Bottomonium masses and radiative transitions

Lattice NRQCD results for
- M1 transitions among S waves (a refinement of [1])
- masses of S, P, D and F waves (and a glimpse beyond)

This work is being done in collaboration with R. M. Woloshyn.

Radiative transitions in bottomonium

FIG. 2 Transitions among $b\bar{b}$ levels. There are also numerous electric dipole transitions $S \leftrightarrow P \leftrightarrow D$ (not shown). Red (dark) arrows denote objects of recent searches.

$J_{PC}$ are shown at the bottom of each figure. States are often denoted by $2S + 1LJ$, with $L = S, P, D, \ldots$. Thus, $L = 0$ states can be $1S_0$ or $3S_1$; $L = 1$ states can be $1P_1$ or $3P_0, 1, 2$; $L = 2$ states can be $1D_2$ or $3D_1, 2, 3$, and so on. The radial quantum number is denoted by $n$.

III. THEORETICAL UNDERPINNINGS

A. Quarks and potential models

An approximate picture of quarkonium states may be obtained by describing them as bound by an interquark force whose short-distance behavior is approximately Coulombic (with an appropriate logarithmic modification of coupling strength to account for asymptotic freedom) and whose long-distance behavior is linear to account for quark confinement. An example of this approach is found in Eichten et al. (1975, 1976, 1978, 1980); early reviews may be found in Appelquist et al. (1978); Grosse and Martin (1980); Novikov et al. (1978); Quigg and Rosner (1979). Radford and Repko (2007) presents more recent results.

Eichten, Godfrey, Mahlke, Rosner, RMP80, 1161 (2008)
Pseudoscalar/vector M1 transitions in the nonrelativistic quark model require

\[ \mathcal{M}(nS \rightarrow n'S) = \int_0^\infty R_{n'}(r)R_n(r)j_0(qr/2)r^2dr \]

Therefore hindered transitions are subtle: recoil, spin, relativistic, . . .
Qualitative success

Near unity; modest momentum dependence.

Small and negative.

(Improved since publication)
Qualitative success

\( \Upsilon (2S) \rightarrow \eta_b (1S) \)

\( \Upsilon (3S) \rightarrow \eta_b (1S) \)
Quantitative problem

BABAR, PRL 101, 071801 (2008) and BABAR, PRL 103, 161801 (2009)
The PACS-CS configurations

- Iwasaki+clover improved action. We use one ensemble of 192 configurations.
- \( m_u = m_d \gtrsim \) physical (\( m_\pi = 156 \text{ MeV} \)) and \( m_s \gtrsim \) physical (\( m_K = 553 \text{ MeV} \)).
- \( 32^3 \times 64 \) lattices with \( \beta = 1.90 \Rightarrow a = 0.0907(14) \text{ fm} \) and \( L = 32a = 2.9 \text{ fm} \).
- Parameters are set using \( m_\pi, m_K \) and \( m_\Omega \) as input.
Tadpole-improved NRQCD action

\[ H = \frac{-\Delta^{(2)}}{2M_0} - c_1 \frac{(\Delta^{(2)})^2}{8M_0^3} + \frac{c_2}{U_0^4 8M_0^2} ig (\Delta \cdot E - E \cdot \Delta) \]
\[ - \frac{c_3}{U_0^4 8M_0^2} \sigma \cdot (\Delta \times E - E \times \Delta) - \frac{c_4}{U_0^4 2M_0} \sigma \cdot B \]
\[ + c_5 \frac{a^2 \Delta^{(4)}}{24M_0} - c_6 \frac{a(\Delta^{(2)})^2}{16nM_0^2} + O(v^6) \]

The stability parameter \( n \) is algorithmic not physical; we use \( n = 4 \).

Tadpole improvement via average link in Landau gauge: \( U_0 = 0.8463 \).

We use tadpole-improved leading order: \( c_i = 1 \) for all \( i \).

The bottom quark bare mass \( M_0 = 1.945 \) is set by fitting the experimental \( \eta_b \) mass. Specifically, the \( \eta_b \) kinetic energy is used:

\[ E(p) - E(0) = \sqrt{p^2 + M_0^2} - M_0 \]

with the three smallest lattice momenta.
Bottomonium propagation

- 16 random U(1) wall sources per configuration.

- Smearing in Coulomb gauge: (l)ocal, (s)meared, (d)oubly-smeared.

\[ O_{\eta b} = \sum_y \chi(x)\phi(x - y)\psi(y) \]

\[ O_{\Upsilon} = \sum_y \chi(x)\sigma_3\phi(x - y)\psi(y) \]

\[ \phi(r) = \left(1 - \frac{r}{2a_0}\right) \exp\left(\frac{-r}{2a_0}\right) \]

with \( a_0 = 1.4 \) (lattice units).

- Constrained multi-exponential fitting to all times except the source:

\[ g_{oo'}(t) = \sum_{n=1}^{N} c_{o'}(n)c_o(n)e^{-E_n(t_f-t_i)} \]
Bottomonium propagation

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O_{\Upsilon} = \sum_y \chi(x) \sigma_3 \phi(x - y) \psi(y)
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\]
Stability of $\eta_b$ and $\Upsilon$ mass fits
We find $m_\Upsilon - m_{\eta_b} = 56 \pm 1$ MeV (statistical error only).
The PDG average is $69.8 \pm 2.8$ MeV; the recent Belle result is $59.3 \pm 1.9^{+2.4}_{-1.4}$ MeV.
Agreement with the variational method

The variational method solves the eigenvalue problem on each time step.

\[ g(t)f_k(t) = \lambda_k(t)g(t_0)f_k(t) \]

where \( g(t) \) is the correlator matrix.

Black symbols are variational. Horizontal lines are 10-term multi-state fits.

Variational results would become more precise with more operators.
Leptonic decay of $\Upsilon$

$$\Gamma[\Upsilon(nS) \rightarrow e^+ e^-] = \frac{16\pi\alpha |\Psi_n(0)|^2}{9 \frac{M^2_{\Upsilon(nS)}}{Z_{\text{match}}} \approx \frac{16\pi\alpha}{9} \frac{c_{\text{local}}^2}{6M^2_{\Upsilon(nS)}}$$

where $\Psi_n(0)$ denotes the wave function at the origin and $Z_{\text{match}}$ relates the lattice vector current to the renormalized continuum current.
Three-point functions

\[\sum_{n} \sum_{n'} c_s^{(V)}(n) A_{nn'}^{(VP)} c_l^{(P)}(n') \exp \left(-E_n^{(V)}(t' - t^{(V)})\right) \exp \left(-E_{n'}^{(P)}(t^{(P)} - t')\right)\]

- \(A_{nn'}^{(VP)}\) is the matrix element of interest.
- Two-point \(c\) and \(E\) values are retained.
- Source is \(V\) or \(P\), and is \(l\) or \(s\) or \(d\).
  Likewise for sink.
  6 “source,sink” used: \(ll, ls, sl, ss, ld, dl\).
- \(P\) momentum is \((0,0,0), (1,0,0)\) or \((2,0,0)\).
  \(V\) momentum is always zero.
- Current insertion is just a Pauli matrix
  (i.e. leading nonrelativistic term).
Three-point functions

\[ \sum_n \sum_{n'} c_s^{(V)}(n) A_{nn'}^{(VP)} c_l^{(P)}(n') \exp \left( -E_n^{(V)}(t' - t^{(V)}) \right) \exp \left( -E_{n'}^{(P)}(t^{(P)} - t') \right) \]

- Three source-sink time separations:
  \( \Delta t = 19 \) and 27 and 15 (since publication).

- 10-term fits not required.
  We use \( nn' = 11, 12, 21, 13, 31, 22 \).

- Excluding \( nn' = 22 \) causes \( \Delta t = 19 \) and 27 to disagree.

- The time fit range is
  \( t_{src} + 2 < t' < t_{snk} - 2 \).
Three-point functions

with $\Delta t = 19$

$V \rightarrow P$

$P \rightarrow V$
Three-point functions

with $\Delta t = 27$

$V \rightarrow P$

$P \rightarrow V$
## Stability of $A^{(VP)}_{nn'}$ fits

<table>
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<tr>
<th>recoil momentum</th>
<th>$\Delta t$</th>
<th>$N_{cf}$</th>
<th>$A^{(VP)}_{11}$</th>
<th>$A^{(VP)}_{12}$</th>
<th>$A^{(VP)}_{13}$</th>
<th>$A^{(VP)}_{21}$</th>
<th>$A^{(VP)}_{31}$</th>
<th>$A^{(VP)}_{22}$</th>
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<td>0.916(2)</td>
<td>-0.043(7)</td>
<td>-0.069(6)</td>
<td>0.090(7)</td>
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<td>0.915(2)</td>
<td>-0.068(2)</td>
<td>-0.050(4)</td>
<td>0.072(4)</td>
<td>0.065(3)</td>
<td>1.11(31)</td>
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<td></td>
<td>12</td>
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<td></td>
<td>27</td>
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<td>0.916(2)</td>
<td>-0.062(7)</td>
<td>-0.056(7)</td>
<td>0.075(7)</td>
<td>0.059(6)</td>
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<td>0.071(3)</td>
<td>0.062(4)</td>
<td>2.1(2.2)</td>
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<td>0.071(4)</td>
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<td>0.908(1)</td>
<td>-0.042(8)</td>
<td>-0.060(8)</td>
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<td>-0.055(6)</td>
<td>0.116(7)</td>
<td>0.066(6)</td>
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<td>10</td>
<td>0.877(1)</td>
<td>-0.030(4)</td>
<td>-0.041(6)</td>
<td>0.101(5)</td>
<td>0.078(6)</td>
<td>1.01(25)</td>
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<tr>
<td></td>
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<td>0.877(1)</td>
<td>-0.031(4)</td>
<td>-0.041(6)</td>
<td>0.102(5)</td>
<td>0.078(6)</td>
<td>1.02(20)</td>
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<td>27</td>
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<td>0.878(1)</td>
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<td>1.0(1.6)</td>
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</tr>
</tbody>
</table>
Qualitative success, quantitative problem
**Possible improvements**

- matching of the vector current: lattice to renormalized continuum.
- relativistic corrections to the transition operator.
- $O(v^6)$ terms.
- radiative corrections to coefficients in the NRQCD Hamiltonian.
- multiple lattice spacings and a continuum limit.

**Other issues**

- Light quarks are close to their physical values.
- The lattice volume is large compared to the physical system.
Masses of higher angular momentum states of bottomonium

- Which $J^{PC}$ states appear as “ground states” on a lattice?
- Which of those states are accessible with present-day methods and existing configurations?
## Creation operators for “ground states”

<table>
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<tr>
<th>$\Lambda^{PC}$</th>
<th>$J^{PC}$</th>
<th>$2S+1L_J$</th>
<th>$\Omega$</th>
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<tr>
<td>$A_1^{++}$</td>
<td>0$^-$</td>
<td>$^1S_0$</td>
<td>1</td>
</tr>
<tr>
<td>$T_1^{--}$</td>
<td>1$^--$</td>
<td>$^3S_1$</td>
<td>${\sigma_1, \sigma_2, \sigma_3}$</td>
</tr>
<tr>
<td>$T_1^{--}$</td>
<td>1$^--$</td>
<td>$^1P_1$</td>
<td>${\Delta_1, \Delta_2, \Delta_3}$</td>
</tr>
<tr>
<td>$A_1^{++}$</td>
<td>0$^+$</td>
<td>$^3P_0$</td>
<td>$\Delta_1\sigma_1 + \Delta_2\sigma_2 + \Delta_3\sigma_3$</td>
</tr>
<tr>
<td>$T_1^{++}$</td>
<td>1$^+$</td>
<td>$^3P_1$</td>
<td>${\Delta_2\sigma_3 - \Delta_3\sigma_2, \Delta_3\sigma_1 - \Delta_1\sigma_3, \Delta_1\sigma_2 - \Delta_2\sigma_1}$</td>
</tr>
<tr>
<td>$E^{++}$</td>
<td>2$^+$</td>
<td>$^3P_2$</td>
<td>$((\Delta_1\sigma_1 - \Delta_2\sigma_2)/\sqrt{2}, (\Delta_1\sigma_1 + \Delta_2\sigma_2 - 2\Delta_3\sigma_3)/\sqrt{6})$</td>
</tr>
<tr>
<td>$T_2^{++}$</td>
<td>2$^+$</td>
<td>$^3P_2$</td>
<td>${\Delta_2\sigma_3 + \Delta_3\sigma_1, \Delta_3\sigma_1 + \Delta_1\sigma_3, \Delta_1\sigma_2 + \Delta_2\sigma_1}$</td>
</tr>
<tr>
<td>$E^{--}$</td>
<td>2$^-$</td>
<td>$^1D_2$</td>
<td>$((D_{11} - D_{22})/\sqrt{2}, (D_{11} + D_{22} - 2D_{33})/\sqrt{6})$</td>
</tr>
<tr>
<td>$T_2^{--}$</td>
<td>2$^-$</td>
<td>$^1D_2$</td>
<td>${D_{23}, D_{31}, D_{12}}$</td>
</tr>
<tr>
<td>$E^{--}$</td>
<td>2$^-$</td>
<td>$^3D_2$</td>
<td>$((D_{23}\sigma_1 - D_{13}\sigma_2)/\sqrt{2}, (D_{23}\sigma_1 + D_{31}\sigma_2 - 2D_{12}\sigma_3)/\sqrt{6})$</td>
</tr>
<tr>
<td>$T_2^{--}$</td>
<td>2$^-$</td>
<td>$^3D_2$</td>
<td>${(D_{22} - D_{33})\sigma_1 + D_{13}\sigma_3 - D_{12}\sigma_2, (D_{33} - D_{11})\sigma_2 + D_{21}\sigma_1 - D_{23}\sigma_3, (D_{11} - D_{22})\sigma_3 + D_{32}\sigma_2 - D_{31}\sigma_1}$</td>
</tr>
<tr>
<td>$A_2^{--}$</td>
<td>3$^--$</td>
<td>$^3D_3$</td>
<td>$D_{12}\sigma_3 + D_{23}\sigma_1 + D_{31}\sigma_2$</td>
</tr>
<tr>
<td>$A_2^{--}$</td>
<td>3$^--$</td>
<td>$^1F_3$</td>
<td>$D_{123}$</td>
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<tr>
<td>$T_2^{++}$</td>
<td>3$^+$</td>
<td>$^1F_3$</td>
<td>${D_{122} - D_{133}, D_{233} - D_{211}, D_{311} - D_{322}}$</td>
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<td>$A_2^{++}$</td>
<td>3$^+$</td>
<td>$^3F_3$</td>
<td>$(D_{221} - D_{331})\sigma_1 + (D_{332} - D_{112})\sigma_2 + (D_{113} - D_{223})\sigma_3$</td>
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<tr>
<td>$T_1^{--}$</td>
<td>4$^-$</td>
<td>$^1G_4$</td>
<td>${D_{2223} - D_{3332}, D_{3331} - D_{1113}, D_{1112} - D_{2221}}$</td>
</tr>
<tr>
<td>$A_1^{--}$</td>
<td>4$^--$</td>
<td>$^3G_4$</td>
<td>$(D_{2223} - D_{3332})\sigma_1 + (D_{3331} - D_{1113})\sigma_2 + (D_{1112} - D_{2221})\sigma_3$</td>
</tr>
<tr>
<td>$E^{++}$</td>
<td>5$^--$</td>
<td>$^1H_5$</td>
<td>$((D_{23111} - D_{13222})/\sqrt{2}, (D_{23111} + D_{13222} - 2D_{12333})/\sqrt{6})$</td>
</tr>
<tr>
<td>$A_2^{++}$</td>
<td>6$^-$</td>
<td>$^1I_6$</td>
<td>$D_{112222} + D_{223333} + D_{331111} - D_{221111} - D_{332222} - D_{113333}$</td>
</tr>
<tr>
<td>$A_1^{--}$</td>
<td>9$^--$</td>
<td>$^1L_9$</td>
<td>$D_{12233333} + D_{23331111} + D_{31112222} - D_{13322222} - D_{21113333} - D_{32221111}$</td>
</tr>
</tbody>
</table>
Simulation details

- same PACS-CS ensemble (198 configurations)
- 64 random-U(1) wall sources per configuration
- gauge-invariant smearing: \( \psi(x) \rightarrow (1 + 0.15\Delta^2)^{8s} \psi(x) \) with \( s = 0, 1, 2 \)
- stout links (Morningstar & Peardon, 2004) for F-wave operators
- a generalized multi-exponential fit:

\[
g(t - t_0) = \sum_{n=1}^{N'} \sum_{s=0}^{2} \sum_{s'=0}^{2} f_s(n) f_{s'}(n) e^{-E_n(t-t_0)} + \sum_{n=N'+1}^{N} \sum_{s=0}^{2} \sum_{s'=0}^{2} f_{s,s'}(n) e^{-E_n(t-t_0)}
\]
Sample $E^{-\cdots}$ correlation functions.
(The lightest meson is $^{3}D_{2}$.)

![Graph showing $E^{-\cdots}$ correlation functions for different values of $s$ and $s'$. The graph includes data points for $s = s' = 0$, $s = s' = 1$, and $s = s' = 2$. The horizontal axis represents Euclidean time, and the vertical axis represents the magnitude of the correlation functions on a logarithmic scale.)
Sample $T_2^{+-}$ correlation functions.
(The lightest meson is $^1F_3$.)
“ground state” bottomonium spectrum
Lattice (with statistical errors) and experiment.

preliminary G wave result: mass($T_{1^{-+}}$) = 10.75±0.07 GeV/$c^2$
quark model expectation: mass(G wave) = 10.52 GeV/$c^2$

(Quarkonium Working Group, hep-ph/0412158, figure 4.10.)
Conclusions

masses:
• A set of quark-antiquark operators for all lattice irreps, $\Lambda^\text{PC}$, has been constructed. These correspond to the 16 bottomonium “ground states” for a lattice simulation, so they are a natural starting point for numerical studies.
• S, P, D and F waves are observed. A first look at a G wave suggests it is also within reach with present-day methods and existing gauge configurations.

M1 transitions:
• These decays are sensitive to a variety of small effects and are thus a valuable challenge for lattice simulations.
• The observed qualitative success is encouraging.
• The observed quantitative discrepancies (relative to experiment) provide the opportunity for future progress.