

$$(\Theta_1 - \Theta_2)\,|_{\text{ends}} = 0,\ \ (\Theta_1' + \Theta_2')\,|_{\text{ends}} = 0,$$

$$(\kappa_1 - \kappa_2)\,|_{\text{ends}} = 0,\ \ (\kappa_1' + \kappa_2')\,|_{\text{ends}} = 0$$

Fixed ends:

$$\vec{\mathbb{X}}|_{\sigma=0} = \vec{0}\,, \vec{\mathbb{X}}|_{\sigma=2\pi} = \vec{R}(0,R),\ \ \vec{\mathbb{P}}|_{\text{ends}} = 0.$$

$$(\Theta_1 - \Gamma^0\Theta_2)\,|_{\text{ends}} = 0,\ \ (\Theta_1' + \Gamma^0\Theta_2')\,|_{\text{ends}} = 0,$$

$$(\kappa_1 - \Gamma^0\kappa_2)\,|_{\text{ends}} = 0,\ \ (\kappa_1' + \Gamma^0\kappa_2')\,|_{\text{ends}} = 0$$

Arvis & LC Gauges

Quantisation of the bosonic String
Dirichlet bc's - New Gauge

J. F. Arvis - 83.

Heuristics: In order to reduce the the # of degrees of freedom one demands cancellations in the Fourier expansions of the corresponding entities.

Ling-Yan Hung - 08.

- Closed bosonic string: all fields are periodic, one demands that

$$\mathbb{P}_0 + T\mathbb{X}_1' = p_0(\tau)\,, \qquad \mathbb{P}_1 + T\mathbb{X}_0' = p_1(\tau)\,.$$

These conditions, good gauge conditions because if we are infinitesimally close that is:

$$p_0(\tau) \Rightarrow p_0(\tau) + \epsilon(\sigma,\tau), \qquad p_1(\tau) \Rightarrow p_1(\tau) + \eta(\sigma,\tau)$$

One can determine corresponding gauge transformations provided we can determine α and β from:

$$Tp_1(\tau)\alpha' + p_0(\tau)\beta' = \epsilon(\tau,\sigma)$$

that is: $p_0^2 \neq p_1^2$

$$Tp_0(\tau)\alpha' + p_1(\tau)\beta' = \eta(\tau,\sigma)$$

residual gauge invariance: solutions when $\epsilon = \eta = 0$.

$$\Rightarrow \alpha' = \beta' = 0 \Rightarrow \alpha = \alpha(\tau),\ \beta = \beta(\tau)$$

Now we use the gauge conditions to eliminate $\mathbb{P}_0, \mathbb{P}_1$,

in the action:

$$S[\mathbb{X},\mathbb{P};\ell,u] = \int d\tau \oint \frac{d\sigma}{2\pi} \left\{ \dot{\mathbb{X}}^\mu \mathbb{P}_\mu - \frac{1}{2}\ell \left[\mathbb{P}^2 + (T\mathbb{X}')^2\right] - u\,\mathbb{X}'^\mu \mathbb{P}_\mu \right\}$$

where we make, decompositions of the periodic functions:

$$\mathbb{X},\mathbb{P} : \mathbb{A} = a + \bar{\mathbb{A}},\ u = u_0 + \bar{u},\ \ell = \ell_0 + \bar{\ell}$$

where a , u_0 , and ℓ_0 are corresponding zero modes

$$S=\int d\tau\Big\{\dot{x}^\mu p_\mu+\oint\frac{d\sigma}{2\pi}\dot{\bar{\mathbb{X}}}_2\bar{\mathbb{P}}_2-\frac{1}{2}\oint\frac{d\sigma}{2\pi}\,\ell\Big[-p_0^2+p_1^2+\mathbb{P}_2^2+(T\mathbb{X}_2')^2\Big]$$

$$+\oint\frac{d\sigma}{2\pi}\Big(-u\,\mathbb{X}_2'\mathbb{P}_2+\Big[\bar{\mathbb{X}}^0\,(Tp_1\ell'+p_0u')+\bar{\mathbb{X}}^1\,(Tp_0\ell'+p_1u')\Big]\Big)\Big\}$$

The variables $(\bar{\mathbb{X}}^0,\bar{\mathbb{X}}^1,\bar{\ell},\bar{u})$, are auxiliary and can be eliminated

$(\bar{\mathbb{X}}_0,\bar{\mathbb{X}}_1)\quad\Rightarrow\ell'=u'=0,\Rightarrow u=u_0,\ \ell=\ell_0,\Rightarrow$

$$S=\int d\tau\Big\{\dot{x}^\mu p_\mu+\oint\frac{d\sigma}{2\pi}\dot{\bar{\mathbb{X}}}_2\bar{\mathbb{P}}_2-\frac{1}{2}\ell_0[p^2+\mathscr{M}^2]-u_0\oint\frac{d\sigma}{2\pi}\bar{\mathbb{X}}_2'\bar{\mathbb{P}}_2\Big\}$$

Where $\mathscr{M}^2=\oint\frac{d\sigma}{2\pi}\Big[\bar{\mathbb{P}}_2^2+\Big(T\bar{\mathbb{X}}_2'\Big)^2\Big]$, and ℓ_0 , corresponds to residual time-reparametrization, while u_0 generates the level matching constraint. One can fix the light-cone gauge for $x^\mu(\tau)$ and the action reduces to the standard light-cone action.

$$(\bar{\ell},\bar{u}) \quad \Rightarrow \quad \begin{pmatrix} \bar{\mathbb{X}}'_0 \\ \bar{\mathbb{X}}'_1 \end{pmatrix} = \frac{1}{-p_0^2+p_1^2} \begin{pmatrix} p_1 & p_0 \\ p_0 & p_1 \end{pmatrix} \begin{pmatrix} \frac{1}{2T}\overline{\left[\mathbb{P}_2^2+(T\mathbb{X}'_2)^2\right]} \\ -\overline{\mathbb{X}'_2\mathbb{P}_2} \end{pmatrix}$$

Fixed ends Gauge Fixing.

$$\mathbb{X}'_0|_{\rm ends}=0\,,\mathbb{P}'_0|_{\rm ends}=0,\; u|_{\rm ends}=0\,,\ell'|_{\rm ends}=0\,,\;\beta|_{\rm ends}=0\,,$$

$$\alpha'|_{\rm ends}=0,\vec{\mathbb{X}}|_{\sigma=0}=\vec{0}\,,\,\vec{\mathbb{X}}|_{\sigma=2\pi}=\vec{R}(0,R),\vec{\mathbb{P}}|_{\rm ends}=\;0.$$

Arvis Gauge: $\quad \mathbb{P}_0+T\mathbb{X}'_1=p_0,\;\; \mathbb{P}_1+T\mathbb{X}'_0=0.$

Good Gauge provided $p_0\neq 0$, residual gauge invariance $\alpha_0(\tau)$

eliminating the auxiliary stuff one gets:

$$S=\int d\tau\left\{\dot{x}^0p_0+\int_0^{2\pi}\frac{d\sigma}{2\pi}\,\dot{\mathbb{X}}_2\mathbb{P}_2-\frac{1}{2}\ell_0\int_0^{2\pi}\frac{d\sigma}{2\pi}\left[-p_0^2+\mathscr{M}^2\right]\right\}$$

$$\mathscr{M}^2=\oint\frac{d\sigma}{2\pi}\left[\mathbb{P}_2^2+\left(T\bar{\mathbb{X}}'_2\right)^2\right],\quad p_0\mathbb{X}'_0=\mathbb{X}'_2\mathbb{P}_2\,,\quad 2Tp_0\mathbb{X}'_1=p_0^2-\left[\mathbb{P}_2^2+(T\mathbb{X}'_2)^2\right].$$

Free ends gauge fixing - No Arvis Gauge LC - good Gauge

One free end, one fixed end No gauge known!

- $D=3, \mathcal{N}=2$ Open Superstring Fixed ends:

  $3D\ D0$-brane, its bc's:

$$\mathbb{X}_0'|_{\text{ends}}=0\,,\mathbb{P}_0'|_{\text{ends}}=0\,,\vec{\mathbb{X}}|_{\sigma=0}=\vec{0}\,,\vec{\mathbb{X}}|_{\sigma=2\pi}=\vec{R}(0,R),\vec{\mathbb{P}}|_{\text{ends}}=0.$$

$$u|_{\text{ends}}=0\,,\ell'|_{\text{ends}}=0\,,\ \left(\Theta_1-\Gamma^0\Theta_2\right)|_{\text{ends}}=0,\ \left(\Theta_1'+\Gamma^0\Theta_2'\right)|_{\text{ends}}=0,$$

$$\left(\kappa_1-\Gamma^0\kappa_2\right)|_{\text{ends}}=0,\ \left(\kappa_1'+\Gamma^0\kappa_2'\right)|_{\text{ends}}=0,\ \beta|_{\text{ends}}=0\,,\alpha'|_{\text{ends}}=0\,.$$

Incidentally the $3D\ D0$ -brane, preserves half of the supersymmetry. If the ends are fixed at the same point then rotational invariance in the plane is preserved and this becomes a standard SUSY Quantum Mechanics with two supercharges. If these strings allow an interaction, it is natural to interpret the end points as $D0$-branes, which would be described by centrally charged $\mathcal{N}=2$ super particle with a mass M saturating the Bogomolnyi bound $M\geq|Z|$ which follows from

$$\{Q^a_\alpha, Q^b_\beta\} = \delta^{ab}(\Gamma^\mu C)_{\alpha\beta}\mathbb{P}_\mu + M\varepsilon^{ab}\varepsilon_{\alpha\beta}$$

Next we will compute the spins of the corresponding superparticle action.

I should also mention that I am skipping over lots of details:

Arvis Gauge:

$$\mathbb{P}_0 + T\Pi_{\sigma 1} = p_0,\ \mathbb{P}_1 + T\Pi_{\sigma 0} = 0,\ \Gamma^+\Theta_1 = 0,\ \Gamma^-\Theta_2 = 0.$$

Show it is a good gauge, study residual gauge invariance eliminate momenta from the action ..

Massive $D=3$, $\mathcal{N}=2$ Superparticle -Hamiltonian Form

- Brink Schwarz Action $S[\mathbb{X},\mathbb{P}] = \int d\tau \Pi^{\mu}_{\tau}\mathbb{P}_{\mu} - \ell\left(\mathbb{P}^2 + m^2\right) + 2im\bar{\Theta}^1\dot{\Theta}^2$

where: $\Pi^{\mu}_{\tau} = \dot{\mathbb{X}}^{\mu} + i\bar{\Theta}^a\Gamma^{\mu}\dot{\Theta}^a$ SUSY $\delta\mathbb{X}^{\mu} = -i\bar{\epsilon}^a\Gamma^{\mu}\Theta^a,\ \delta\Theta^a = \epsilon^a$

κ - symmetry

$$\delta_\kappa\mathbb{X}^{\mu} = i\bar{\kappa}^a\Gamma^{\mu}\Theta^a \qquad \delta_\kappa\ell = 2i\dot{\bar{\Theta}}^a\kappa^a$$

$$\delta_\kappa\Theta^1 = \Gamma\cdot\mathbb{P}\kappa^1 - m\kappa^2 \qquad \delta_\kappa\Theta^2 = \Gamma\cdot\mathbb{P}\kappa^2 + m\kappa^1$$

Reparametrization Invariant $\delta\mathbb{X} = \alpha(\mathbb{P} - i\bar{\Theta}^a\Gamma\dot{\Theta}^a) \quad \delta\Theta^a = \alpha\dot{\Theta}^a, \quad \delta\ell = \frac{1}{2}\dot{\alpha}$

Noether Charges

$$\mathcal{P}_{\mu} = \mathbb{P}_{\mu},\ \mathcal{J}^{\mu} = [\mathbb{X}\wedge\mathbb{P}]^{\mu} + im\bar{\Theta}^1\Gamma^{\mu}\Theta^2 + \frac{i}{2}\bar{\Theta}^a\Theta^a\mathbb{P}^{\mu}$$

$$Q^1 = \sqrt{2}(\Gamma\cdot\mathbb{P}\Theta^1 + m\Theta^2),\ Q^2 = \sqrt{2}(\Gamma\cdot\mathbb{P}\Theta^2 - m\Theta^1)$$

LC $\quad \Gamma^{+}\Theta^a = 0, \quad x^{+} = \tau, \quad \Theta^a = \frac{1}{\sqrt{2\sqrt{2}p_{-}}}\begin{pmatrix}\vartheta^a \\ 0\end{pmatrix}, p_{+} = -H = \frac{1}{2p_{-}}(p^2 + m^2)$

$$L = \dot{x}^{-}p_{-} + \dot{x}p + \frac{i}{2}\vartheta^a\dot{\vartheta}^a - H$$ Normal Canonical System

Quantization: $\quad [x,p] = [x^{-},p_{-}] = i, \quad (\vartheta^1)^2 = (\vartheta^2)^2 = \frac{1}{2}$

- SUSY: Central Charge $\{Q^a_\alpha, Q^b_\beta\} = \delta^{ab}(\Gamma^\mu C)_{\alpha\beta}\mathbb{P}_\mu + m\varepsilon^{ab}\varepsilon_{\alpha\beta}$

$C = \begin{pmatrix} 0, 1 \\ -1, 0 \end{pmatrix}, \mathcal{P}^2 = -m^2$ Helicity Operator , $\mathcal{P} \cdot \mathcal{J} = i\frac{m}{2}\vartheta^1\vartheta^2$

$\vartheta^1 = \frac{1}{\sqrt{2}}\sigma_3 \otimes \tau_1, \ \vartheta^2 = \frac{1}{\sqrt{2}}\sigma_3 \otimes \tau_3 \qquad \frac{\mathcal{P} \cdot \mathcal{J}}{m} = \frac{1}{4}(1 \otimes \tau_2)$

Quantization of Brink Schwarz Superparticle leads to Anyons!

helicity: 1/4 , that is semions!

Now if the effective action for a pair of such $D0$ -branes is in the non-relativistic limit, a supersymmetric matrix mechanics with two SUSYs, as it is suggested by critical superstring theory, then the spin-1/4 nature of the $D0$ -branes should also be apparent from their statistical phase $\exp(i2\pi s)$, under their interchange.

The model of supersymmetric matrix mechanics was considered by

C. Pedder, J. Sonner, and D. Tong ~ 08.

who computed the statistical phase as a Berry phase and the spin is 1/4

Conclusions

- Presented the Hamiltonian form of the Closed String. Showed absence of Lorentz anomalies in LC in $D = 3$. Outlined the spectrum and helicity content -irrational anyons
- Oulined the Hamiltonian form of $D = 3, \mathcal{N} = 2$, Superstring and persistence of irrational helicities.
- Hamiltonian Form of the Open String leads systems of gauge invariant bc'c.
- For Superstrings we also manufactured systems of bc's invariant under kappa-transformations.
- Emphasized the use of Arvis Gauge for closed and fixed ends cases. Comments regarding the $D = 3, \mathcal{N} = 2$ superstrings with fixed ends.