Gradient formula for the beta function of 2d quantum field theory

Anatoly Konechny
Heriot-Watt University

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Outline

- Some history and motivation
- Preliminaries (notations etc.)
- Formulation of the result
- Derivation 1
- Local renormalization operation
- Derivation 2
- Discussion

RG Transformations are governed by a beta function vector field:

$$\mu \frac{d\lambda^i}{d\mu} = \beta^i(\lambda)$$

Gradient flow conjecture (Wallace and Zia, 1974):

$$\beta^i(\lambda) = -G^{ij}(\lambda) \frac{\partial S(\lambda)}{\partial \lambda^j}$$

This implies the monotonic decrease of S

$$\mu \frac{dS}{d\mu} = \beta^i \frac{\partial S}{\partial \lambda^i} = -G_{ij}\beta^i \beta^j \leq 0$$

Hence

- No limiting cycles
- Critical exponents are always real ($\partial_i \beta^j$ is symmetric).

Friedan, 1980 formulated and proved a gradient formula up to two loops for particular classes of 2D General Sigma Models (GSM).
A crucial ingredient for a gradient formula for GSM is the dilaton field Fradkin, Tseytlin, 1985 .
Callan, Friedan, Martinec, Perry, 1985 showed that at one loop the vanishing beta functions equations are equivalent to critical points of a certain functional.
A gradient formula for GSM was checked to two loops by Tseytlin, 1986.
In string theory - $\beta^G = \beta^B = \beta^\Phi = 0$ - equations of motion.
Gradient formula - action principle.

Zamolodchikov, 1986 c-theorem:

$$\mu\frac{dc}{d\mu} = \beta^i\partial_i c = -g_{ij}\beta^i\beta^j \leq 0$$

and showed

$$\partial_i c = -g_{ij}\beta^j$$

in the leading order near fixed points. Is $S = c$ and $G_{ij} = g_{ij}$?
Osborn, 1990 showed that a (modified) gradient formula exists for GSM to all orders in α' provided one can cosntruct a certain integration measure for the zero modes.
Osborn, 1991 proved a formula

$$\partial_i c = -g_{ij}\beta^j - b_{ij}\beta^j$$

for 2D QFT's subject to some power counting restrictions. New ingredient: b_{ij} - an antisymmetric tensor on the space of theories. One can find a nonvanishing b_{ij} at the next to leading order in conformal perturbation theory Freedman, Headrick and A. Lawrence, 2005.

The derivation of Osborn is essentially nonperturbative.

The main ingredient: Wess-Zumino consistency conditions on local Weyl transformations in the presence of curved metric and sources.

A gradient formula was proven for boundary RG flows (Friedan, A.K. 2003) The proof involves some power counting assumptions.

In the present work (Friedan, A.K. 2009) we generalize Osborn's formula not imposing any power counting restrictions.

Preliminaries

Stress energy tensor: Consider Euclidean 2D QFT's with conserved $T_{\mu\nu}(x)$

$$\delta \ln Z = \frac{1}{2} \iint d^2x \, \langle \delta g_{\mu\nu} T^{\mu\nu}(x) \rangle$$

For conformally flat metrics $g_{\mu\nu}(x) = \mu^2(x)\delta_{\mu\nu}$ the change of local scale $\mu(x)$ is generated by the trace $\Theta(x) \equiv g^{\mu\nu}T_{\mu\nu}(x)$

$$\mu(x)\frac{\delta \ln Z}{\delta \mu(x)} = \langle \Theta(x) \rangle$$

For $\mu(x) = \mu =$ Const

$$\mu \frac{\partial}{\partial \mu} \langle \mathcal{O}_1(x_1) \ldots \rangle_c = \int d^2x \, \langle \Theta(x) \mathcal{O}_1(x_1) \ldots \rangle_c$$

Action principle: Schwinger

λ^i - coupling constants coupling to local operators $\phi_i(x)$ so that

$$\frac{\partial}{\partial\lambda^i}\langle\mathcal{O}_1(x_1)\dots\rangle_c = \int d^2x\,\langle\phi_i(x)\mathcal{O}_1(x_1)\dots\rangle_c$$

$\lambda^i \mapsto$ local sources $\lambda^i(x)$

$$\frac{\delta\ln Z}{\delta\lambda^i(x)} = \langle\phi_i(x)\rangle$$

Correlation functions

$$\langle\phi_{i_1}(x_1)\phi_{i_2}(x_2)\dots\Theta(y_1)\Theta(y_2)\dots\rangle_c = \frac{\delta}{\delta\lambda^{i_1}(x)}\ \dots\ \mu(y_m)\frac{\delta}{\delta\mu(y_m)}\ln Z$$

In renormalizable theory the action principle implies

$$\Theta(x) = \beta^i\phi_i(x) \qquad \text{up to contact terms}$$

We prove a general gradient formula

$$\partial_i c + (g_{ij} + \Delta g_{ij})\beta^j + b_{ij}\beta^j = 0$$

where

$$c = 4\pi^2(x^\mu x^\nu x^\alpha x^\beta - x^2 g^{\mu\nu} x^\alpha x^\beta - \frac{1}{2}x^2 x^\mu g^{\nu\alpha} x^\beta)\langle T_{\mu\nu}(x)\, T_{\alpha\beta}(0)\rangle_{/\Lambda|x|=1}$$

$$g_{ij} = 6\pi^2\Lambda^{-4}\,\langle\, \phi_i(x)\,\phi_j(0)\,\rangle_{c\,/\Lambda|x|=1}$$

$$b_{ij} = \partial_i w_j - \partial_j w_i$$

$$w_i = 3\pi\int d^2x\, x^2\theta(1-\Lambda|x|)\langle\phi_i(x)\Theta(0)\rangle_c$$

$$\Delta g_{ij} = \int d^2x\,[G_\Lambda(x) - G_0(x)](\mathcal{L}_\beta - \mu\frac{\partial}{\partial\mu})\langle\phi_i(x)\phi_j\rangle$$

$$G_\Lambda(x) = 3\pi x^2\theta(1-\Lambda|x|)$$

may need subtractions as the integral may diverge in the IR; to be discussed more later.

The assumptions

- The stress-energy tensor is conserved
- The action principle holds; one and two-point functions of ϕ_i and $T_{\mu\nu}$ are once differentiable w.r.t. couplings
- At large distances the global conformal symmetry is not spontaneously broken

$$\lim_{|x|\to\infty} |x|^3 \langle J_\mu(x) T_{\alpha\beta}\rangle_c = 0$$

for any vector field $J_\mu(x)$

The proof

We can express

$$\partial_i c + g_{ij}\beta^j + b_{ij}\beta^j = -r_i$$

via correlators involving the pure contact term field $D(x) = \Theta(x) - \beta^i \phi_i(x)$

$$r_i = \int d^2y \int d^2x \, G_\Lambda(x) \left[\langle \phi_i(y) \, D(x) \, \Theta(0) \rangle - \langle D(y) \, \phi_i(x) \, \Theta(0) \rangle\right]$$

Each term on the LHS can be expressed via integrated 3 point functions of scalar fields ϕ_i, Θ

$$c = -\int d^2x \;\; G_\Lambda(x) \, \langle \, \Theta(x) \, \Theta(0) \, \rangle_c$$

$$\partial_i c = -\int d^2y \int d^2x \;\; G_\Lambda(x) \, \langle \, \phi_i(y) \, \Theta(x) \, \Theta(0) \, \rangle_c$$

$$
\begin{aligned}
g_{ij}\beta^j &= -\Lambda\frac{\partial}{\partial\Lambda}\int d^2x\, G_\Lambda(x)\langle\phi_i(x)\Theta(0)\rangle_c \\
&= \mu\frac{\partial}{\partial\mu}\int d^2x\, G_\Lambda(x)\langle\phi_i(x)\Theta(0)\rangle_c \\
&= \int d^2y \int d^2x\; G_\Lambda(x)\,\langle\,\Theta(y)\,\phi_i(x)\,\Theta(0)\,\rangle_c
\end{aligned}
$$

b_{ij} is defined in terms of integrated 3 point functions of $\langle\phi_i(x)\phi_j(y)\Theta(z)\rangle_c$

Local renormalization operation

Operations are local first order differential operators defined on functionals of the sources $\lambda^i(x)$ and metric $\mu(x)$. Pure contact operations are operations that vanish when restricted to flat metric and constant sources; they can be used to store contact terms present in operator identities.

We call by ordinary fields $O(x) = O^i\phi_i$ with no extra contact terms. For such fields the distributional correlators are obtained from those of fields $\phi_i(x)$ and are thus accurately represented by the corresponding operations.

The generating functional satisfies a differential equation

$$[\mu(x)\frac{\delta}{\delta\mu(x)} - \beta^i(\lambda(x))\frac{\delta}{\delta\lambda^i(x)} - \mathcal{D}(x)]\ln Z = 0$$

where $\mathcal{D}(x)$ is a pure contact operation. The form of $\mathcal{D}(x)$ is constrained by locality and general covariance.

Imposing in addition a 'loose' power counting restriction

$$\mathcal{D}(x) = \frac{1}{2}\mu^2 R_2(x)C(x) + \partial_\mu \lambda^i(x) J_i^\mu(x) + \partial^\mu \left[W_i(x)\partial_\mu \lambda^i\right] + \frac{1}{2}\partial_\mu \lambda^i \partial^\mu \lambda^j G_{ij}(x)$$

where $C(x)$, $W_i(x)$, $G_{ij}(x)$ are ordinary spin-0 fields, and $J_i^\mu(x)$ is an ordinary spin-1 field, and where the 2-d curvature is given by

$$\mu^2 R_2(x) = -2\partial^\mu \partial_\mu \ln \mu(x)\,.$$

The operators $C(x)$, $W_i(x)$, $G_{ij}(x)$ have dimension near zero, slightly irrelevant. Suitable for describing perturbative NLSM.
Strict power counting: $C(x)$, $W_i(x)$, $G_{ij}(x)$ must be all proportional to the identity operator. Holds near unitary fixed points.
We will not assume any power counting restrictions. The results can be later specialized to situations where power counting applies.

The operation $\mathcal{D}(x)$ must satisfy the WZ consistency constraints

$$[\Theta(x) - \beta(x) - \mathcal{D}(x),\, \Theta(y) - \beta(y) - \mathcal{D}(y)] \ln Z = 0$$

One can use the operation $\mathcal{D}(x)$ to extract correlators involving the pure contact term field $D(x) = \Theta(x) - \beta^i \phi_i(x)$.
The Callan-Symanzik equations for correlators at finite separation take the form

$$\mu \frac{\partial}{\partial \mu} \langle\, \phi_{i_1}(x_1) \dots \Theta(y_1) \dots \rangle_c = \beta^i \frac{\partial}{\partial \lambda^i} \langle\, \phi_{i_1}(x_1) \cdots \Theta(y_1) \dots \rangle_c$$
$$+ \langle\, \Gamma \phi_{i_1}(x_1) \dots \Theta(y_1) \dots \rangle_c + \dots + \langle\, \phi_{i_1}(x_1) \dots [-\partial_\mu J^\mu(y_1)] \dots \rangle_c$$
$$+ \dots$$

where

$$\Gamma \phi_{i_1}(x_1) = \partial_{i_1} \beta^i \phi_i(x_1) - \partial_\mu J^\mu_{i_1}(x_1)\,.$$

The vector fields $J_i^\mu(x)$, $J^\mu(y)$ can be extracted from $\mathcal{D}(x)$. One can show that $J^\mu(y) = \partial^\mu C(y)$ and

$$\mu\frac{\partial}{\partial\mu}T_{\alpha\beta}(x) = [\partial_\alpha\partial_\beta - \delta_{\alpha\beta}\partial^2]C(x)$$

- admixtures of trivially conserved currents.

The Wess-Zumino consistency conditions imply an operator identity

$$\partial_\mu J^\mu(x) = \beta^j\partial_\mu J_j^\mu(x)$$

which is true up to contact terms. This is a generalization of what is known as Curci-Paffuti relation for NLSM's.

The proof continued

We would like to express

$$r_i = \int d^2y \int d^2x \, G_\Lambda(x) \, [\langle \phi_i(y) \, D(x) \, \Theta(0) \rangle - \langle D(y) \, \phi_i(x) \, \Theta(0) \rangle]$$

$$G_\Lambda(x) = 3\pi x^2 \theta(1 - \Lambda |x|)$$

in the form $r_i = M_{ij}\beta^j$. Using $\mathcal{D}(x)$ one expresses r_i via two point functions involving the currents J_i^μ and J^μ. The obstruction one encounters is the term

$$r_{i,1} = \int dx^2 \, G_\Lambda(x) \langle \partial_\mu J_i^\mu(x) \Theta(0) \rangle_c$$

A new sum rule

The stress-energy conservation implies the (distributional) identity

$$x^2\langle\, \partial_\mu J_i^\mu(x)\,\Theta(0)\,\rangle_c =$$

$$\partial_\mu\left[x^2\langle\, J_i^\mu(x)\,\Theta(0)\,\rangle_c - 2x_\alpha x^\beta\langle\, J_i^\alpha(x)\,T_\beta^\mu(0)\,\rangle_c + x^2\langle\, J_i^\alpha(x)\,T_\alpha^\mu(0)\,\rangle_c\right]$$

Assuming

$$\lim_{|x|\to\infty} |x|^3\langle\, J_i^\mu(x)\,T_{\alpha\beta}(0)\,\rangle_c = 0$$

we obtain a sum rule

$$\int d^2x\, x^2\,\langle\, \partial_\mu J_i^\mu(x)\,\Theta(0)\,\rangle_c = 0$$

This allows us to write

$$r_{i,1} = \int dx^2\,[G_\Lambda(x) - G_0(x)]\langle\partial_\mu J_i^\mu(x)\Theta(0)\rangle_c$$

$$G_\Lambda(x) - G_0(x) = 3\pi x^2 \theta(\Lambda|x| - 1)$$

The correlator in the integrand is now at a finite separation and we can use $\Theta = \beta^i \phi_i$. With an infrared cutoff L we can write

$$r_{i,1} = \int_{|x|<L} dx^2 \, [G_\Lambda(x) - G_0(x)] \langle \partial_\mu J_i^\mu(x) \phi_j(0) \rangle_c \beta^j$$

Removing cutoff needs care as the integral may be divergent. The divergence must be orthogonal to the beta function and a subtraction does no harm to the gradient formula. This is the only place where the IR condition on the decay of $T_{\alpha\beta}$ is used. Together with another contribution, which is proportional to β^i by virtue of the WZ condition, we obtain

$$\partial_i c + g_{ij}\beta^j + b_{ij}\beta^j = -r_{i,1} - r_{i,2} = -\Delta g_{ij}\beta^j$$

$$\Delta g_{ij} = \mathrm{Sym}_{i,j} \int dx^2 \, [G_\Lambda(x) - G_0(x)] \langle \partial_\mu J_i^\mu(x) \phi_j(0) \rangle_c$$

$$= \int d^2x \, [G_\Lambda(x) - G_0(x)] (\mathcal{L}_\beta - \mu \frac{\partial}{\partial \mu}) \langle \phi_i(x) \phi_j \rangle$$

with an IR subtraction implemented.

The proof can be done with an IR cutoff systematically implemented and removed in the end. The IR convergence of all terms except Δg_{ij} follows from the IR assumptions.

END OF PROOF

Discussion

The infrared condition

$$\lim_{|x|\to\infty} |x|^3 \langle\, J_i^{\mu}(x)\, T_{\alpha\beta}(0)\,\rangle_c = 0$$

is needed as follows from an example. Consider a free compact boson

$$S[R, g_{\mu\nu}] = \frac{1}{8\pi}\int d^2x\, (\lambda\sqrt{g} g^{\mu\nu}\partial_\mu X \partial_\nu X + QX\sqrt{g}R_2)$$

on a topological plane

$$\int d^2x\, \sqrt{g} R_2 = 0$$

The zero mode integral is then well defined. The beta function vanishes.

The anomaly can be computed to be

$$D(x) = \frac{1}{2}C(\lambda)\sqrt{g}R_2(x) + J^\mu_\lambda(x)\partial_\mu\lambda + \frac{1}{2}g_{\lambda\lambda}\partial_\mu\lambda\partial^\mu\lambda + \partial_\mu(w_\lambda\partial^\mu\lambda)$$

where

$$C(\lambda) = \frac{1}{12\pi} + \frac{Q^2}{4\pi\lambda}\,, \quad J^\mu_\lambda(x) = -\frac{Q}{4\pi\lambda}\partial^\mu X(x)\,, \quad g_{\lambda\lambda} = \frac{1}{64\pi\lambda^2}$$

The violation of gradient formula comes from the mixed anomaly

$$\langle T(z)J_{\lambda,z}(0)\rangle = -\frac{Q^2}{4\pi\lambda^2}\frac{1}{z^3}$$

The special conformal symmetry is broken by the boundary condition at infinity.
In NLSM the spherical topology is needed to have a factor $e^{-\Phi}$ in the zero modes integration measure which is crucial for the gradient formula Osborn, 1991

Dilaton couplings and gradient formula for NLSM

To make the Callan-Symanzik equations for $T_{\alpha\beta}$ correlators fully covariant introduce the dilaton couplings according to

$$\frac{\delta \ln Z}{\delta \lambda_D^i(x)} = \frac{1}{2}\mu^2 R_2(x)\frac{\delta \ln Z}{\delta \lambda^i(x)}$$

The C.-S. equations for NLSM take the form

$$\mu\frac{\partial}{\partial\mu}\langle T_{\mu\nu}(y_1)T_{\alpha\beta}(y_2)\dots\rangle_c = \beta^i\frac{\partial}{\partial\lambda^i}\langle T_{\mu\nu}(y_1)T_{\alpha\beta}(y_2)\dots\rangle_c$$
$$+\langle\Gamma^C_{\mu\nu}(y_1)T_{\alpha\beta}(y_2)\dots\rangle_c + \langle T_{\mu\nu}(y_1)\Gamma^C_{\alpha\beta}(y_2)\dots\rangle_c + \dots$$
$$= (\beta^i\frac{\partial}{\partial\lambda^i} + \beta_D^i\frac{\partial}{\partial\lambda_D^i})\langle T_{\mu\nu}(y_1)T_{\alpha\beta}(y_2)\dots\rangle_c$$

$$\Gamma^C_{\mu\nu}(x) = (\partial_\mu\partial_\nu - g_{\mu\nu}\partial_\alpha\partial^\alpha)C(x)\,, \quad C(x) = \beta_D^i\phi_i(x)$$

$$\mu\frac{\partial c}{\partial\mu} = (\beta^i\frac{\partial}{\partial\lambda^i} + \beta^i_D\frac{\partial}{\partial\lambda^i_D})c$$

It follows from the definition

$$\frac{\partial c}{\partial\lambda^i_D} = 2\int d^2x\, G_\Lambda(x)\langle\Theta(x)\partial_\mu\partial^\mu\phi_i(0)\rangle_c$$

Integrating by parts we get

$$\frac{\partial c}{\partial\lambda^i_D} = -g^D_{ij}\beta^j$$

$$g^D_{ij} = 2\int d^2x\,[G_0(x) - G_\Lambda(x)]\langle\phi_j(x)\partial_\mu\partial^\mu\phi_i(0)\rangle_c$$

It further follows that

$$\beta^i_D \frac{\partial c}{\partial \lambda^i_D} = \beta^i \Delta g_{ij} \beta^j$$

and we see that $\Delta g_{ij} \neq 0$ when c depends onλ^i_D. Contracting the gradient formula with β^i gives the Zamolodchikov formula

$$\mu \frac{\partial c}{\partial \mu} = (\beta^i \frac{\partial}{\partial \lambda^i} + \beta^i_D \frac{\partial}{\partial \lambda^i_D})c = -\beta^i g_{ij} \beta^j$$

We see that fixed points $\beta^i = 0$ are in one-to-one correspondence with the extrema

$$\frac{\partial c}{\partial \lambda^i} = \frac{\partial c}{\partial \lambda^i_D} = 0$$

Further directions

- Explicit calculations for NLSM and current-current perturbations of WZW theory. Hard to use/check the IR condition and sum rule perturbatively.
- Implications for string theory should be understood. We worked throughout with normalized connected correlators.
- What is the geometry on the spaces of 2D QFT's?