Heavy quark expansion in QCD
and $b \to c$ near zero recoil

N. Uraltsev
The $1/m_Q$ expansion in QCD can be constructed for a few important cases where we now know quite a bit about its terms.

The key information is provided by the Small Velocity heavy quark sum rules, including spin sum rules first established for heavy quarks in QCD.

Allowed to predict values of $\bar{\Lambda}$, $\mu^2_\pi$, $\rho^3_D$, ... based on the hyperfine mass splitting $M_{B^*} - M_B \approx 47$ MeV.
A similar analysis has been extended motivated by the formfactor $F(0)$ in $B \rightarrow D^* \ell \nu$ near zero recoil

Arrived at three apparently isolated, yet linked through the HQE, observations:

- Large negative *inelastic* corrections to $F(0)$ driving it down to $F(0) \approx 0.86$
- Large value of the nonlocal correlators of $\bar{Q} \vec{\pi}^2 Q$ and $\bar{Q} \vec{\sigma} \vec{B} Q$ in $B$ mesons from the hyperfine splitting $\Delta M^2$ in $B$ vs. $D$
  
  The enhanced negative corrections in $F(0)$ are related to the ‘discrepancy’ in the hyperfine splitting ratio between charm and beauty mesons
- This implies enhanced inclusive yield of *radials* in $b \rightarrow c \ell \nu$
  
  Resolve ‘$\frac{1}{2} > \frac{3}{2}$’ paradox
  
  Account for the missing semileptonic channels
  
  Predict significance of the $\frac{3}{2}^+$ ‘$D$-wave’

As a byproduct we find significant corrections to the ground-state factorization; relevant for precision inclusive decays
\( \mathbf{V_{cb} at zero recoil} \)

\[
dw (B \to D^*+\ell\bar{\nu}) \sim G_F^2 \cdot |V_{cb}|^2 \cdot |\bar{p}| \cdot |F_{B\to D^*}(\bar{p})|^2
\]

\(|V_{cb}|\) requires \(F_{B\to D^*}(\bar{p})\) – it is shaped by bound-state physics

At \(\bar{p}=0\) \((\bar{p}_e = -\bar{p}_\nu)\) 
\[\text{almost nothing happened!}\]

Without \textit{isotopic} effects (in the heavy quark limit) \(F_{\bar{p}=0} = 1\):

\[
F_{n/p}(0) = 1 + \frac{0}{m_{c,b}} + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{m_{c,b}^2}\right) + \mathcal{O}\left(\frac{\Lambda_{QCD}^3}{m_{c,b}^3}\right) + \ldots
\]

No \(1/m_{b,c}\)-corrections

(cf. Ademollo-Gatto) \hspace{2cm} \text{(cf. Ademollo-Gatto)}

1986 Voloshin, Shifman
1990 Luke
Challenge to theory: corrections to $F(0) = 1$ are driven by $1/m_c$, potentially significant!

Originally (before 05/1994) were thought to be only about -0.02. Neubert

In fact, deviations from the symmetry limit in QCD are considerably larger in QCD. Shifman, N.U., Vainshtein

There have been folklore around sum rules for $F_{D^*}$ often incorrect...
The QCD approach

\[ T^{\text{zr}}(q_0) = \int d^3 x \int dx_0 \, e^{-i q_0 x_0} \frac{1}{2 M_B} \langle B \left| \frac{1}{3} i T \left\{ \bar{c} \gamma_k \gamma_5 b(x) \bar{b} \gamma_k \gamma_5 c(0) \right\} \right| B \rangle \]

\[ q_0 = M_B - M_D^* - \epsilon \]
The QCD approach

\[ T^{zr}(q_0) = \int d^3 x \int dx_0 \, e^{-iq_0 x_0} \frac{1}{2M_B} \langle B | \frac{1}{3} i T \{ \bar{c} \gamma \gamma_5 b(x) \bar{b} \gamma \gamma_5 c(0) \} | B \rangle \]

\[ q_0 = M_B - M_{D^*} - \epsilon \]

\[ T^{zr}(\epsilon) \] can be calculated in the short-distance expansion at \( |\epsilon| \gg \Lambda_{QCD} \), expansion parameters are \( \frac{\mu_{\text{hadr}}}{\epsilon}, \frac{\mu_{\text{hadr}}}{2m_c + \epsilon}, \frac{\mu_{\text{hadr}}}{2m_b - \epsilon} \) and \( \alpha_s \) at the related scale.

\[ l_0(\mu) = -\frac{1}{2\pi i} \int_{|\epsilon| = \mu} T^{zr}(\epsilon) \, d\epsilon \]

hence OPE for \( l_0(\mu) \)
The QCD approach

\[ T^{\text{zr}}(q_0) = \int d^3 x \int d x_0 \, e^{-i q_0 x_0} \frac{1}{2 M_B} \langle B | \frac{1}{3} i T \{ \bar{c} \gamma_k \gamma_5 b(x) \bar{b} \gamma_k \gamma_5 c(0) \} | B \rangle \]

\[ q_0 = M_B - M_{D^*} - \epsilon \]

\[ l_0(\mu) = -\frac{1}{2\pi i} \int \frac{T^{\text{zr}}(\epsilon)}{\epsilon} d\epsilon \]

\( T^{\text{zr}}(\epsilon) \) can be calculated in the short-distance expansion at \( |\epsilon| \gg \Lambda_{\text{QCD}} \),

expansion parameters are \( \frac{\mu_{\text{hadr}}}{\epsilon} \), \( \frac{\mu_{\text{hadr}}}{2m_c+\epsilon} \), \( \frac{\mu_{\text{hadr}}}{2m_b-\epsilon} \) and \( \alpha_s \) at the related scale

hence OPE for \( l_0(\mu) \)

Using analytic properties we can shrink the contour onto the physical cut \( \epsilon > 0 \), then by optical theorem

\[ l_0(\mu) = \frac{1}{\pi} \int_0^\mu \text{Im} \ T^{\text{zr}}(\epsilon) \, d\epsilon = |F_{D^*}|^2 + \sum_{\epsilon < \mu} |F_{B \rightarrow n}|^2 \]
\[ \sum_{f \neq D^*} |F_{B \rightarrow f}|^2 \equiv \mathcal{W}_{\text{inel}}(\mu) \]

OPE:

\[ F_{D^*} = \sqrt{l_0(\mu) - \mathcal{W}_{\text{inel}}(\mu)} < \sqrt{l_0(\mu)} \]

\[ l_0(\mu) = \xi_{A^{\text{pert}}}(\mu) - \Delta \frac{1}{m_Q^2} - \Delta \frac{1}{m_Q^3} - \Delta \frac{1}{m_Q^4} - \ldots \]

\[ \Delta \frac{1}{m_Q^2} = \frac{\mu_G^2}{3m_c^2} + \frac{\mu_{\pi}^2 - \mu_G^2}{4} \left( \frac{1}{m_c^2} + \frac{1}{m_b^2} + \frac{2}{3m_c m_b} \right) \]

\[ \Delta \frac{1}{m_Q^3} = \frac{\rho_D^3 - \frac{1}{3} \rho_{LS}^3}{4m_c^3} + \frac{\rho_D^3 + \rho_{LS}^3}{6m_c m_b} \left( \frac{1}{m_c} - \frac{1}{2m_b} \right) \]

\ldots
\[ F_{D^*} = \sqrt{\xi_A^{\text{pert}} - \Delta_{\text{power}} - w_{\text{inel}}} \]

\[ \sqrt{\xi_A^{\text{pert}}} \approx 0.98 \text{ at } \mu \approx 0.8\text{ GeV}, \quad -\Delta \frac{1}{m_Q^2} - \Delta \frac{1}{m_Q^3} \approx -0.13 \]

\[ F_{D^*} \leq 0.92 \quad \text{– upper bound} \]

It turns out the sum of the excitation probabilities (wavefunction overlap deficit) should be quite significant,

\[ w_{\text{inel}} \gtrsim 0.14 \quad F_{D^*} \approx 0.86 \quad \text{– prediction} \]
How well is the OPE under control?

- Perturbative corrections: small numerically

\[ \sqrt{\xi_A} \approx 1 - 0.022 + (0.005 - 0.004) + 0.0022 - 0.0014 + \ldots \]

applies only to Wilsonian \( \xi_A^{\text{pert}}(\mu) \)!

\[ \alpha_s = 0.3 \]

2d order BLM
full \( \mathcal{O}(\alpha_s^2) \)
3d order BLM

Assume \( \mu = \varepsilon_M \) around 0.8 GeV

\[ \sqrt{\xi_A} \approx 0.98 \text{ at } \alpha_s(m_b) = 0.22 \]

uncertainty 1% seems reasonably conservative
• Power corrections to $l_0$:

Take the low-end expectation values $\mu_\pi^2 \simeq 0.4$ GeV$^2$, $\rho_D^3 \simeq 0.15$ GeV$^3$:

$$-\Delta \frac{1}{m_Q^2} \simeq -0.095 \quad -\Delta \frac{1}{m_Q^3} \simeq -0.028 \quad -\Delta \frac{1}{m_Q^4} \simeq 0.02 \quad -\Delta \frac{1}{m_Q^5} \simeq 0.01$$

• $\alpha_s$-corrections to the coefficients of power-suppressed terms

Calculated for $\mu_\pi^2$, correction is extremely small. Expect mild effect for $\mu_G^2$; in $1/m_Q^3$ even a 30% renormalization would not produce a significant change

$$\sqrt{1 - \Delta^A} \lesssim 0.94 \pm 0.01$$

the upper bound seems safe at 1% level

Would expect formfactor about 0.92 \textit{if no overlap deficit were there}
Sum of inelastic probabilities

Wavefunction overlap deficit in the language of Quantum Mechanics

Required to turn the upper bound for $F_{D^*}$ into an estimate

$$I_1(\mu) = -\frac{1}{2\pi i} \oint |\epsilon| = \mu T^{\text{zt}}(\epsilon) \epsilon \, d\epsilon = \sum_{\epsilon < \mu} \epsilon_n |F_{B\rightarrow n}|^2$$

$$w_{\text{inel}}(\mu) = \frac{I_1(\mu)}{\bar{\epsilon}(\mu)} \quad \bar{\epsilon}(\mu) - \text{‘average excitation energy’ (up to } \mu)$$

$I_1(\mu)$ is calculated in the OPE similar to $I_0(\mu)$:

$$I_1 = \frac{-(\rho^3_{\pi G} + \rho^3_A)}{3m_c^2} + \frac{-2\rho^3_{\pi\pi} - \rho^3_{\pi G}}{3m_cm_b} + \frac{\rho^3_{\pi\pi} + \rho^3_{\pi G} + \rho^3_S + \rho^3_A}{4} \left( \frac{1}{m_c^2} + \frac{2}{3m_cm_b} + \frac{1}{m_b^2} \right) + \mathcal{O}\left(\frac{1}{m_Q^3}\right)$$

BPS limit $\delta$ BPS $^1$ $\delta$ BPS $^2$
Inelastic piece

\[ I_1^{(\text{BPS})} = \frac{-\left(\rho_{\pi^G}^3 + \rho_A^3\right)}{3m_c^2} + O\left(\frac{1}{m_c^3}\right) \]

\( (\rho_{\pi^G}^3 + \rho_A^3) - \rho_{LS}^3 \) determines hyperfine splitting to order \( 1/m_Q^2 \)

Extract comparing \( B \) and \( D \) mesons

\( \delta^{(2)} I_1 \) is positive; \( \delta^{(1)} I_1 \) comes with small coefficient \( 1/3m_cm_b \) and the minimum is very shallow:

\[ I_1/I_1^{(\text{BPS})} = 1 - (1 - \nu_{3/2}) \frac{m_c^2}{m_b^2} + [...]^2 \quad \nu_{3/2} > 0 \]

\[ \nu_{\text{inel}} = \frac{0.48 \text{ GeV}^3 + \kappa 0.35 \text{ GeV}^3}{3m_c^2 \epsilon_{\text{rad}}} \simeq 14\% \]

\( |\kappa| \lesssim 0.15 \)

A 6% decrease in \( F_{D^*} \)

The way to evaluate \( \sum |F_{B\to n}|^2 \) through \( I_1 \) in the ’t Hooft model yields almost exact number
Excited states should predominantly be resonances (*radial excitations*)
Continuum is $1/N_c$-suppressed and is usually smaller

$D^{(*)}\pi$ inelastic contributions can be estimated for soft pions

The complete estimate yields a significant contribution:

$$g_{D^*D\pi} = 4.9 \text{ GeV}^{-1}$$

$$(\Gamma_D = 96 \text{ KeV})$$

$$g_{B^*B\pi}/g_{D^*D\pi} = 1, 0.8, 0.6, 0.4$$

Altogether we expect $D^{(*)}\pi$ piece to yield around 4% in

$$\sum |F_{B\rightarrow n}|^2,$$

about a fourth of the resonance-based estimate

$$\delta F_{D^*} \simeq -2\%.$$
QCD lower bound:

\[ F_{D^*} < 0.92 \quad F_{D^*} < 0.9 \quad \text{including continuum estimate} \]

The unbiased predicted value

\[ F_{D^*} \lesssim 0.86 \]

The central number has about 2\% accuracy it may lower if \( \mu_\pi^2 \) turns out larger

Central value goes down for increasing \( \mu_\pi^2 \) yet the corrections from higher power terms also increase
$B \to D \ell \nu$ near zero recoil

Experimentally challenging theoretically advantageous

N.U. 2003

\[
\langle D(p_2) | \bar{c} \gamma_\nu b | B(p_1) \rangle = f_+ (p_1 + p_2) \nu + f_- (p_1 - p_2) \nu
\]

A single amplitude

\[
J_0 = (M_B + M_D) f_+(0) + (M_B - M_D) f_-(0)
\]

at $\vec{q} = 0$

HQ limit:

\[
f_+ = \frac{M_B + M_D}{2 \sqrt{M_B M_D}}, \quad f_- = -\frac{M_B - M_D}{M_B + M_D} f_+
\]

\[
\frac{J_0}{2 \sqrt{M_B M_D}} = 1 - a_2 \left( \frac{1}{m_c} - \frac{1}{m_b} \right)^2 - a_3 \left( \frac{1}{m_c} - \frac{1}{m_b} \right)^2 \left( \frac{1}{m_c} + \frac{1}{m_b} \right) + \ldots
\]

Power corrections are well under control and small

Any amplitude with massless leptons depends, however solely on $f_+$,

(only the combination of $f_+$ and $f_-$ has no $1/m$ corrections)

\[
F_+ \equiv \frac{2 \sqrt{M_B M_D}}{M_B + M_D} f_+
\]

has $1/m_Q$ corrections since nothing forbids this in $\vec{J}$

Not a drawback in the era of dynamics
\[ F_+ = 1 + \left( \frac{\Lambda}{2} - \Sigma \right) \left( \frac{1}{m_c} - \frac{1}{m_b} \right) \frac{M_B-M_D}{M_B+M_D} - \mathcal{O} \left( \frac{1}{m_Q}^2 \right) \]

\[ \Lambda = M_B - m_b, \quad \Sigma = \ldots \]

From inclusive decays and exact sum rules we know \( \frac{\Lambda}{2} - \Sigma \) (positive, but small \( \propto \frac{\mu^2 - \mu^2_G}{3\mu_{\text{hadr}}} \))

Moreover, we know all power corrections here are small at small \( \mu^2 - \mu^2_G \)

\[ \frac{2\sqrt{M_B M_D}}{M_B + M_D} f_+(0) = 1.04 \pm 0.01 \pm 0.01 \]

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All orders in \( 1/m \) in ‘BPS’, to \( 1/m^2 \cdot 1/\text{BPS}^2, \ \alpha_s^1 \)

The bulk 3% is the perturbative factor, only 1% comes from power terms
Comprehensive heavy quark expansion can say much for nonperturbative effects in certain cases.

Inequalities or positivity properties are essential, require a physical renormalization scheme. Conceptually similar to the lattice, yet may differ in details.

Exploit physical behavior of the correlators in Minkowsky domain.

- $F_{D^*} \approx 0.86$; uncertainty about 2% at known $\mu_\pi^2$, $\rho_D^3$ plus effect of higher-order power terms. Central value goes down for larger $\mu_\pi^2$, yet higher power corrections become significant.

- Large inelastic contribution decreasing $F_{D^*}$ by 6% close to the BPS estimate.

- Large nonlocal correlators $\rho^3$ from hyperfine splittings ‘Discrepancy’ in the splitting for $B$ and $D$ is settled.

- Large inclusive yield of ‘radials’, $7 \div 10\%$ of $\Gamma_{\text{sl}}$; resolve $\frac{1}{2}$ vs. $\frac{3}{2}$ problem.

May provide missing semileptonic channels.
Hyperfine splitting in $D$ vs. $B$

$$M_{B^*}^2 - M_B^2 \simeq M_{D^*}^2 - M_D^2 \simeq M_{K^*}^2 - M_K^2 \simeq M_\rho^2 - M_\pi^2$$

If these were exact and if perturbative corrections could be discarded

$$-(\rho_{LS}^3 + \rho_{\pi G}^3 + \rho_A^3) \simeq 2\Lambda \mu_G^2 (1 + \kappa)$$

$$-(\rho_{\pi G}^3 + \rho_A^3) \approx 0.48 \text{ GeV}^3$$

The final outcome is $\kappa \approx -0.15$ and $-(\rho_{\pi G}^3 + \rho_A^3) \approx 0.48 \text{ GeV}^3$
Experimental determination

We can take $V_{cb}$ extracted from inclusive decays and calculate $F_{D^*}$

\begin{align*}
F_{D^*} & \simeq 0.810 \pm 0.007 \pm 0.026 \pm \delta_{\text{incl}} & \text{BaBar 2008} \\
& 0.849 \pm 0.011 \pm \delta_{\text{incl}} & \text{HFAG Average 2009} \\
& 0.836 \pm 0.015 \pm \delta_{\text{incl}} & \text{HFAG 08, CLEO / ALEPH excluded}
\end{align*}

(inclusive value taken without error bars)

Finally $F_{D^*}$ moved practically to the value obtained in the heavy quark expansion in QCD, $0.86 \pm 0.02$
Experimental determination

For \( B \to D \ell \nu \):

\[
F_+(0) \approx 1.021 \pm 0.019 \pm 0.041 \pm \delta_{\text{incl}}
\]

Good agreement with the dynamic heavy quark expansion in QCD

Using \( |V_{cb}| \) from \( \Gamma_{sl}(B) \) I predicted

\[
|V_{cb}| G(1) = 43.7 \times 10^{-3}
\]

There is no tension between inclusive and exclusive \( V_{cb} \)

rather a remarkable agreement