My Collaborators

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Part I: Introduction
What is an M2-brane?

- **Example from EM**: A charged particle moving along a 1 dimensional worldline is a source of 1-form field $A_\mu$.

- In supergravity, a $p$-brane is a $(p + 1)$ space-time dimensional object sourcing the $(p + 1)$-form gauge field.

- In 11d SUGRA, the only antisymmetric tensor field is the 3-form $A^{(3)}$. The corresponding field strength is a 4-form $F^{(4)} = dA^{(3)}$.

  - **Maxwell eq. for an electric source**: $d \star F^{(4)} = \star \delta^{(3)}$

    $\Rightarrow$ Elec. charge is localised in 3 ($= 2 + 1$) spacetime dim. $\Rightarrow$ M2-brane.

  - **Maxwell eq. for a magnetic source**: $dF^{(4)} = \star \delta^{(6)}$

    $\Rightarrow$ Mag. charge is localised in 6 ($= 5 + 1$) spacetime dim. $\Rightarrow$ M5-brane.
Motivation

- How many conformal field theories (CFTs) do we know in $(2 + 1)$ dimensions?
- What are the worldvolume theories of a stack of $N$ M2-branes in M-theory?
- Understand Chern-Simons (CS) theories better
- Algebraic Geometry and Quiver Gauge Theories
Motivation: AdS/CFT

- **Well-known:** String theory in $\text{AdS}_5 \times S^5 \leftrightarrow (3+1)d \mathcal{N} = 4 \text{ SYM}
- **Known:** String theory in $\text{AdS}_5 \times \text{SE}^5 \leftrightarrow (3+1)d \mathcal{N} = 1 \text{ SCFT}

**Long standing problem:**

- M-theory in $\text{AdS}_4 \times \text{SE}^7 \leftrightarrow$ which field theories?
- Different $\text{SE}^7$'s leads to CFTs
- Such field theories live on $\mathcal{N}$ M2-branes at the tip of the CY cone over $\text{SE}^7$
- **(2+1)d SUSY CS-matter theories** (Martelli-Sparks, Hanany-Zaffaroni, etc.)
Part II: $\mathcal{N} = 2$ CS-Matter Theories

- Theories with $\mathcal{N} = 1$ SUSY in $(2 + 1)d$ have no holomorphy properties
  ⇒ We cannot control their non-perturbative dynamics
- Start with $\mathcal{N} = 2$ SUSY (4 supercharges) in $(2 + 1)d$.
  This may get enhanced to higher SUSY.
An $\mathcal{N} = 2$ CS-Matter Theory

- **Gauge group:** $\mathcal{G} = \prod_{a=1}^{G} U(N)_a$

- **The 3d $\mathcal{N} = 2$ vector multiplet $V_a$.** Can be obtained from a dimensional reduction of 4d $\mathcal{N} = 1$ vector multiplet.
  - A one-form gauge field $A_a$, a real scalar field $\sigma_a$ (from the components of the vector field in the compactified direction), a two-component Dirac spinor $\chi_a$, a real auxiliary scalar fields $D_a$.
  - All fields transform in the adjoint representation of $U(N)_a$.

- **The chiral multiplet.** It consists of matter fields $\Phi_{ab}$, charged in the gauge groups $U(N)_a$ and $U(N)_b$.
  - Complex scalars $X_{ab}$, Fermions $\psi_{ab}$, Auxiliary scalars $F_{ab}$.
**$\mathcal{N} = 2$ CS-Matter Lagrangian**

- The action consists of 3 terms:
  \[
  S = S_{\text{CS}} + S_{\text{matter}} + S_{\text{potential}}.
  \]

- CS terms in Wess–Zumino gauge:
  \[
  S_{\text{CS}} = \sum_{a=1}^{G} \frac{k_a}{4\pi} \int \text{Tr} \left( A_a \wedge dA_a + \frac{2}{3} A_a \wedge A_a \wedge A_a - \bar{\chi}_a \chi_a + 2D_a \sigma_a \right),
  \]
  where $k_a$ are called the **CS levels**. Gauge fields are non-dynamical.

- The matter term is
  \[
  S_{\text{matter}} = \int d^3x \ d^4\theta \sum_{\Phi_{ab}} \text{Tr} \left( \Phi_{ab}^+ e^{-V_a} \Phi_{ab} e^{V_b} \right).
  \]

- The superpotential term is
  \[
  S_{\text{potential}} = \int d^3x \ d^2\theta W(\Phi_{ab}) + \text{c.c.}.
  \]
What Is Special in $2 + 1$ dimensions?

- The Yang–Mills coupling has mass dimension $1/2$ in $(2 + 1)$ dimensions
  - All theories are strongly coupled in the IR

- The CS levels $\kappa_\alpha$ are integer valued
  (so that the path integral is invariant under large gauge transformation)
  - **Non-renormalisable theorem (NRT):** Each $\kappa_\alpha$ is not renormalised beyond a possible finite 1-loop shift [Witten '99]

- The action are classically marginal ($\kappa_\alpha$ have mass dimension 0)

- NRT $\Rightarrow$ The action is also quantum mechanically exactly marginal
  (Any quantum correction is irrelevant in the IR or can be absorbed by field redef.) [Gaiotto-Yin '07]

- The theory is **conformally invariant** at the quantum level
The Mesonic Moduli Space

The vacuum equations:

- F-terms: \( \partial X_{ab} W = 0 \)
- 1st D-terms: \( \sum_{b=1}^{G} X_{ab}X_{ab}^\dagger - \sum_{c=1}^{G} X_{ca}^\dagger X_{ca} + [X_{aa}, X_{aa}^\dagger] = 4k_{a}\sigma_{a} \)
- 2nd D-terms: \( \sigma_{a}X_{ab} - X_{ab}\sigma_{b} = 0 \).

Note that the fields \( X_{ab}, \sigma_{a} \) are matrices, and no summation convention.

Space of solutions of these eqns are called the mesonic moduli space, \( \mathcal{M}^{\text{mes}} \).

The F-terms and the LHS of the 1st D-terms are familiar in 3+1 dimensions

The RHS of 1st D-terms and 2nd D-terms are new in 2+1 dimensions.
Quiver Gauge Theories

What is a quiver gauge theory?

- It is a gauge theory associated with a directed graph with nodes and arrows.
  - Each node represents each factor in the gauge group $G$.
  - Each arrow going from a node $a$ to a different node $b$ represents a field $X_{ab}$ in the bifundamental rep. $(N, \overline{N})$ of $U(N)_a \times U(N)_b$.
  - Each loop on a node $a$ represents a field $\phi_a$ in the adjoint rep. of $U(N)_a$.

**Drawback:** A quiver diagram does NOT fix the superpotential.

For a $(2 + 1)d$ CS quiver theory, need to assign the CS levels $k_a$ to each node.
Abelian CS Quiver Theories

- The fields $X_{ab}, \sigma_a$ are just complex numbers.
- The vacuum equations do the following things:
  - Set all $\sigma_a$ to a single field, say $\sigma$. It is a real field.
  - Impose the following condition on the CS levels: $\sum_a k_a = 0$.
- Define the CS coefficient: $k \equiv \gcd(\{k_a\})$. 
Let’s consider first the abelian case $N = 1$.

- Solving the vacuum equations in 2 steps:
  1. Solving F-terms. The space of solutions of F-terms is the Master space, $\mathcal{F}^b$.
  2. Further solving D-terms: Modding out $\mathcal{F}^b$ by the gauge symmetry.

- Among the original gauge symmetry $U(1)^G$, one is a diagonal $U(1)$; it does not couple to matter fields $\rightarrow$ We are left with $U(1)^{G-1}$.

- Up to this point, the process is the same for a (3+1)d theory living on a D3-brane probing $\text{CY}_3$. 

Moduli Space of a CS Quiver Theory

- 1st D-terms:

\[ \sum_{b=1}^{G} X_{ab} X_{ab}^\dagger - \sum_{c=1}^{G} X_{ca}^\dagger X_{ca} + [X_{aa}, X_{aa}^\dagger] = 4k_a \sigma \]

- The CS levels induce FI-like terms: \(4k_a \sigma\).

- This gives a fibration of CY\(_3\) over \(\mathbb{R}\) \(\Rightarrow\) Total space is CY\(_4\).

- The mesonic moduli space \(M^{\text{mes}}\) is a CY\(_4\).

- Remaining D-terms gauge redundancy: \(U(1)^{G-2}\) (baryonic directions)

- Therefore, the mesonic moduli space can be written as

\[ M^{\text{mes}}_{N=1,k} = \left( \mathcal{F}^b / / U(1)^{G-2} \right) / \mathbb{Z}_k \]

- For higher \(N\), the moduli space is

\[ M^{\text{mes}}_{N,k} = \text{Sym}^N \left( M^{\text{mes}}_{N=1,k} \right) \]
Part III: Brane Tilings
What is known in 3+1 dimensions?

- SCFTs on D3-branes probing CY₃ are best described in terms of brane tilings [Hanany et al. from '05]
- The gravity dual of each theory is on the AdS₅ × Y₅ background (Y₅ being a 5 dimensional Sasaki-Einstein manifold)
- **Example:** The $\mathcal{N} = 4$ Super Yang-Mills (Y₅ is a 5-sphere $S^5$)
Example: The $\mathcal{N} = 1$ conifold theory [Klebanov-Witten '98]

- **2n sided face** = $U(N)$ gauge group with $nN$ flavours
- **Edge** = A chiral field charged under the two gauge group corresponding to the faces it separates
- **$D$ valent node** = A $D$-th order interaction term in superpotential
Graph is bipartite: Nodes alternate between clockwise (white) and anticlockwise (black) orientations of arrows.

Black (white) nodes connected to white (black) only

Odd sided faces are forbidden by anomaly cancellation condition

White (black) nodes give $+\ (-)\ sign\ in\ the\ superpotential$

**Conifold theory:** \[ W = \text{Tr}(X_{12}^1 X_{21}^1 X_{12}^2 X_{21}^2 - X_{12}^1 X_{21}^2 X_{12}^2 X_{21}^1) \]
Assign a CS level to each gauge group (node in quiver & face in the tiling).

Rules above still work!

Each brane tiling (with specified CS levels) defines a unique Lagrangian for an $\mathcal{N} = 2$ CS theory (4 supercharges) in 2+1 dimensions.

The tiling has an interpretation of a network of D4-branes and NS5-brane ending on the NS5-brane in Type IIA. (Imamura & Kimura ’08)

Largest known family of SCFTs in (2 + 1) dimensions!
Example: The ABJM Theory [Aharony, Bergman, Jafferis, Maldacena ’08]

- **Gauge group**: $U(N) \times U(N)$. **CS levels**: $(k, -k)$.
- **Superpotential**: $W = \text{Tr}(X_{12}X_{21}X_{12}X_{21} - X_{12}X_{21}X_{12}X_{21})$.
- **The case of** $N = 1$ and $k = 1$: $W = 0$
  - The F-terms admit any complex solutions of $X_{12}^i, X_{21}^i$ $(i = 1, 2)$
  - The Master space is $\mathcal{F}^b = \mathbb{C}^4$
  - The mesonic moduli space is $\mathcal{M}_{N=1}^{\text{mes}} = \mathcal{F}^b / U(1)^{G-2} = \mathbb{C}^4$
  - The moduli space generated by $X_{12}^i, X_{21}^i$ (each has scaling dimension $1/2$)
  - These are free scalar fields
Example: A Conifold \((\mathcal{C}) \times \mathbb{C}\) Theory

- **Gauge group:** \(U(1) \times U(1)\). **CS levels:** \((1, -1)\).

- **Superpotential:** 
  \[
  W = \text{Tr} \left( \phi_1 (X_{12}^2 X_{21}^1 - X_{12}^1 X_{21}^2) + \phi_2 (X_{21}^2 X_{12}^1 - X_{21}^1 X_{12}^2) \right)
  \]

- The \(\mathbb{C}\) is parametrised by \(\phi_1 = \phi_2\), and the \(\mathcal{C}\) is generated by \(X_{12}^i, X_{21}^i\).

- **Non-trivial scaling dimensions:** \(1/2\) for \(\phi\)'s and \(3/4\) for \(X\)'s (by symmetry argument).

- These values agree with a computation on the gravity dual (volume minimisation of \(SE^7\)). This is a (weak) test of AdS/CFT.
Toric Structures

- The moduli space of $N = 1$ theories admits a **toric structure**, due to the $U(1)$ quotients in $\mathcal{M}_{N=1,k=1}^{\text{mes}} = \mathcal{F}^b // U(1)^{G-2}$

- The toric data of the moduli space are collected in the toric diagram, which is unique up to a $GL(3, \mathbb{Z})$ transformation

- There is a prescription (called the **forward algorithm**) in going from brane tilings to toric diagrams

The toric diagram of $\mathbb{C}^4$  

The toric diagram of $\mathbb{C} \times \mathbb{C}$
Part IV: Toric Phases
There are some models which have different brane tilings, but have the same mesonic moduli space in the IR.

These models are said to be toric dual to each other. Each of these models is referred to as toric phase.

In \((3 + 1)d\), toric duality is understood to be Seiberg duality \((\text{Feng, Hanany, He, Uranga; Beasley, Plesser '01})\). This is however not clear in \((2 + 1)d\).

The following quantities are matched between toric phases:

- Mesonic moduli spaces & toric diagrams
- Chiral operators & partition functions (Hilbert series)
- Global symmetries
- Scaling dimensions (R-charges) of chiral operators
Phases of The $\mathbb{C}^4$ Theory

- Phase I: The ABJM model ($k_1 = -k_2 = 1$)

  Note: In $(3 + 1)d$, these two pictures correspond to the conifold theory.

- Phase II: The Hanany-Vegh-Zaffaroni (HVZ) model ($k_1 = -k_2 = 1$)
The toric diagram of $\mathbb{C}^4$

The lift of a point in toric diagram due to CS levels $(1, -1)$

The $(3 + 1)d$ conifold theory  The $(2 + 1)d$ ABJM model
Phases of The Conifold $(\mathcal{C}) \times \mathbb{C}$ Theory

- Phase I: $k_1 = -k_2 = 1, k_3 = 0$

- Phase II: $k_1 = -k_2 = 1$
  Note: In $(3 + 1)d$, these two pictures correspond to the $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ theory.

- Phase III: $k_1 = 0, k_2 = -k_3 = 1$
The toric diagram of $\mathbb{C} \times \mathbb{C}$

The toric diagram of $\mathbb{C} \times \mathbb{C}$

The lift of points in toric diagram due to CS levels $(1, -1)$

The $(3+1)d \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ theory

The $(2+1)d \mathbb{C} \times \mathbb{C}$ theory
Phases of The $D_3$ Theory

- Phase I: $k_1 = k_2 = -k_3 = -k_4 = 1$

- Phase II: $k_1 = -k_2 = 1$, $k_3 = 0$
  Note: In $(3+1)d$, these are of the SPP theory.

- Phase III: $k_1 = -k_2 = k_3 = -k_4 = 1$
The toric diagram of $D_3$

The toric diagram of $D_3$

The lift of points in toric diagram due to CS levels $(1, -1, 0)$

The $(3+1)d$ SPP theory

The $(2+1)d$ $D_3$ theory
Part V: Fano 3-folds
What are Fano surfaces?

- Fano $n$-folds are $n$ dim complex manifolds admitting positive curvatures
- Fano 2-folds are $\mathbb{P}^1 \times \mathbb{P}^1$ and the del Pezzo surfaces $dP_n$ (which are $\mathbb{P}^2$ blown-up at $0 \leq n \leq 8$ points). Only $\mathbb{P}^1 \times \mathbb{P}^1$ and $dP_{n=0,1,2,3}$ are toric.
- There are precisely 18 different smooth toric Fano 3-folds (Batyrev '82). Their toric diagrams are known (http://malham.kent.ac.uk/grdb/FanoForm.php).
- Study theories on M2-branes probing a cone over Fano 3-folds
- **Problem**: Translate toric data to brane tilings
The $M^{1,1,1}$ theory

- Gauge group: $U(1) \times U(1) \times U(1)$. The CS levels: $\vec{k} = (1, 1, -2)$
- The mesonic global symmetry is $\mathcal{G} = SU(3) \times SU(2) \times U(1)_R$
- The scaling dimensions of quiver fields $X^i_{12}, X^i_{23}, X^i_{31}$ are $7/9, 7/9, 4/9$.
- The operators are in the rep $(3n, 0; 2n)_{2n}$ of $\mathcal{G}$. This can be computed directly from the field theory side (using Hilbert series) and confirms the known KK spectrum.
The $M^{1,1,1}$ theory from a cone over $\mathbb{P}^2 \times \mathbb{P}^1$

The toric diagram of the $M^{1,1,1}$ theory ($\mathbb{P}^2 \times \mathbb{P}^1$)

- The 4 blue points form the toric diagram of $\mathbb{P}^2$
- The 2 black points together with the blue internal point form the toric diagram of $\mathbb{P}^1$
The $Q^{1,1,1}/\mathbb{Z}_2$ theory

- **Phase I:** $k_1 = -k_2 = -k_3 = k_4 = 1$

- **Phase II:** $k_1 = k_2 = -k_3 = -k_3' = 1$
The $Q^{1,1,1}/\mathbb{Z}_2$ theory from a cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

The toric diagram of the $Q^{1,1,1}/\mathbb{Z}_2$ theory ($\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$).

- The mesonic global symmetry is $SU(2)^3 \times U(1)_R$.
- The mesonic operators are in the rep $(2n; 2n; 2n)_{2n}$ of $SU(2)^3 \times U(1)_R$. 
The $dP_n \times \mathbb{P}^1$ theories

- The $dP_1 \times \mathbb{P}^1$ theory, $\vec{k} = (1, 1, -1, -1)$

- The $dP_2 \times \mathbb{P}^1$ theory, $\vec{k} = (1, 1, -1, 0, -1)$
The $dP_n \times \mathbb{P}^1$ theories (continued)

- The $dP_3 \times \mathbb{P}^1$ theory, $\vec{k} = (0, 0, 0, 0, -1, 1)$

- The toric diagrams of (i) $dP_1 \times \mathbb{P}^1$, (ii) $dP_2 \times \mathbb{P}^1$, (iii) $dP_3 \times \mathbb{P}^1$
Conclusions

- All theories described are conjectured to live on the worldvolume of M2-branes probing the $\text{CY}_4$, which is also the mesonic moduli space
- Infinite families of SCFTs
- A variety of scaling dimensions
- Toric duality