Structure of gauge and gravity amplitudes in string and field theory

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based on work done with
S. Badger, N. Berkovits, N.E.J. Bjerrum-Bohr,
P. Damgaard, M.B. Green, J. Russo
Recently we have experienced fantastic progress in the evaluation of on-shell gauge and gravity amplitudes in field theory.

Both on-shell S-matrix for $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA showed a rather remarkable simplicity compared to the Feynman graph approach.
Recently we have experienced fantastic progress in the evaluation of on-shell gauge and gravity amplitudes in field theory.

Both on-shell S-matrix for $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA showed a rather remarkable simplicity compared to the Feynman graph approach.

In this talk I will discuss some properties of these maximally supersymmetric theories in various dimensions $4 \leq D \leq 10$.

Unfortunately the implementation of supersymmetry in perturbation in string theory and in field theory is not completely well understood and needs to be reconsidered.
Motivations

In $\mathcal{N} = 4$ SYM this simplicity is expected because of the extended supersymmetries and the structure of the gauge interaction of the theory:

- precise rôle of supersymmetry in perturbation
- sub-leading color corrections

The simplicity of the $\mathcal{N} = 8$ SUGRA S-matrix calls for an explanation:

- how much is due to supersymmetry?
- how much to gauge (diffeomorphism) invariance?
- Connection with string theory analysis
1. Tree-level amplitudes

2. One loop amplitudes

3. $\mathcal{N} = 4$ super-Yang-Mills in various dimensions

4. $\mathcal{N} = 8$ supergravity in various dimensions

5. Conclusion & Outlook
Part I

Tree-level amplitudes
In non-Abelian gauge theories the full tree amplitudes can be decomposed onto color ordered amplitudes

\[ A_{\text{tree}}^n(1, \ldots, n) = g_{\text{YM}}^{n-2} \sum_{\sigma \in S_{n-1}} \text{tr}(\lambda^{\sigma(1)} \cdots \lambda^{\sigma(n)}) A_{\text{tree}}^n(\sigma(1), \ldots, \sigma(n)) \]

All the information about the amplitude is in the color ordered partial amplitudes

\[ A_{n,\sigma}^{\text{tree}} \equiv A_{n}^{\text{tree}}(\sigma(1), \ldots, \sigma(n)); \quad \sigma \in S_n \]
[Kawai–Lewellen–Tye] showed that (super-)Gravity amplitudes are given by the left/right product of gauge theory amplitudes

\[
M_{4}^{\text{tree}}(1, \ldots, 4) = -\kappa^2 s_{12} A_{4}^{\text{tree}}(1, 2, 3, 4) \tilde{A}_{4}^{\text{tree}}(1, 2, 4, 3)
\]

\[
M_{5}^{\text{tree}}(1, \ldots, 5) = \kappa^3 s_{12} s_{34} A_{5}^{\text{tree}}(1, 2, 3, 4, 5) \tilde{A}_{5}^{\text{tree}}(2, 1, 4, 3, 5) + \kappa^3 s_{13} s_{24} A_{5}^{\text{tree}}(1, 3, 2, 4, 5) \tilde{A}_{5}^{\text{tree}}(3, 1, 4, 2, 5)
\]

\[
M_{n}^{\text{tree}}(1, \ldots, n) = \kappa^{n-2} \sum_{\sigma, \tilde{\sigma} \in S_{n-1}} P_{n-3}^{\sigma, \tilde{\sigma}}(s_{ij}) A_{n, \sigma}^{\text{tree}} \tilde{A}_{n, \tilde{\sigma}}^{\text{tree}}
\]

To avoid unphysical double poles then \( P_{n-3}^{\sigma, \tilde{\sigma}}(s_{ij}) \) is an homogeneous polynomial of degree \( n - 3 \) in the kinematic invariants

Again all the information about the amplitude is in the color ordered factor \( A_{n, \sigma}^{\text{tree}} \).
Feynman rules gives $n!$ different amplitudes but all the partial amplitudes $A_{n,\sigma}^{\text{tree}}$ are not independent.
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Reflection property

$$A_n^{\text{tree}}(1, \ldots, n) = (-1)^n A_n^{\text{tree}}(n, \ldots, 1)$$
Feynman rules gives $n!$ different amplitudes but all the partial amplitudes $A_{n,σ}^\text{tree}$ are not independent.

Cyclicity property

$$A_n^\text{tree}(1, \ldots, n) = A_n^\text{tree}(2, \ldots, n, 1)$$

Cyclicity reduces this to $(n − 1)!$ independent amplitudes.
Feynman rules gives $n!$ different amplitudes but all the partial amplitudes $A_{n,\sigma}^{\text{tree}}$ are not independent

 Photon decoupling identity (no self coupling between photons)

\[ \sum_{\sigma \in \mathcal{S}_n} A_{n}^{\text{tree}}(\sigma(1), \ldots, \sigma(n)) = 0 \]
Feynman rules gives \( n! \) different amplitudes but all the partial amplitudes \( A_{\text{tree}}^{n,\sigma} \) are not independent.

What is the minimal information contained in these amplitudes? What is the minimal number of amplitudes to be computed.
Open string tree-level amplitudes

For understanding the structure of the tree-level amplitudes it is useful to embed them in string theory.
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The tree-level SYM amplitude are obtained as the $\alpha' \to 0$ limit of the open string amplitudes

$$A_{\text{SYM}}^n(1, \ldots, n) = \lim_{\alpha' \to 0} A(1, \ldots, n)$$

$$A(1, \ldots, n) = \left\langle U^{(1)}(z_1) U^{(n-1)}(z_{n-1}) U^{(n)}(z_n) \prod_{i=2}^{n-2} \int dz_i V^{(i)} \right\rangle$$

where $U^i$ are unintegrated conformal weight 0 vertex operators and $V^i$ are conformal weight one vertex operators.

This form of the amplitude is generic to all string theories (bosonic, RNS, pure spinor) and the following argument apply to any string theories.
Cyclicity: \((n - 1)!\) amplitudes

\[ A(1, \ldots, n) = \int_{\text{ordered}} \prod_{i=2}^{n-2} dz_i \prod_{1 \leq i < j \leq n} |z_i - z_j|^{2 \alpha' k_i \cdot k_j} \sum_{(\zeta_j) \in \{0, 1, z_i\}} L_k \prod_{i=2}^{n-2} \frac{1}{z_j - \zeta_j} \]

Ordered integral \(\Re(z_1) < \cdots < \Re(z_n)\)

Cyclicity: \((n-1)!\) independent amplitudes

\(PSL(2, \mathbb{R})\) invariance \(z_1 = 0, z_{n-1} = 1\) and \(z_n = +\infty\).
We deform the contour of integration

\[ e^{-2i\alpha'\pi k_2(k_1+k_3)} = e^{-2i\alpha'\pi(k_2\cdot k_3)} \]

The real part of the monodromy relation leads to the stringy version of the Kleiss-Kuijf relations: \((n-2)!\) independent amplitudes

\[ \mathcal{A}_n(\beta_1, \ldots, \beta_r, 1, \alpha_1, \ldots, \alpha_s, n) = (-1)^r \times \]

\[ \Re \left[ \prod_{1 \leq i < j \leq r} e^{2i\pi\alpha'(k_{\beta_i} \cdot k_{\beta_j})} \sum_{\sigma \subset \text{OP} \{\alpha\} \cup \{\beta^T\}} \prod_{i=1}^r \prod_{j=0}^s e^{(\alpha_j, \beta_i)} \mathcal{A}_n(1, \{\sigma\}, n) \right], \]

exp\((\alpha, \beta) = \exp(2i\pi\alpha' k_\alpha \cdot k_\beta)\) if \(\Re(z_\beta - z_\alpha) > 0\) or 1 otherwise
Monodromies: BCJ: \((n - 3)!\) amplitudes

We deform the contour of integration

\[
\text{[Bjerrum-bohr, Damgaard, Vanhove]}
\]

The imaginary part of the monodromy relation:

\((n - 3)!\) independent amplitudes

\[
0 = \Im \left[ \prod_{1 \leq i < j \leq r} e^{2i\pi\alpha'(k_{j'i} \cdot k_{j'j})} \sum_{\sigma \subseteq \Omega \{\alpha\} \cup \{\beta^T\}} \prod_{i=1}^{r} \prod_{j=1}^{s} e^{\alpha_i, \beta_j} A_n(1, \{\sigma\}, n) \right].
\]

Proves the \([Bern, Carrasco, Johanson, '08]\) relations in the field theory limit
Minimal Basis for tree-level amplitudes

There is a \textit{minimal} basis of amplitude $\mathcal{B}$ of size $(n - 3)!$

\[
\mathcal{B}^\sigma = \mathcal{A}(1, \sigma(2), \ldots, \sigma(n - 2), n - 1, n); \quad \sigma \in \mathfrak{S}_{n-3}
\]

$c_\sigma$ are rational functions of degree $(c_\sigma) = 0$ in the $\sin(2\alpha' \pi p \cdot q)$

For gravity using that $V_{\text{grav}} = V_{\text{open}} \otimes \tilde{V}_{\text{open}}$ gives

\[
\mathcal{M}_n^{\text{closed}} = \frac{\kappa^{n-2}}{(\alpha')^{n-3}} \sum_{\sigma, \tilde{\sigma} \in \mathfrak{S}_{n-3}} g_{\sigma\tilde{\sigma}} \mathcal{B}^\sigma \tilde{\mathcal{B}}^{\tilde{\sigma}}
\]

where the matrix $g_{\sigma\tilde{\sigma}}$ is symmetric and degree $(g_{\sigma\tilde{\sigma}}) = n - 3$. 
In higher-loop amplitudes

When reconstructing the loop amplitudes from the maximal cuts the basis of functions allows to avoid the bad shifts

- The symmetric form makes more obvious that the large-$z$ behaviour of $\mathcal{N} = 8$ gravity amplitude is the square of $\mathcal{N} = 4$ SYM amplitudes
- Have good large-$z$ behaviour in the cut in loops
Basis of one-loop scalar integral functions

In $D = 4 - 2\epsilon$ one expands the amplitudes on a basis of scalar integral functions with massive external legs

$$\mathcal{M}^{(4-2\epsilon)}_{n;1} = \sum_i b_{o,i} \, I^{(i)}_{\square} + \sum_i t_i \, I^{(i)}_{\triangleright} + \sum_i b_{u,i} \, I^{(i)}_{\circ} + c_{\text{rational pieces}}$$

This basis of scalar integral functions captures the IR and UV divergences of the one-loop amplitudes [Bern, Dunbar, Dixon, Kosower]
The Reduction formulas

On-shell integrals are reduced to the scalar integral functions by cancelling one power of loop momentum with one propagator

\[ \int d^D \ell \, \frac{2(\ell \cdot k_1)}{\ell^2 (\ell - k_1)^2} \times (\ldots) = \int d^D \ell \left( \frac{1}{(\ell - k_1)^2} - \frac{1}{\ell^2} \right) \times (\ldots) \]

These reduction formulas are well adapted to the soft and collinear singularities of QCD/\( \mathcal{N} = 4 \) SYM amplitudes.
The no triangle property in $\mathcal{N} = 8$

Since $\mathcal{N} = 4$ super-Yang-Mills amplitudes have $n - 4$ powers of loop momenta they are reducible to boxes only.
The no triangle property in $\mathcal{N} = 8$

$\mathcal{N} = 8$ amplitudes $2n - 8$ powers of loop momenta should contain boxes, triangles, bubbles and rational terms.

$n$-legs

$2n-8$

$+$ rational terms
The no triangle property in $\mathcal{N} = 8$

Explicit computations by [Bjerrum-Bohr et al., Bern et al.] showed that the amplitudes reduce to scalar box integral functions like for $\mathcal{N} = 4$ SYM.

\[ l^{2(n-4)} \]

= 

\[ \text{Diagram} \]
Explicit computations by [Bjerrum-Bohr et al., Bern et al.] showed that the amplitudes reduce to scalar box integral functions like for $\mathcal{N} = 4$ SYM

This result was unexpected because the counting was based on reduction formula that did not take into account all the cancellations occurring in gravity and did not reflect the softer IR behaviour of gravity amplitudes.
The no triangle property of $\mathcal{N} = 8$ amplitudes

- Gravity does not have color factor
  - Summation over all the permutations at one-loop
  - Sum over all the planar and nonplanar diagrams at higher loop order
- Gauge invariance $\varepsilon_{\mu\nu} \rightarrow \varepsilon_{\mu\nu} + \partial_\mu \nu_\nu + \partial_\nu \nu_\mu$

Unordered amplitudes are more than just the sum over all ordering of color ordered amplitudes. All the various ordering have the same tensorial structure

$$M = \sum_r t_r \int_0^\infty \frac{dT}{T} T^{2n-D/2} \int_0^1 \prod_{i=1}^{n-1} d\nu_i \mathcal{P}(\partial Q_n) e^{-T \sum_{r,s} (k_r \cdot k_s) G_{r,s}^{(1)}}$$

$$Q_n = \sum_{i<j} (k_i \cdot k_j) [(\nu_i - \nu_j)^2 - |\nu_i - \nu_j|^2]$$
The no triangle property of $\mathcal{N} = 8$ amplitudes

Loop momentum is a total derivative $k_i \cdot \ell \sim \partial_{\nu_i} Q_n$ which can be freely integrated

$$\int_0^\infty \frac{dT}{T} \int_0^1 d^{n-1} \nu \partial_{\nu_i} Q_n (\cdots) = - \int_0^\infty \frac{dT}{T} \int_0^1 d^{n-1} \nu Q_n \partial_{\nu_i} (\cdots)$$

$\partial_{\nu_i} \partial_{\nu_j} Q_n \sim (k_i \cdot k_j) [\delta(\nu_i - \nu_j) - 1]$ does not contain any loop momenta

two powers of loop momentum are cancelled at each steps

[Bjerrum-Bohr, Vanhove]
The no triangle property of $\mathcal{N} = 8$ amplitudes

This implies new reduction formulas for unordered integrals

[Bjerrum-Bohr, Vanhove]

These reduction formulas reflect that the graviton amplitudes have softer IR singularities than for QCD

The dimension shifted contributions cancel in the total amplitude

Gauge invariance implies that one can push all the ‘triangles’ into total derivative which cancel in the total amplitude (no boundary contributions)
The no triangle property of $\mathcal{N} = 8$ amplitudes

For $\mathcal{N} = 8$ sugra amplitude the no triangle property arises because the amplitude has $n - 4$ powers of $\ell^2$.
The new reduction formula applies to QED one-loop amplitudes as well.

The $n$-photon one-loop amplitude in massless QED has the structure:

$$
\begin{align*}
\text{2n-legs} & = \text{diagram 1} + \text{diagram 2} + \text{rational terms}
\end{align*}
$$

- Using the ordered reduction formula the amplitude should contain all the scalar integral functions.
The \( n \)-photon one-loop amplitude in massless QED has the structure

- For \( n = 4 \) external photons [Mahlon; Bern et al.]
  
  \[
  A^{QED}_{4\gamma} = I_{\text{box}} + I_{\text{Triangle}} + I_{\text{bubble}}
  \]

- For \( n = 6 \) external photons
  
  [Bernicot, Guillet; Binoth, Heinrich, Gehrmann, Mastrolia]

  \[
  A^{QED}_{6\gamma} = I_{\text{box}} + I_{\text{Triangle}}
  \]
No triangle property in massless QED

The new reduction formula applies to QED one-loop amplitudes as well

The $n$-photon one-loop amplitude in massless QED has the structure

For $2n \geq 8$ external photons

\[
A_{2n \geq 8}^{QED} = I_{\text{box}}
\]

This is due to the very good high-energy behaviour of the tree amplitudes

\[
\lim_{z \to \infty} A_{e^+ e^- \to n \gamma}^{\text{tree}}(\rho_a + z \rho_b, \rho_b - z \rho_a) \sim \frac{z}{z^{n-2}}
\]
Part III

The multiloop amplitudes in $\mathcal{N} = 4$ super-Yang-Mills
The coupling constant of $\mathcal{N} = 4$ SYM has dimension

$$[g_{YM}^2] = (\text{length})^{D-4}$$

Power counting gives that the 4-point $L$-loop amplitude $\mathcal{A}_{4;L}$ has superficial UV divergence ($\Lambda$ UV cut-off)

$$[\mathcal{A}_{4;L}^{(D)}] = \Lambda^{(D-4)L}$$

UV finite in $D < 4$

- could be logarithmically diverging in four dimensions

Supersymmetry improves this power counting
The coupling constant of $\mathcal{N} = 4$ SYM has dimension

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$$[\mathcal{A}_{4;L}^{(D)}] = \Lambda^{(D-4)L-4} t_8 F^4$$

UV finite in $D < 4 + \frac{4}{L}$

- Off-shell $\mathcal{N} = 2$ superspace is enough to assure finiteness in four dimensions by factorizing a $t_8 F^4$ term

[Mandelstam; Howe, Stelle, et al.; Brink, Lindgren, Nilsson]

- does not explain the non-renormalisation theorems for $F^4$ for $L \geq 2$
- does not lead to the correct divergences structure at $L = 2$ in $D = 7$ and $L = 3$ in $D = 6$
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$$[\mathcal{A}_{4;L}^{(D)}] = \Lambda^{(D-4)L-4-2\gamma_L} \partial^2 \gamma_L \partial^2 \mathcal{F}^{4} \quad L \geq 2$$

UV finite in $D < 4 + \frac{4 + 2\gamma_L}{L}$

- Perturbative computations by [Bern, Dixon, Dunbar, Perelstein, Rozowski] indicates a better behaviour with $\gamma_L = 1$

- confirmed by $\mathcal{N} = 3$ superspace arguments in $D = 4$ [Howe, Stelle]
The coupling constant of $\mathcal{N} = 4$ SYM has dimension

$$[g_{YM}^2] = (\text{length})^{D-4}$$

Power counting gives that the 4-point $L$-loop amplitude $\mathcal{A}_{4;L}$ has superficial UV divergence ($\Lambda$ UV cut-off)

$$[\mathcal{A}^{(D)}_{4;L}] = \Lambda^{(D-4)\frac{L-4-2\gamma_L}{L} \partial^2 \gamma_L} t_b F^4 \quad L \geq 2$$

UV finite in $D < 4 + \frac{4 + 2\gamma_L}{L}$

- Perturbative computations by [Bern, Dixon, Perelstein, Rozowski] indicates a better behaviour with

$$\gamma_L = 1$$

- Validity of this rule to all orders?
- What about the color factors dependence?
We consider the representation of the $\mathcal{N} = 4$ superconformal algebra.

There are true BPS states which do not develop an anomalous dimension, and fake F-term developing an anomalous dimension that can be written as D-terms.\cite{Drummond, Heslop, Howe, Kerstan}

For $D < 10$ one can construct the dimension-2 supercurrent operator

$$T_{ij} = \text{tr}(\varphi_i \varphi_j) = t_{ij} + \delta_{ij} \mathcal{K}$$

- $\frac{1}{2}$-BPS states: There are single-trace operators and multi-trace operators: $t_8 \text{tr}(F^4)$ and $t_8 (\text{tr}F^2)^2$, both true BPS states.
- $\frac{1}{4}$-BPS states: true F-term are at least double trace operators: $\partial^2 t_8 (\text{tr}F^2)^2$. The single trace operator is a fake F-term since $\partial^2 t_8 \text{tr}(F^4) \sim \int d^{16}\theta \mathcal{K}$
- $\frac{1}{8}$-BPS states: True F-term are at least triple traces, and $\partial^4 t_8 (\text{tr}F^2)^2 \sim \int d^{16}\theta T_{ij} T^{ij}$ is a fake F-term.
Superconformal multiplet of $\mathcal{N} = 4$ algebra

We consider the representation of the $\mathcal{N} = 4$ superconformal algebra. There are true BPS states which do not develop an anomalous dimension, and fake F-term developing an anomalous dimension that can be written as D-terms. [Drummond, Heslop, Howe, Kerstan]

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- \(\frac{1}{8}\)-BPS states: True F-term are at least triple traces, and \(\partial^4 t_8 (\text{tr} F^2)^2 \sim \int d^{16} \theta T_{ij} T^{ij}\) is a fake F-term.
We consider the four points open string amplitudes within the (non-minimal) pure spinor formalism of [Berkovits].

An $L$-loop open string amplitude with $H$-handles and $B$-boundaries has $3(L - 1)$ real moduli with $L = B + 2H - 1$

$$\mathcal{A}_L^4 \sim g_s^{L-1} \int d^{3L-3}\tau \langle \prod_{i=1}^{3L-3} (\mu_i|b) V_1 \cdots V_4 \rangle$$

SYM vertex operator

$$V = \int d^2 z : (\partial x^m A_m + \partial \theta^\alpha A_\alpha + d_\alpha W^\alpha + F_{mn}(\lambda \gamma^{mn} w)) e^{ik \cdot X} :$$

the dimension 3/2 gaugino superfield

$$W^\alpha \sim \chi^\alpha + \theta_\beta F^{\alpha\beta} + \cdots$$
Zero mode saturation gives for $L \leq 6$ at a generic point in the moduli space

$$\mathcal{A}_L^4 \sim g_s^{L-1} \int d^{3L-3} \tau (\mu_\tau | \partial \chi)^{L-2} \int d^{16} \theta \theta^{12-2L} \mathcal{W}^4$$

But contributions from the boundary of the moduli space can generate reduction of number of derivatives
\[
\text{Tr}(T_1T_2T_3T_4) \frac{k_1 + k_2}{k_1 \cdot k_2} \\
\text{Tr}([T_1, T_2])\text{Tr}([T_3, T_4]) \frac{k_1 + k_2}{k_1 \cdot k_2} = 0
\]

- Single trace operator: there are \( L - 2 \) open string poles

\[
\mathcal{A}_L^4 \sim g_s^{L-1} N^L k^2 t_8 \text{tr} F^4
\]
\[
\text{Tr}(T_1 T_2 T_3 T_4) \frac{k_1 + k_2}{k_1 \cdot k_2}
\]
\[
\text{Tr}([T_1, T_2]) \text{Tr}([T_3, T_4]) \frac{k_1 + k_2}{k_1 \cdot k_2} = 0
\]

- Single trace operator: there are \( L - 2 \) open string poles

\[
\mathcal{A}_L^4 \sim g_s^{L-1} N^L k^2 t_8 \text{tr} F^4
\]

The \( \partial^2 \text{tr}(F^4) \) operator gets all loop order correction from \( L \geq 2 \)
Double trace operator:

\[ \mathcal{A}_L^4 \sim g_s^{L-1} N^{L-1} k^L t_8 (\text{tr} F^2)^2 \]
\[ \text{Double trace operator:} \]

\[
\mathcal{A}_L^4 \sim g_s^{L-1} N^{L-1} k^L t_8 (\text{tr} F^2)^2
\]

\[ \nabla_2^2 t_8 (\text{tr} F^2)^2 \text{ operator is not renormalized above } L = 2. \]

\[ \nabla^4 t_8 (\text{tr} F^2)^2 \text{ operator gets perturbative contributions from } L \geq 3. \]
Gravitational corrections

Adding an handle suppresses two boundaries and lead of $1/N^2$ corrections

The leading low-energy limit $\ell_s \to 0$ of the $L$-loop open string amplitude is given by

$$A_L^4 \sim g_s^{L-1} N^L \left( 1 + \frac{c_1}{N^2} + \frac{c_2}{N^4} + \cdots \right) k^2 t_8 \text{tr}(F^4)$$

$c_i$ is given by pure SYM and mixed SUGRA+SYM contributions
The leading UV divergence for \( \mathcal{N} = 4 \) SYM 4-point amplitudes is given by the single trace operator \( \partial^2 t_8 \text{tr} F^4 \)

\[
\mathcal{A}_{4;L}^{(D)} = \Lambda^{(D-4)L-4-2\gamma_L} \partial^2 \gamma_L t_8 F^4 \quad \text{for } L \geq 2
\]

\( D < D_c = 4 + \frac{6}{L} \)
The leading UV divergence for $\mathcal{N} = 4$ SYM 4-point amplitudes is given by the single trace operator $\partial^2 t_8 \text{tr} F^4$

$$D < D_c = 4 + \frac{6}{L}$$

The double trace $\partial^{\beta_L} t_8 (\text{tr} F^2)^2$ the critical dimension for UV divergences

$$D_c = 4 + \frac{4 + 2[L/2]}{L}, \quad \beta_L = L \quad \text{for } L \leq 4$$

$$D_c = 4 + \frac{8}{L}, \quad \beta_L = 4 \quad \text{for } L \geq 3$$

Results in agreement with computations by [Bern, Dixon, Roiban, Carrasco, Johansson, to appear]
Critical dimensions

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<th>$L=3$</th>
<th>$L=4$</th>
<th>$L=5$</th>
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<td>$s^\gamma_L t_8 \text{tr}(F^4)$</td>
<td>$D_c = 8$</td>
<td>$D_c = 7$</td>
<td>$D_c = 6$</td>
<td>$D_c = \frac{11}{2}$</td>
<td>$D_c = \frac{26}{5}$</td>
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<td>$\gamma_2 = 1$</td>
<td>$\gamma_3 = 1$</td>
<td>$\gamma_4 = 1$</td>
<td>$\gamma_5 = 1$</td>
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<tr>
<td>$s^\beta_L t_8 (\text{tr} F^2)^2$</td>
<td>$D_c = 8$</td>
<td>$D_c = 7$</td>
<td>$D_c = \frac{20}{3}$</td>
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<td>$\beta_3 = 2$</td>
<td>$\beta_4 = 2$</td>
<td>$\beta_5 = 2$</td>
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</table>

For $L \geq 4$ the UV divergence is dominated by the single trace term

$$\Lambda^{(D-4)L-6} \partial^2 t_8 \text{tr}(F^4) \quad L \geq 2$$

the double-trace term is subleading

$$\Lambda^{(D-4)L-8} \partial^4 t_8 (\text{tr} F^2)^2 \quad L \geq 3$$
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