Can the supersymmetric backgrounds be classified?

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Branes, Supergravity and M-theory
Paul K Townsend 60th Birthday conference

Cambridge, July 2012

R/95/2
9501068
January, 1995

The eleven-dimensional supermembrane revisited

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ABSTRACT

It is argued that the type IIA 10-dimensional superstring theory is actually a compactified 11-dimensional supermembrane theory in which the fundamental supermembrane is identified with the the solitonic membrane of 11-dimensional supergravity. The charged extreme black holes of the 10-dimensional type IIA string theory are interpreted as the Kaluza-Klein modes of 11-dimensional supergravity and the dual sixbranes as the analogue of Kaluza-Klein monopoles. All other p-brane solutions of the type IIA superstring theory are derived from the 11-dimensional membrane and its magnetic dual fivebrane soliton.

What does classification mean?

- Solve the Killing spinor equations
  Solution to the KSE means to express the fluxes in terms of the geometry and determine the conditions on the geometry imposed by supersymmetry
- Solve the field equations and Bianchi identities
  Some of the field equations and Bianchi identities are implied by the KSEs but not all. To find solutions the remaining have to be imposed in addition to the KSEs
- Boundary conditions, global properties
  These are crucial for the physical interpretation of the solutions, ie string and M-theory solitons, (flux) compactifications, AdS/CFT gravitational duals, black holes,

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What are the rules?

- Solve the equations in general without additional restrictions on the fields apart from those imposed by the KSE and field equations
- Consider theories with the most general field content and matter couplings: in lower dimensions such models emerge from string and M-theory compactifications
- Explore all fractions of supersymmetry that can be preserved

Solution of KSEs

The KSEs of supergravities take the form

$$\begin{aligned} \mathcal{D}\epsilon \equiv \nabla\epsilon + \Sigma(F,g)\epsilon &= 0 \\ \mathcal{A}(F,g)\epsilon &= 0 \end{aligned}$$

- The gravitino KSE is a parallel transport equation. The holonomy of $\mathcal{D}$ for generic backgrounds is in GL rather than in SO for connection like Levi-Civita
  - Standard techniques, like a Berger type of theory, do not apply
- The KSEs have a gauge symmetry which includes $Spin(n-1,1)$ for n-dimensional theories and $Spin(n-1,1)\times G$ for gauged supergravities
- The KSEs can always be solved in all cases!
  But the quality of the solutions depends on the simplicity of the orbits of the gauge group on the space of spinors and their products [Gran, Gutowski, GP, Roest].

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Example

Consider the KSE of a Euclidean 6-dimensional gauge theory

$$F_{CD}\Gamma^{CD}\epsilon = 0$$

Since $Spin(6) = SU(4)$ and ϵ in the **4** or $\bar{\mathbf{4}}$ (Weyl) representations.

- The **4** representation is identified with $\Lambda^{\mathrm{ev}}(\mathbb{C}^3)$ and the gamma matrices are

$$\Gamma_\alpha = \sqrt{2}e_\alpha\wedge\,, \;\; \Gamma_{\bar\alpha} = \sqrt{2}e_\alpha \lrcorner\,, \quad \Gamma_\alpha\Gamma_{\bar\beta} + \Gamma_{\bar\beta}\Gamma_\alpha = 2\delta_{\alpha\bar\beta}$$

- The gauge group is $SU(4)$ and has a single type of non-trivial orbit on **4** with isotropy group $SU(3)$, and so ϵ can be chosen as $\epsilon = 1$. This leads to a linear system

$$F_{CD}\Gamma^{CD}1 = F_{\bar\alpha\bar\beta}\Gamma^{\bar\alpha\bar\beta}1 + \delta^{\alpha\bar\beta}F_{\alpha\bar\beta}1 = 0 \Longrightarrow F_{\bar\alpha\bar\beta} = 0\,, \quad \delta^{\alpha\bar\beta}F_{\alpha\bar\beta} = 0$$

- Although $\epsilon = 1$ is the optimal choice, the calculation could have been done with $\epsilon = a1 + b^{\alpha\beta}e_{\alpha\beta}$ and the linear system would have had a solution! But the $SU(3)$ covariance would have been hidden!

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$D = 4, \mathcal{N} = 1$ supergravity

KSEs of lower dimensional supergravities have been extensively investigated, starting from those of Maxwell-Einstein systems in 4D [Tod] and 5D [Gauntlett, Gutowski, Hull, Pakis, Reall].

Consider 4D $\mathcal{N} = 1$ supergravity with any number of scalar and (non-abelian)vector multiplets and general couplings.

- The Dirac spinors are $\Lambda^*(\mathbb{C}^2)$.
- $\mathrm{hol}(\mathcal{D}) \subseteq Pin_c(3,1)$; gauge group $Spin_c(3,1) = SL(2,\mathbb{C}) \cdot U(1)$.

The solution of the KSEs yields [Gran, Gutowski, GP]

- $N = 1$, Killing spinor $\epsilon_1 = 1 + e_1$: solutions admit a null Killing vector field and all other conditions on the geometry are known.
- $N = 2$,
  - Killing spinor $\epsilon_2 = a\,1 + \bar{a}\,e_1$: solutions are pp-waves, ie admit a parallel null vector field, and include the stringy cosmic strings.
  - Killing spinors $\epsilon_2 = b\,e_{12} - \bar{b}\,e_2$: solutions are domain walls with sections either AdS_3 or $\mathbb{R}^{2,1}$.
- $N = 3$: $N = 3$ backgrounds locally admit 4 supersymmetries.
- $N = 4$: These are locally isometric to either AdS_4 or $\mathbb{R}^{3,1}$

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Heterotic supergravity

Heterotic supergravity: fields, a metric g, a closed 3-form field strength H, $dH = 0$, and dilaton Φ.
The Killing spinor equations of Heterotic supergravities are

$$\hat{\nabla}_\mu \epsilon = \nabla_\mu \epsilon - \frac{1}{8} H_{\mu\nu\rho} \Gamma^{\nu\rho} \epsilon = 0 \,,$$
$$\Gamma^\mu \partial_\mu \Phi \epsilon - \frac{1}{24} H_{\mu\nu\rho} \Gamma^{\mu\nu\rho} \epsilon = 0 \,, \quad \epsilon \in \Delta^+_{\mathbf{16}}$$

- Holonomy of the supercovariant connection: $\mathrm{hol}(\hat{\nabla}) \subseteq Spin(9,1)$.

  $$[\hat{\nabla}, \hat{\nabla}]\epsilon = \hat{R}\epsilon = 0$$

  So either parallel spinors have a non-trivial isotropy group in $Spin(9,1)$ or $\hat{R} = 0$ and the solutions are group manifolds.
- The KSEs of Heterotic supergravity have been solved in all cases [Gran, Lohrmann, GP; Gran, Roest, Sloane, GP].

Solution of KSE for $dH = 0$.

L	$\mathrm{Stab}(\epsilon_1, \dots, \epsilon_L)$	N
1	$Spin(7) \ltimes \mathbb{R}^8$	1
2	$SU(4) \ltimes \mathbb{R}^8$	$-,\ 2$
3	$Sp(2) \ltimes \mathbb{R}^8$	$-,\ -,\ 3$
4	$(\times^2 SU(2)) \ltimes \mathbb{R}^8$	$-,\ -,\ -,\ 4$
5	$SU(2) \ltimes \mathbb{R}^8$	$-,\ -,\ -,\ -,\ 5$
6	$U(1) \ltimes \mathbb{R}^8$	$-,\ -,\ -,\ -,\ -,\ 6$
8	$\mathbb{R}^8$	$-,\ -,\ -,\ -,\ -,\ -,\ -,\ 8$
2	G_2	$-,\ 2$
4	$SU(3)$	$-,\ 2,\ -,\ 4$
8	$SU(2)$	$-,\ 2,\ -,\ 4,\ -,\ 6,\ -,\ 8$
16	$\{1\}$	$8,\ 10,\ 12,\ 14,\ 16$

$SU(3)$

M admits 4 $\hat{\nabla}$-parallel 1-forms λ^a, and a 2-form ω and a (3,0)-form χ, fundamental forms of $SU(3)$.

$$i_a\omega = 0\,, \quad i_a\chi = 0\,, \quad \mathcal{L}_a\omega = 0\,, \quad \mathcal{L}_a\chi = k_a\chi$$

The Lie algebra $\mathfrak{g}$ of vector fields associate to λ^a is

$$\mathbb{R}^{3,1}\,, \quad \mathfrak{sl}(2,\mathbb{R})\oplus\mathbb{R}\,, \quad \mathfrak{su}(2)\oplus\mathbb{R}\,, \quad \mathfrak{cw}_4$$

The spacetime locally is $M = P(G, B^6; \pi)$, Lie $G = \mathfrak{g}$ equipped with connection λ^a and B^6 a Hermitian manifold with metric $d\tilde{s}^2_{(6)}$ and Hermitian form $\omega_{(6)} = \omega$. Then

$$ds^2 = \eta_{ab}\lambda^a\lambda^b + \pi^* d\tilde{s}^2_{(6)}\,, \quad H = CS(\lambda) + \pi^*\tilde{H}_{(6)}\,, \quad \tilde{H}_{(6)} = -i(\partial - \bar{\partial})\omega$$

$\mathfrak{g}$ abelian: B^6 is a Calabi-Yau with torsion, ie hol$(\hat{\tilde{\nabla}}) \subseteq SU(3)$. Moreover

$$\tilde{\theta}_{\omega_{(6)}} = 2\tilde{d}\Phi\,, \quad \partial_a\Phi = 0\,, \quad \mathcal{F} \equiv d\lambda - \lambda^2 \in \mathfrak{su}(3)\,.$$

where $\theta = \star(\omega \wedge \star d\omega)$ is the Lee form.

$\mathfrak{g}$ non-abelian: B^6 is Hermitian and $\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq U(3)$. Moreover

$$\hat{\tilde{\rho}} = k_a \mathcal{F}^a \ , \quad \tilde{\theta}_{\omega_{(6)}} = 2\tilde{d}\Phi \ , \quad \partial_a \Phi = 0 \ , \quad \mathcal{F}^a \in \mathfrak{u}(3) \ .$$

The complex trace of $\mathcal{F}$ is related k which is dual to the structure constants of $\mathfrak{g}$.

- The geometry of the remaining cases is similarly known
- The half supersymmetric solutions associated with $\mathbb{R}^8$ and $SU(2)$ holonomies have been classified [GP]
  - The $\mathbb{R}^8$ solutions are superpositions of pp-waves and fundamental strings propagating in $\mathbb{R}^8$.
  - The $SU(2)$ solutions are constructed from certain instantons on 4-dimensional hyper-Kähler manifolds.

$D = 11$ supergravity

The KSEs are

$$\mathcal{D}_M\epsilon = \nabla_M\epsilon - \left(\frac{1}{288}\Gamma_M{}^{L_1L_2L_3L_4}F_{L_1L_2L_3L_4} - \frac{1}{36}F_{ML_1L_2L_3}\Gamma^{L_1L_2L_3}\right)\epsilon = 0$$

- The Dirac rep of $Spin(10,1)$ is identified with $\Lambda^*(\mathbb{C}^5)$ and a suitable real section is chosen for the Majorana rep
- $\mathrm{hol}(\mathcal{D}) \subseteq SL(32,\mathbb{R})$ [Hull; Duff, Liu]
- There are two types of orbits of $Spin(10,1)$ in the Majorana rep with isotropy groups $SU(5)$ and $Spin(7) \ltimes \mathbb{R}^9$. [Bryant; Figueroa-O'Farrill]. Representatives are $1 + e_{12345}$ and $1 + e_{1234}$, respectively.
- The isotropy group of two generic spinors in $Spin(10,1)$ is $\{1\}$.

How much susy is preserved?

- As $\mathrm{hol}(\mathcal{D}) \subseteq SL(32, \mathbb{R})$, any number of spinors has a non-trivial isotropy group in $SL(32, \mathbb{R})$ and so backgrounds preserving any number of supersymmetries may be allowed.
- The supersymmetry algebra with brane charges [Townsend] is

$$\{Q_\alpha, Q_\beta\} = \Gamma^M_{\alpha\beta} P_M + \Gamma^{M_1 M_2}_{\alpha\beta} b_{M_1 M_2} + \Gamma^{M_1 \dots M_5}_{\alpha\beta} b_{M_1 \dots M_5}$$

As there is an isomorphism between the space symmetric bi-spinors and the space of 1- 2- and 5-forms, brane charges can always be chosen such that there are states which can preserve any number of supersymmetries.
- The number of supersymmetries preserved [Duff] are

1, 2, 3, 4 , 5, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 32

- Backgrounds preserving more than 24 supersymmetries are homogenous [Figueroa-O'Farrill].

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Maximal Supersymmetry

For maximal supersymmetry

$$[\mathcal{D}_M, \mathcal{D}_N]\epsilon = \sum_n \mathcal{R}^{[n]}_{MN}\Gamma^{[n]}\epsilon = 0 \Longrightarrow \mathcal{R}^{[n]}_{MN} = 0$$

Then

$$\mathcal{R}^{[n]}_{MN} = 0 \Longrightarrow \nabla F = 0\ , \quad R \sim F^2\ , \quad i_X i_Y i_Z F \wedge F = 0$$

- The last equation is a Plücker relation and implies that F is simple

The maximally supersymmetric backgrounds [Figueroa-O'Farrill, GP] are locally isometric to

$$AdS_4 \times S^7\ (F^2 < 0)\ ,\ \ AdS_7 \times S^4\ (F^2 > 0)$$
$$\text{plane wave}\ (F^2 = 0)\ ,\ \ R^{10,1}\ (F = 0)$$

$N = 31$ Supersymmetry

The Killing spinors span a hyperplane which has a normal ν in the space of spinors. Using the gauge symmetry set either $\nu = 1 + e_{12345}$ or $\nu = 1 + e_{1234}$. Then choose η^i such that

$$\epsilon^i = A^i{}_j \eta^j \,, \quad \det A \neq 0 \,, \quad (\nu, \eta^i) = 0$$

Then

$$[\mathcal{D}_M, \mathcal{D}_N]\epsilon^i = A^i{}_j \sum_n \mathcal{R}^{[n]}_{MN} \Gamma^{[n]} \eta^j = 0 \Longrightarrow \sum_n \mathcal{R}^{[n]}_{MN} \Gamma^{[n]} \eta^j = 0$$

Using also the field equations and Bianchi identities

$$\Gamma^M \mathcal{R}_{MN} = 0 \;\text{ and }\; \sum_n \mathcal{R}^{[n]}_{MN} \Gamma^{[n]} \eta^j = 0 \Longrightarrow \mathcal{R}^{[n]}_{MN} = 0$$

- All backgrounds with 31 supersymmetries are maximally supersymmetric [Gran, Gutowski, GP, Roest]
- All solutions with 30 superymmetries are also maximally supersymmetric [Gran, Gutowski, GP]

$N \geq 1$ Supersymmetry

- The KSEs for $N = 1$ backgrounds have been solved [Gauntlett, Pakis, Gutowski] using spinor bilinears.
  In the spinorial approach [Gillard, Gran, GP], the KSEs are solved utilizing the representatives of the two orbits
  • $\epsilon = f(1 + e_{12345})$: Solutions admit a time-like Killing vector field and its orbit space has a $SU(5)$ structure
  • $\epsilon = 1 + e_{1234}$: Solutions admit a null Killing vector field, the spacetime has a $Spin(7) \ltimes \mathbb{R}^9$ structure and include backgrounds like pp-wave propagating in a $Spin(7)$ manifold.
- There are two classes of $N \geq 2$ solutions
  • Killing spinors with non-trivial isotropy group in $Spin(10, 1)$: Spacetime admits one or more Killing vector field and structure group the isotropy group of the spinors
  • Killing spinors with trivial isotropy group in $Spin(10, 1)$: Spacetime again admits one or more Killing vector fields and the tangent bundle is trivial. Backgrounds probably severely restricted.

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Isotropy groups

The isotropy groups of Majorana spinors in $Spin(10,1)$ are subgroups of either $SU(5)$ or $Spin(7) \ltimes \mathbb{R}^9$.

N_H	H
1	$Spin(7) \ltimes \mathbb{R}^9$
2	$Spin(7), SU(5), SU(4) \ltimes \mathbb{R}^9, G_2 \times \mathbb{R}^9$
3	$Sp(2) \times \mathbb{R}^9$
4	$SU(4), G_2, SU(2) \times SU(3), (SU(2)^2) \ltimes \mathbb{R}^9, SU(3) \ltimes \mathbb{R}^9$
5	$SU(2) \ltimes \mathbb{R}^9$
6	$Sp(2), U(1) \ltimes \mathbb{R}^9$
8	$SU(3), SU(2)^2, SU(2) \ltimes \mathbb{R}^9$
10	$SU(2)$
12	$U(1)$
16	$SU(2), \mathbb{R}^9$
32	$\{1\}$

IIB Supergravity

The KSEs are

$$\mathcal{D}_M\epsilon = \nabla_M\epsilon + \Sigma_M(F,g)\epsilon = 0\,, \quad \mathcal{A}\epsilon = 0$$

- The spinors are in the Weyl rep of $Spin(9,1)$ which is identified with $\Lambda^{\rm ev}(\mathbb{C}^5)$.
- $\mathrm{hol}(\mathcal{D}) \subseteq SL(32,\mathbb{R})$ [Tsimpis, GP]
- There are 3 different orbits of $Spin(9,1)$ in the Weyl rep with isotropy groups
  - $Spin(7)\ltimes\mathbb{R}^8$ and spinor representative $1+e_{1234}$ with isotropy groups ,
  - $SU(4)\ltimes\mathbb{R}^8$ and spinor representative $a1+be_{1234}$, a,b complex
  - G_2 and spinor representative $f(1+e_{1234})+ig(e_{15}+e_{2345})$, f,g real
- The isotropy group of two generic spinors in $Spin(9,1)$ is $\{1\}$

Maximal Supersymmetry

For maximal supersymmetry

$$[\mathcal{D}_M, \mathcal{D}_N]\epsilon = \sum_n \mathcal{R}^{[n]}_{MN}\Gamma^{[n]}\epsilon = 0 \Longrightarrow \mathcal{R}^{[n]}_{MN} = 0$$

$$\sum_n \mathcal{A}^{[n]}\Gamma^{[n]}\epsilon = 0 \Longrightarrow \mathcal{A}^{[n]} = 0 \Longrightarrow F_{(1)} = F_{(3)} = 0$$

Then

$$\mathcal{R}^{[n]}_{MN} = 0 \Longrightarrow \nabla F_{(5)} = 0\,, \quad R \sim F^2_{(5)}\,, \quad i_X i_Y i_Z F^M_{(5)} \wedge F_{(5)M} = 0$$

- $F_{(5)}$ are the structure constants of 10-D Lorentzian 4-Lie algebra and the only solution is that $F_{(5)}$ is the sum of two simple 5-forms.

The maximally supersymmetric backgrounds [Figueroa-O'Farrill, GP] are locally isometric to

$$AdS_5 \times S^5\,, \quad \text{plane wave}\,, \quad R^{10,1}\ (F = 0)$$

$N \neq 32$ Supersymmetry

- Backgrounds with near maximal supersymmetry
  - All solutions with $N > 28$ supersymmetries are maximally supersymmetric [Gran, Gutowski, GP, Roest]
  - There is a unique solution, a plane wave [Bena, Roiban], with strictly $N = 28$ supersymmetries
- $N = 1$ solutions: The backgrounds with $Spin(7) \ltimes \mathbb{R}^8$ and $SU(4) \ltimes \mathbb{R}^8$ invariant Killing spinors admit a null Killing vector field while those with a G_2 invariant Killing spinor a time like vector field. The KSEs have been solved in all cases. [Gran, Gutowski, GP]
- $N > 1$ solutions: These again can be separated in those admitting Killing spinors with either a non-trivial or a trivial isotropy group in $Spin(9, 1)$. It is expected that the geometry of the latter is rather restricted

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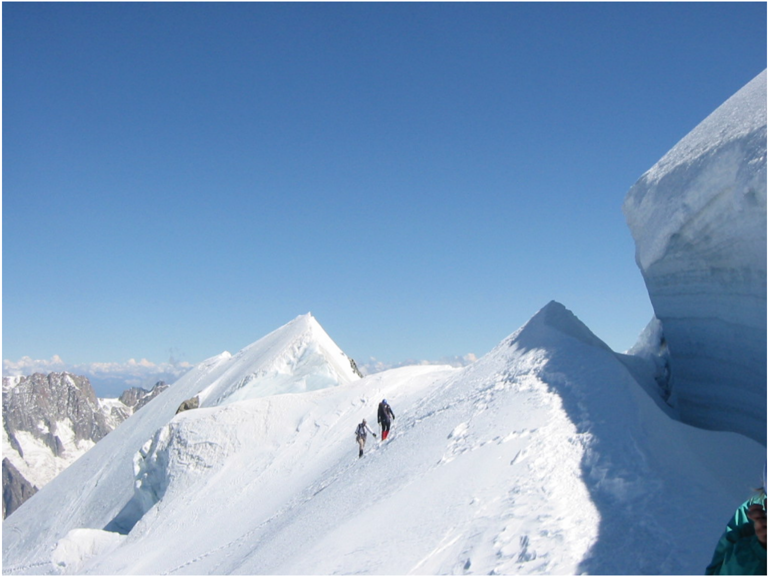
Summary

- The classification of supersymmetric solutions of 10- and 11-D supergravities is not complete but there has been significant progress
- For Heterotic supergravity, the problem has been solved at a satisfactory level but it remains to explore various applications
- In type II and 11-D theories, the KSEs can be solved, and backgrounds with maximal and nearly maximal supersymmetry have been classified. However many aspects remain unresolved, for example there is no proof of the fractions of supersymmetry preserved by supersymmetric backgrounds and an understanding of their geometries
- Many applications have also been explored like the search of AdS/CFT backgrounds, black holes, near horizon geometries, and possibly there will be many more applications in the future.



Branes, supergravity and M-theory - a wonderful journey of ideas

with some challenging sections and Paul's inspiration has been invaluable!

Happy Birthday Paul!