Amplitudes using Pure Spinors

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Plan of the Talk

- N = 8 supergravity, perturbative results.
- Motivation of pure spinor approach.
- Pure spinor particle.
- Conjecture of amplitude prescription.
- Low energy matching of L = 1, 2, 3 and 4.
- L ≥ 5.

N = 8 SUGRA, review of perturbative results, I

- The first non-trivial amplitudes are 4-point.
- The results by explicit computation:

L	Skeleton	4-point	Low energy	Pages
1	1	1	$\mathcal{R}^4$	-
2	1	2	$\partial^4\mathcal{R}^4$	-
3	2	9	$\partial^6\mathcal{R}^4$	1
4	5	50	$\partial^8\mathcal{R}^4$	258
5	16	439	???	???

Green, Schwarz, Brink 1982

Bern, Dixon, Dunbar, Perelstein, Rozowsky 1998

Bern, Carrasco, Dixon, Johansson, Kosower, Roiban 2007

Bern, Carrasco, Dixon, Johansson, Roiban 2009

- The low energy limit is not manifest, 'surprising' cancelations beyond two loops.
- Need to calculate the whole amplitude to get the low energy dependence (= UV-properties).

N = 8 SUGRA, review of perturbative results, II

- Matches the formula: $D_c^{(L)} = 4 + \frac{6}{L}$
- If true, yields that N = 8 is perturbatively finite.

Using the general formula for supergravity yields:

$\partial^{2a}\mathcal{R}^4\Lambda^{L(D-2)-6-2a}$

$$D_c^{(L)} = 2 + \frac{6+2a}{L}$$

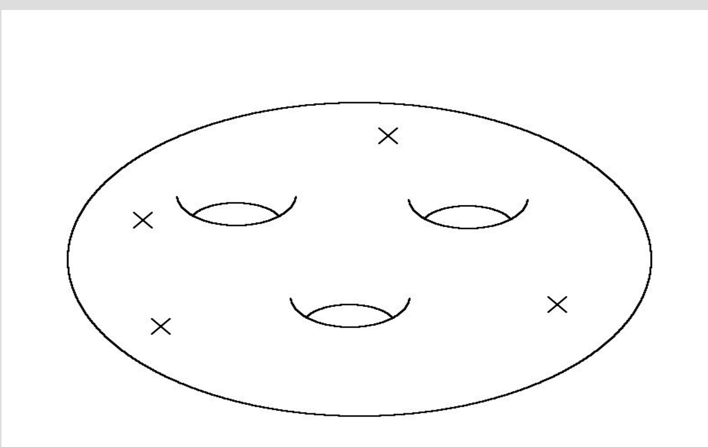
is a = L for L ≥ 5?

Motivation – pure spinor approach

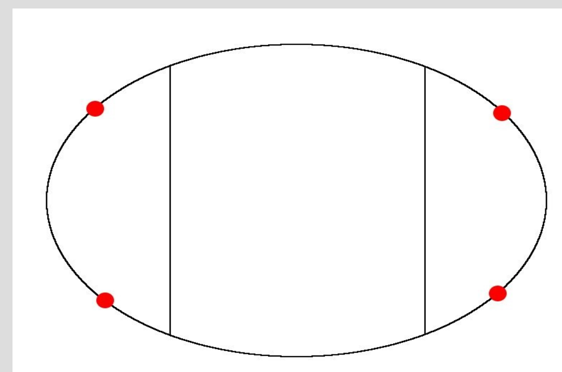
- Compute loop amplitudes with manifest Poincaré and Spacetime supersymmetry.
- Gives explicit low energy dependence (of 4-pt functions) without explicit computations.

Motivation – particle approach

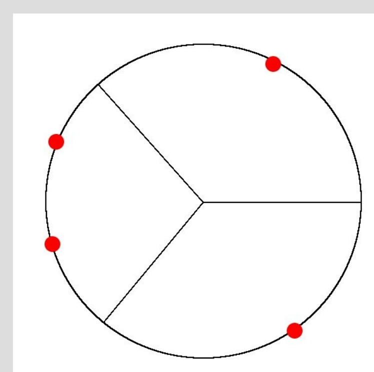
- Get the low energy dependence of the different skeletons.
- The distribution of external vertices:



$\Rightarrow$



$\partial^8 \mathcal{R}^4$



$\partial^6 \mathcal{R}^4$

$$\mathcal{Y}(\dots)\mathcal{R}^4 \sim \partial^6 \mathcal{R}^4 + \dots$$

Pure Spinor Particle

- Action (non-minimal, D=10): Berkovits 2001 (minimal)

$$S = \int d\tau \left(\dot{X}P + \dot{\theta}p + \dot{\lambda}w + \bar{w}\dot{\bar{\lambda}} - s\dot{r} - \hat{p}\dot{\hat{\theta}} + \hat{w}\dot{\hat{\lambda}} + \dot{\hat{\bar{\lambda}}}\hat{\bar{w}} + \dot{\hat{r}}\hat{s} - \frac{1}{2}P^2 \right)$$

- Hamiltonian and BRST-charge:

$$H = \frac{1}{2}P^2 \qquad Q = \lambda d + \bar{w}r \qquad d_\alpha = p_\alpha + \frac{1}{2}P_m\left(\gamma^m\theta\right)_\alpha$$

- All fields other then *X* are locally constants.

- Discuss the unhatted variables as the hatted ones behaves in the same way.

Pure spinors - basics

- Q is not nilpotent: $[Q,Q] = P^m(\lambda\gamma_m\lambda)$
- Pure spinor constraint: $(\lambda\gamma^m\lambda) = 0$
- Eleven independent components.
- Introduces a gauge invariance: $\delta_\epsilon w_\alpha = \epsilon^m(\gamma_m\lambda)_\alpha$
- Gauge invariant combinations:

$$N_{mn} = \frac{1}{2}(w\gamma_{mn}\lambda) \qquad \bar{J} = \bar{w}\bar{\lambda} - sr$$

$$J = \lambda w \qquad \bar{N}_{mn} = \frac{1}{2}\bar{w}\gamma_{mn}\bar{\lambda} - \frac{1}{2}s\gamma_{mn}r$$

$$S = s\bar{\lambda}$$

$$S_{mn} = \frac{1}{2}s\gamma_{mn}\bar{\lambda}$$

The b-ghost

- In the non-minimal formalism one can construct a b-ghost: $[Q,b]=H$

$$
\begin{aligned}
b &= \frac{1}{2}\left(\frac{G^{\alpha}\bar{\lambda}_{\alpha}}{(\lambda\bar{\lambda})}+\frac{\bar{\lambda}_{\alpha}r_{\beta}H^{[\alpha\beta]}}{(\lambda\bar{\lambda})^{2}}+\frac{\bar{\lambda}_{\alpha}r_{\beta}r_{\gamma}K^{[\alpha\beta\gamma]}}{(\lambda\bar{\lambda})^{3}}+\frac{\bar{\lambda}_{\alpha}r_{\beta}r_{\gamma}r_{\delta}L^{[\alpha\beta\gamma\delta]}}{(\lambda\bar{\lambda})^{4}}\right)\\
G^{\alpha} &= -\frac{1}{2}P^{m}\left(\gamma_{m}d\right)^{\alpha}\,, \qquad H^{[\alpha\beta]}=-\frac{1}{384}\gamma^{\alpha\beta}_{mnp}\left[(d\gamma^{mnp}d)-24N^{mn}P^{p}\right]\\
K^{[\alpha\beta\gamma]} &= \frac{1}{192}\gamma^{[\alpha\beta}_{mnp}\left(\gamma^{m}d\right)^{\gamma]}N^{np}\,, \qquad L^{[\alpha\beta\gamma\delta]}=\frac{1}{12244}\gamma^{[\alpha\beta}_{mnp}\gamma^{m}{}_{qr}{}^{\gamma\delta]}N^{np}N^{qr}
\end{aligned}
$$

- Depends on fields in the denominator, will pose a (interesting) problem later on.
- Observe that it depends on the momentum.

Vertex Operators

- N = 2 case (IIA-SUGRA in 10 dimensions):

$$V\left(X,\theta,\hat{\theta}\right) \quad = \quad P^m G_{mn} P^n + d_\alpha W^\alpha{}_\beta \hat{d}^\beta - d_\alpha \hat{E}^\alpha_m P^m - P^m E_{m\alpha} \hat{d}^\alpha + \ldots$$

Fields involved are linearized superfields

- Observe the dependence of the momentum.
- The Riemann tensor arises in the superfields as:

$$\begin{aligned} G_{mn} &\sim \theta^2\hat{\theta}^2\mathcal{R} \\ W^\alpha{}_\beta &\sim \theta\hat{\theta}\mathcal{R} \\ \hat{E}^\alpha_m &\sim \theta\hat{\theta}^2\mathcal{R} \\ E_{\alpha m} &\sim \theta^2\hat{\theta}\mathcal{R} \end{aligned}$$

- How to define amplitudes?

Pure spinor string

- Yields a theory with ghost number anomaly of 3 and is connected to N = 2 topological string.
- Amplitudes defined as for the bosonic string.
- Try to get a particle approach of the above.

The scalar particle using first quantised approach

- Action of a free particle:

$$S = \int d\tau \left(P\dot{X} - \frac{1}{2}P^2 \right)$$

- Vertex operator:

$$V_0\left(k_r, \tau_r\right) = e^{ik_r X(\tau_r)}$$

Formalism by: Dai,Siegel 2007

Amplitudes, I

- Definitions:

  Moduli: Length of lines.

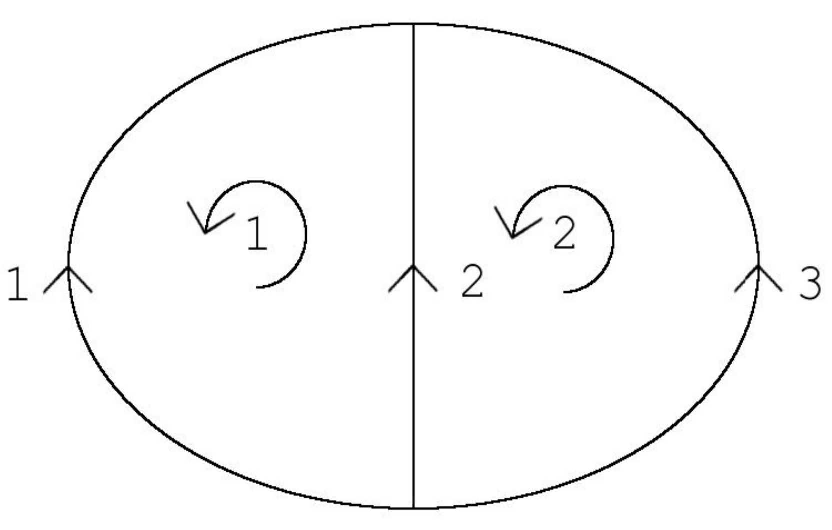
  b-cycles: Internal loops.

  one-forms:

$$\omega_I(\tau) = {a_I}^i d\tau_i$$
$${a_I}^i = \pm 1$$

- Period matrix:

$$\Omega_{IJ} = \oint_{b_I} \omega_J$$
$$\Delta = \det \Omega_{IJ}$$



$$\omega_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \omega_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\Omega_{IJ} = \begin{pmatrix} T_1 + T_2 & -T_2 \\ -T_2 & T_2 + T_3 \end{pmatrix}$$

$$\Delta = T_1 T_2 + T_1 T_3 + T_2 T_3$$

Amplitudes, II

- The N-point and L-loop amplitude:

$$A_{F_L}(\{k_r\}) \;=\; \int_0^\infty dT_1 \ldots dT_{3L-3} \int_{F_L} \prod_{s=1}^{N} d\tau_s I\left(\{T_i\},\{\tau_r\},\{k_r\}\right)$$

Where

$$\begin{aligned} I\left(\{T_i\},\{\tau_r\},\{k_r\}\right) &= \int dx_0 \int d^L \ell e^{\int d\tau \frac{1}{2}(\ell^I \omega_I/d\tau)^2} e^{-\sum_{r<s} k_r k_s G(\tau_r,\tau_s)} e^{ix_0(\sum k_r)} \\ &= \frac{1}{\Delta^{D/2}} e^{-\sum_{r<s} k_r k_s G(\tau_r,\tau_s)} \delta\left(\sum k_r\right) \end{aligned}$$

$$G(\tau_r,\tau_s) = -\frac{1}{2}\left|\int_c ds\right| + \frac{1}{2}\int_c \omega_I \left(\Omega^{-1}\right)^{IJ} \int_c \omega_J$$

Dai,Siegel 2007

Observe the similarities with the bosonic string.

Amplitudes – a conjecture, I

- Use the correspondence between the bosonic and pure spinor string amplitude to formulate the amplitudes using the scalar particle:

$$A(s,t,u) = \int_0^\infty dT_1 \ldots dT_{3L-3} \int_{F_L} \prod_{r=1}^{4} d\tau_r \, K(\{T_i\},\{\tau_r\},\{k_r\})$$

$$K(\{T_i\},\{\tau_r\},\{k_r\}) = \int \mathcal{D}X\mathcal{D}P\mathcal{D}\Phi\mathcal{D}\hat{\Phi} \left(\mathcal{N}\hat{\mathcal{N}} \prod_{i=1}^{3(L-1)} \left(\int_0^{T_i} \frac{d\tau}{T_i} b \int_0^{T_i} \frac{d\tau}{T_i} \hat{b} \right) V(k_1,\tau_1)\ldots V(k_4,\tau_4)\, e^{-S} \right)$$

- Not an exact correspondence for any amplitudes, due to 0/0-singularities.

Amplitudes – a conjecture, II

- For four-point amplitudes with few loops the regulator needed is:

$$\mathcal{N} = e^{q_1 \int_{F_L} d\tau [Q,\chi^1] + q_2 \int_{F_L} d\tau [Q,\chi^2]}$$
$$= e^{-q_1 \int_{F_L} d\tau (\lambda\bar{\lambda} - \theta r) - q_2 \int_{F_L} d\tau (N_{mn}\bar{N}^{mn} + J\bar{J} - \frac{1}{2}(d\gamma_{mn}\lambda)S^{mn} - \lambda dS)}$$

- Only source of zero modes for *s*.
- The functional integral yields integration over zero modes in the end.

Measure of zero modes

- We can separate the fields into world-line scalars: $(X^m, \theta^\alpha, \lambda^\alpha, \bar{\lambda}_\alpha, r_\alpha)$

  vectors: $(P_m, p_\alpha, w_\alpha, \bar{w}^\alpha, s^\alpha)$

  these have 1 and L zero modes respectively.
- The measure can be derived to be of the form:

$$\mathcal{D}\Phi_0 = \lambda^{3(L-1)}\, d^{10L}N\, d^{10L}\bar{N}\, d^L J\, d^L \bar{J}\, d^{11L}S\, d^{16L}d\, d^{11}\bar{\lambda}\, d^{11}\lambda\, d^{11}r\, d^{16}\theta$$

Berkovits, 2005

Counting zero modes, I

$$\mathcal{N} = e^{q_1 \int_{F_L} d\tau [Q,\chi^1] + q_2 \int_{F_L} d\tau [Q,\chi^2]}$$
$$= e^{-q_1 \int_{F_L} d\tau (\lambda\bar{\lambda} - \theta r) - q_2 \int_{F_L} d\tau \left(N_{mn}\bar{N}^{mn} + J\bar{J} - \frac{1}{2}(d\gamma_{mn}\lambda)S^{mn} - \lambda d S\right)}$$

$$\mathcal{D}\Phi_0 = \lambda^{3(L-1)}\, d^{10L}N\, d^{10L}\bar{N}\, d^L J\, d^L \bar{J}\, d^{11L}S\, d^{16L}d\, d^{11}\bar{\lambda}\, d^{11}\lambda\, d^{11}r\, d^{16}\theta$$

- The only source of zero modes of the field *s* is the regulator.
- In the regulator *s* is multiplied with the field *d*.
- Therefore, the b-ghost insertions and vertices has to contribute 5L zero modes of *d*, 5 for each b-cycle in the diagram.

Counting zero modes, II

- Observe also that the difference between the number of θ and *r* is 5.
- Therefore, a special component is picked out in the functional integral:

$$\left\langle \lambda^3 \hat{\lambda}^3 \mathcal{O} \right\rangle \Big|_{\theta^5 \hat{\theta}^5} \sim D^5 \hat{D}^5 \mathcal{O}$$

$$D_\alpha = \partial_\alpha + \frac{1}{2} \left(\gamma^m \theta\right)_\alpha \partial_m$$

Observe that due to the form of the regulator one can trade insertions of *r*'s with *D*'s.

Connection between period matrix and b-insertions

- If the fields in the b-ghosts only contributes with zero modes yields:

$$\prod_{i=1}^{3L-3}\int_0^{T_i}\frac{d\tau}{T_i}\,b|_{zero} = b^{I_1J_1}\dots b^{I_{3L-3}J_{3L-3}}\frac{\partial\Omega_{I_1J_1}}{\partial T_1}\dots\frac{\partial\Omega_{I_{3L-3}J_{3L-3}}}{\partial T_{3L-3}}$$

$$b|_{zero} = b^{IJ}\omega_I\omega_J/(d\tau^2)$$

- Can only contribute the maximal number of *d* insertions (= 6L - 6) if

$$\sum_{i=1}^{3L-3} c_i\frac{\partial\Omega_{IJ}}{\partial T_i}\neq 0$$

- Depends crucially on the skeleton.

Low energy limit

- Only interested in leading low energy limit of the amplitude.
- Made by $e^{ik_r X} \to 1$ and introducing a large momentum cutoff (/small T cutoff).
- Can only be taken in dimensions larger then the critical dimension where the amplitude has a logarithmic UV-divergence

$$D \geq D_c^{(L)}$$

- Observe that this will determine the most UV-divergent piece of the amplitude.

One-loop, four-point

- Here we have one unintegrated vertex, three integrated vertex operators and one b-insertion.
- Maximal number of *d*-insertions are 2 + 3 = 5 which is the needed number:

$$A^{(1)} \sim \left\langle \lambda^3 \hat{\lambda}^3 D \hat{D} A_\alpha W^3 \right\rangle\Big|_{\theta^5 \hat{\theta}^5} \Lambda^{D-8} \sim D^8 \hat{D}^8 G^4|_{\theta=\hat{\theta}=0} \Lambda^{D-8} \sim \mathcal{R}^4 \Lambda^{D-8}$$

- Which shows that it is half BPS interaction and logarithmic divergent in 8 dimensions.

$$\mathcal{R}^4 = t_8 t_8 \mathcal{R}\mathcal{R}\mathcal{R}\mathcal{R}$$

Mafra 2006 (String)

Two-loop, I

- Here all vertices are integrated and one has three b-insertions.
- Maximal number of *d*-insertions from the b-ghosts are 3 for each loop:

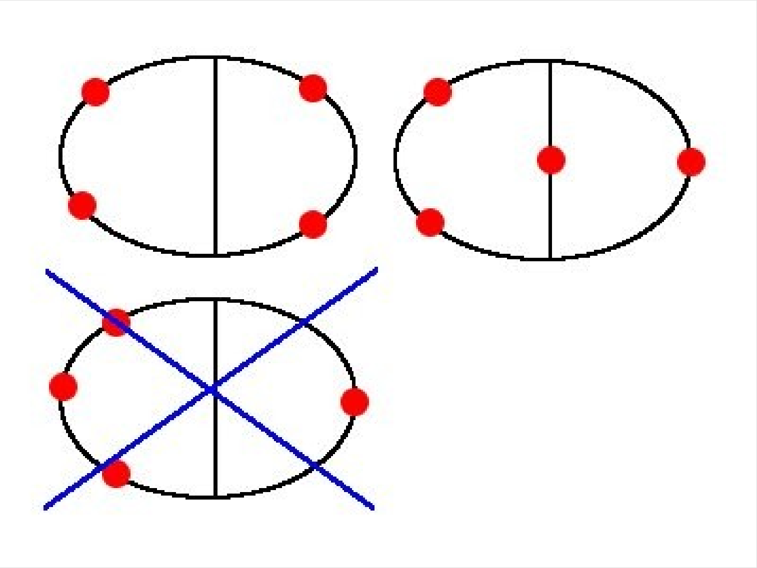
$$\prod_{i=1}^{3} b^{I_i J_i} \frac{\partial \Omega_{I_i J_i}}{\partial T_i} = b^{11} \left(b^{11} - 2b^{12} + b^{22} \right) b^{22} \sim b^{11} b^{12} b^{22}$$

- To get the appropriate number, the vertices has to contribute with the maximal number of *d*-terms and two vertices on each loop.

Two-loop, II

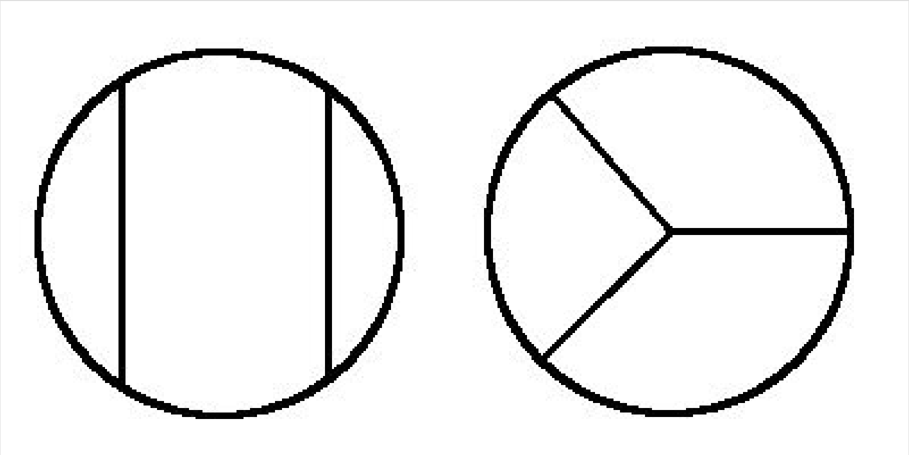
$$A^{(2)} \sim \left\langle \lambda^3 \hat{\lambda}^3 D^3 \hat{D}^3 W^4 \right\rangle\Big|_{\theta^5 \hat{\theta}^5} \Lambda^{2(D-7)} \sim D^{12}\, \hat{D}^{12} G^4 \Big|_{\theta=\hat{\theta}=0} \Lambda^{2(D-7)} \sim \partial^4 \mathcal{R}^4\, \Lambda^{2(D-7)}$$

Berkovits, Mafra 2006 (String)



- Shows that it is a quarter BPS interaction and is logarithmic divergent in 7 dimensions.

Three-loop, I



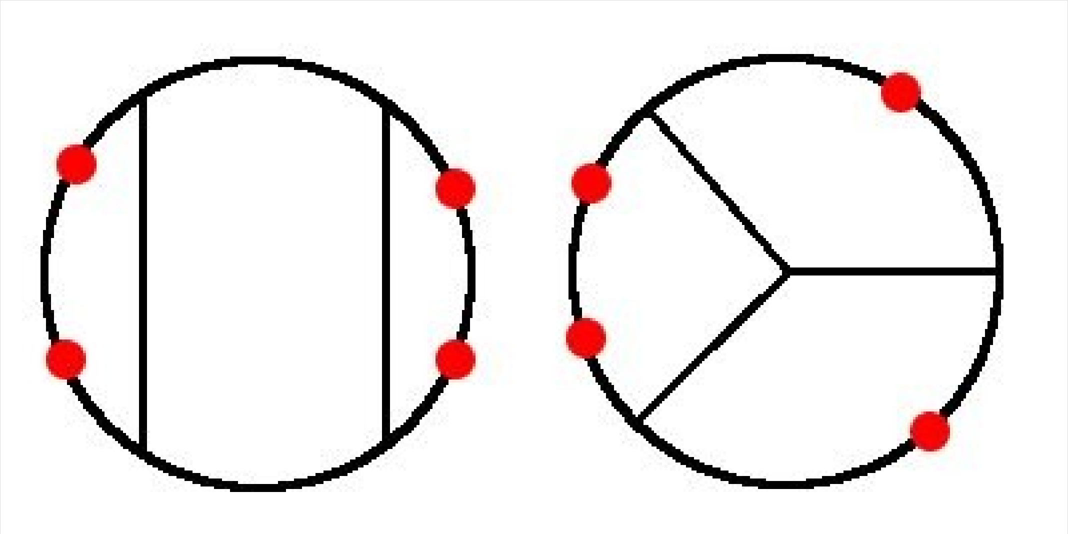
$$\Omega^L_{IJ} = \begin{pmatrix} T_1 + T_2 & -T_2 & 0 \\ -T_2 & T_2 + T_3 + T_5 + T_6 & -T_3 \\ 0 & -T_3 & T_3 + T_4 \end{pmatrix} \quad \Omega^M_{IJ} = \begin{pmatrix} T_1 + T_4 + T_5 & -T_5 & -T_4 \\ -T_5 & T_2 + T_5 + T_6 & -T_6 \\ -T_4 & -T_6 & T_3 + T_4 + T_6 \end{pmatrix}$$

- Number of independent components are 5 for the Ladder and 6 for the Mercedes.

Three-loop, II

- For the Ladder the b-ghost can only contribute with 3 + 5 + 3 *d* zero modes.
- The vertices has to be attached in pairs to the first and last loop.
- For the Mercedes the b-ghost can contribute with 4 + 4 + 4 *d* zero modes.
- One vertex has to be attached to each loop thus one is 'free'.

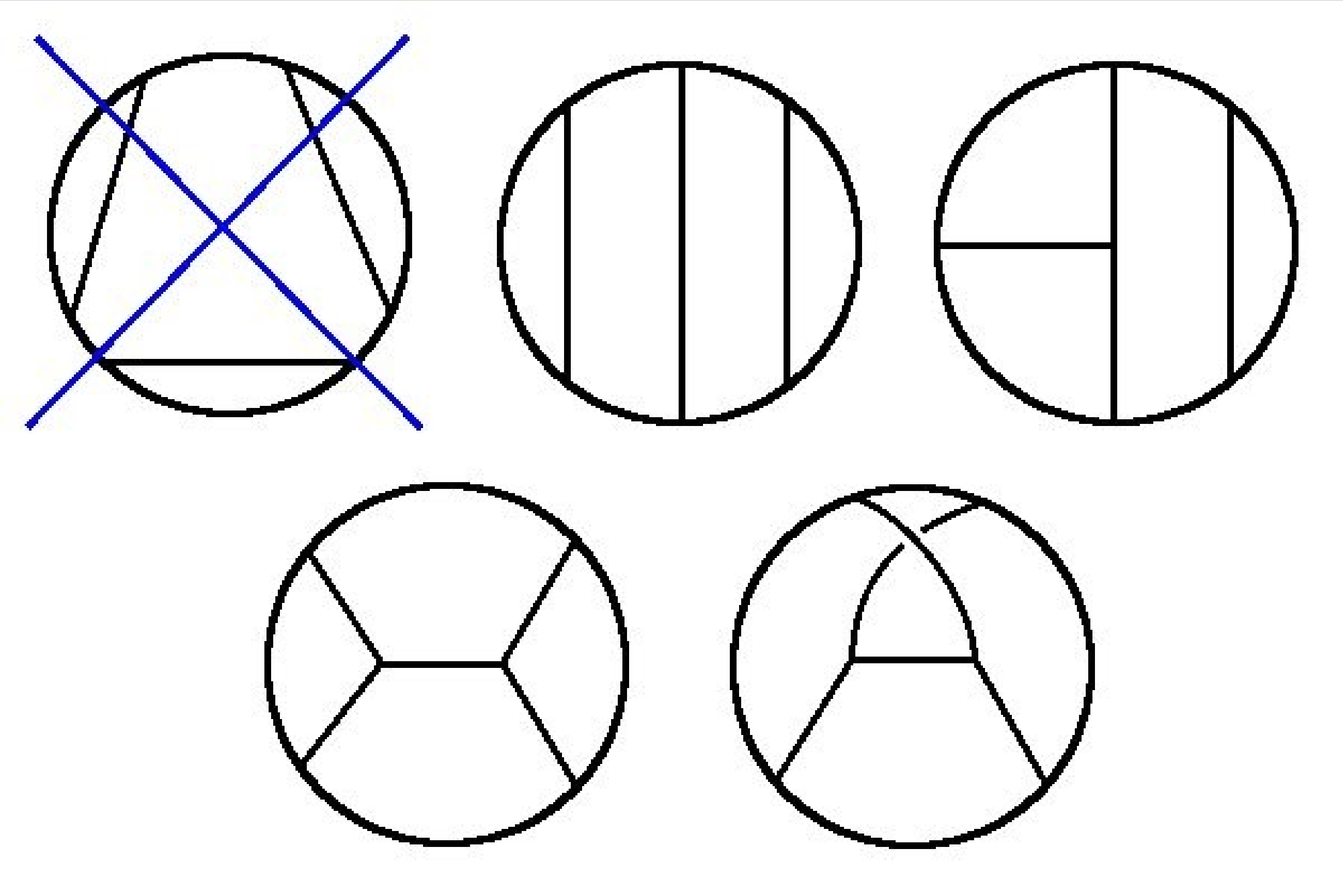
Three-loop, III



$$A_M^{(3)} \sim \left\langle \lambda^3 \hat{\lambda}^3 D^6 D^6 E \hat{E} W^2 \right\rangle\Big|_{\theta^5 \hat{\theta}^5} \Lambda^{3(D-6)} \sim D^{14} \hat{D}^{14} G^4 \Big|_{\theta=\hat{\theta}=0} \Lambda^{3(D-6)} \sim \partial^6 \mathcal{R}^4 \Lambda^{3(D-6)}$$

- Shows that the interaction is an eight BPS and logarithmic divergent in 6 dimensions.

Four-loop, I

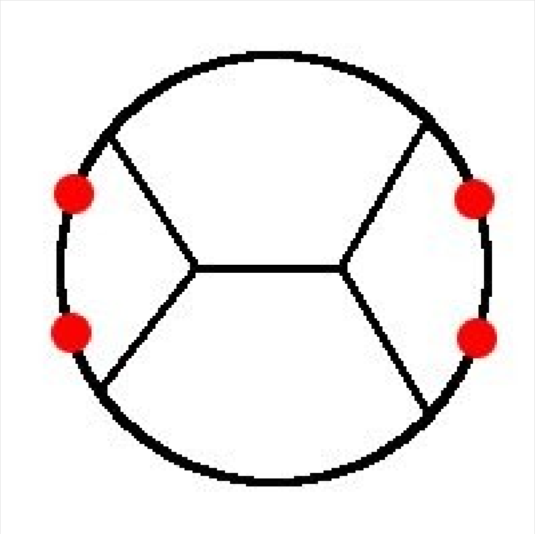


Four-loop, II

- The Ladder has to have two pairs of vertices at the first and fourth loop.
- The third diagram has to have one vertex each for the first two loops and two vertices at the last loop.
- The fourth diagram has to have one vertex at the first and last loop and two 'free' vertices.
- The fifth diagram has to have two fixed vertices in 'opposite pairs'.

Four-loop, III

$$A_H^{(4)} \sim \left\langle \lambda^3 \hat{\lambda}^3 D^9 D^9 E^2 \hat{E}^2 \right\rangle\Big|_{\theta^5 \hat{\theta}^5} \Lambda^{4D-22} \sim D^{16} \hat{D}^{16} G^4 \Big|_{\theta=\hat{\theta}=0} \Lambda^{4D-22)} \sim \partial^8 \mathcal{R}^4 \Lambda^{4D-22)}$$

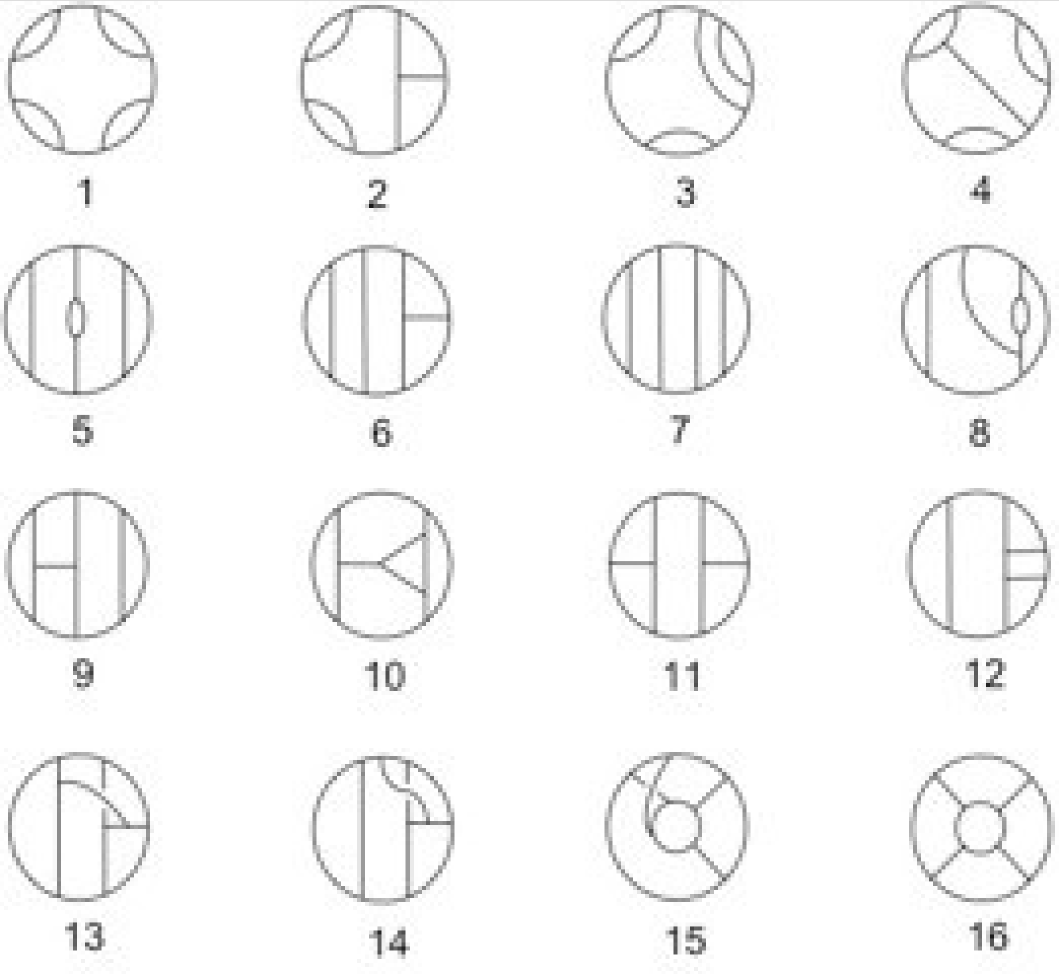


- Shows that this is a non-BPS interaction and is logarithmic divergent in 5.5 dimensions.
- Same behaviour for the non-planar skeleton.

Summary so-far

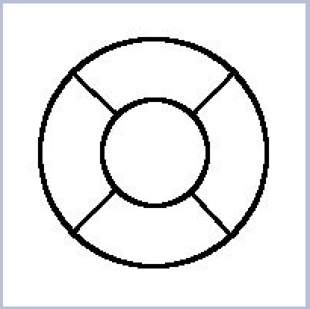
- Match known results using pen and paper.
- Matches the formula $D_c^{(L)} = 4 + 6/L$ which if true for all loops yields that N = 8 is finite.
- The leading order term at four loops is non-BPS.
- As the leading term is non-BPS term for four loops one would expect that it also have contributions from higher orders in perturbation theory.

Five-loop, I



- The interesting skeletons are number 15 and 16.
- Consider the 16'th in more detail.

Five-loop, II



- By using the no-triangle and no-bubble rule of one-loop subdiagrams one get that one do not need to insert any external vertices in the diagram.

- Has 12 insertions of b-ghost which yields at most 24 *d* terms from the second term, but need 25. Where is the missing *d*?

Five-loop, III

- Has a 0/0-singularity: $\left(\frac{r}{(\lambda\bar{\lambda})}\right)^{12}$

- Needs to be regularised.

Berkovits, Nekrasov 2006
Grassi, Vanhove 2009
Aisaka, Berkovits 2009

- Minimal use of the small-lambda regulator shows that one *r* is traded to one *d* as:

$$r \rightarrow \lambda\,\bar{\lambda}\,d$$

- This is the missing *d* term from the b-ghost insertions.

Five-loop, IV

- The vertices all can contribute by the term:

$$V = P^m G_{mn} P^n$$

- One can also show that one can integrate over the positions of the external vertices yielding that the leading low energy term is proportional to the skeleton itself:

$$A^{(5)} \sim \left\langle \lambda^3 \hat{\lambda}^3 D^{11} D^{11} G^4 \right\rangle\Big|_{\theta^5 \hat{\theta}^5} \Lambda^{5D-24} \sim D^{16} \hat{D}^{16} G^4 \Big|_{\theta=\hat{\theta}=0} \Lambda^{5D-24} \sim \partial^8 \mathcal{R}^4 \Lambda^{5D-24}$$

- The term is equal to the term at four loops and is logarithmic divergent in 24/5 dimensions.

Beyond five loops

- One can show by repeated use of $r \to \lambda \bar{\lambda} d$ and $dd \sim P$ that the term from the b-ghost leading to the leading low energy dependance is proportional to:

$$d^{5L} \hat{d}^{5L} P^{4(L-5)}$$

- This yields that the vertices can contribute by the term $V = P^m G_{mn} P^n$ and the leading low energy dependance of the amplitude equals:

$$\begin{aligned} A^{(L)} &\sim \left\langle \lambda^3 \hat{\lambda}^3 D^{11} \hat{D}^{11} G^4 \right\rangle \Big|_{\theta^5 \hat{\theta}^5} \Lambda^{L(D-2)-14} \sim D^{16}\, \hat{D}^{16}\, G^4\, \Lambda^{L(D-2)-14} \\ &\sim \partial^8 \mathcal{R}^4\, \Lambda^{L(D-2)-14} \end{aligned}$$

Properties of the amplitudes

- Observe that the amplitudes are logarithmic divergent in

$$D_c^{(L)} = 2 + \frac{14}{L} \quad L \geq 4$$

Which is different from the formula:

$$D_c^{(L)} = 4 + \frac{6}{L} \quad 2 \leq L \leq 4$$

- The former formula predicts a logarithmic divergence at 7 loops in 4 dimensions.

Vanhove 2010,
Berkovits 2010

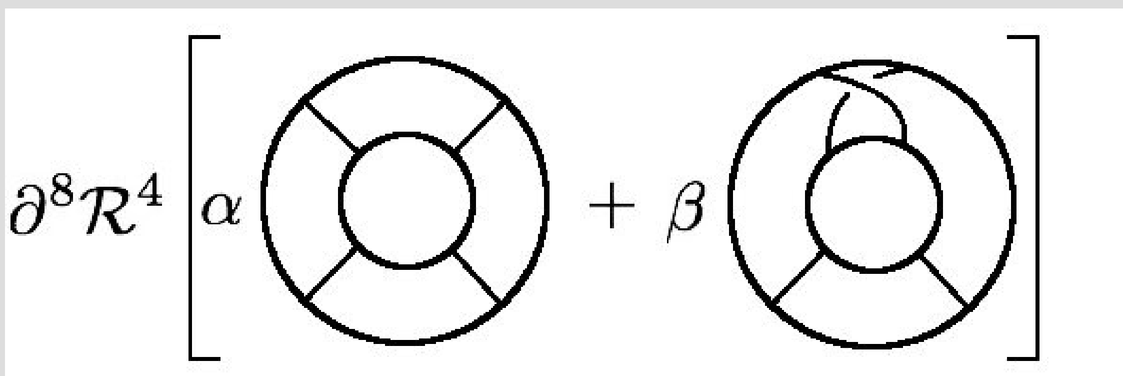
Howe, Lindström 1981
Bossard, Howe, Stelle 2009
Green, Russo, Vanhove 2010

Summary

- Introduced a pure spinor particle approach to computing amplitudes.
- Yields manifest UV-dependence of different skeletons.
- Matches the low energy behaviour of the computations by Bern et al.
- Predict a d*8 R**4 contribution at five loops and beyond.
- Can also be done for SYM yielding a difference between single and double trace operators.

Open questions

- Predicts a d*8 R**4 term, non-zero at five loops?



- Small-lambda regulator?
- Pure spinor $\leftrightarrow$ (RNS and GS)?
- ...