Brane Tilings, M2-Branes and Chern-Simons Theories

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- Special thanks to: Yang-Hui He, Alexander Shannon, and Alberto Zaffaroni



Part I: Introduction

What is an M2-brane?

- Example from EM: A charged particle moving along a 1 dimensional worldline is a source of 1-form field A_μ.
- In supergravity, a p-brane is a $(p+1)$ space-time dimensional object sourcing the $(p+1)$-form gauge field.
- In 11d SUGRA, the only antisymmetric tensor field is the 3-form $A^{(3)}$. The corresponding field strength is a 4-form $F^{(4)} = dA^{(3)}$.
  - Maxwell eq. for an electric source: $\underbrace{d \overbrace{* F^{(4)}}^{7-\text{form}}}_{8-\text{form}} = *\delta^{(3)}$

    $\Rightarrow$ Elec. charge is localised in 3 $(= 2 + 1)$ spacetime dim. $\Rightarrow$ M2-brane.
  - Maxwell eq. for a magnetic source: $\underbrace{dF^{(4)}}_{5-\text{form}} = *\delta^{(6)}$

    $\Rightarrow$ Mag. charge is localised in 6 $(= 5 + 1)$ spacetime dim. $\Rightarrow$ M5-brane.

Motivation

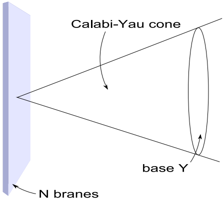
- How many conformal field theories (CFTs) do we know in $(2+1)$ dimensions?
- What are the worldvolume theories of a stack of N M2-branes in M-theory?
- Understand Chern-Simons (CS) theories better
- Algebraic Geometry and Quiver Gauge Theories

Motivation: AdS/CFT

- **Well-known:** String theory in $\mathrm{AdS}_5 \times \mathrm{S}^5 \quad \leftrightarrow \quad (3+1)$d $\mathcal{N}=4$ SYM
- **Known:** String theory in $\mathrm{AdS}_5 \times \mathrm{SE}^5 \quad \leftrightarrow \quad (3+1)$d $\mathcal{N}=1$ SCFT

Long standing problem:

- M-theory in $\mathrm{AdS}_4 \times \mathrm{SE}^7 \quad \leftrightarrow \quad$ which field theories?
- Different SE^7's leads to CFTs
- Such field theories live on N M2-branes at the tip of the CY cone over SE^7
- **(2+1)d SUSY CS-matter theories** (Martelli-Sparks, Hanany-Zaffaroni, etc.)



Part II: $\mathcal{N} = 2$ CS-Matter Theories

- Theories with $\mathcal{N} = 1$ SUSY in $(2+1)$d have no holomorphy properties

  $\Rightarrow$ We cannot control their non-perturbative dynamics

- Start with $\mathcal{N} = 2$ SUSY (4 supercharges) in $(2+1)$d.

  This may get enhanced to higher SUSY.

An $\mathcal{N} = 2$ CS-Matter Theory

- Gauge group: $\mathcal{G} = \prod_{a=1}^{G} U(N)_a$
- The 3d $\mathcal{N} = 2$ vector multiplet V_a. Can be obtained from a dimensional reduction of 4d $\mathcal{N} = 1$ vector multiplet.
    - A one-form gauge field A_a , a real scalar field σ_a (from the components of the vector field in the compactified direction) , a two-component Dirac spinor χ_a , a real auxiliary scalar fields D_a.
    - All fields transform in the adjoint representation of $U(N)_a$:
- The chiral multiplet. It consists of matter fields Φ_{ab}, charged in the gauge groups $U(N)_a$ and $U(N)_b$.
    - Complex scalars X_{ab} , Fermions ψ_{ab} , Auxiliary scalars F_{ab} .

$\mathcal{N} = 2$ CS-Matter Lagrangian

- The action consists of 3 terms: $S = S_{\rm CS} + S_{\rm matter} + S_{\rm potential}$.
- CS terms in Wess–Zumino gauge:

$$S_{\rm CS} = \sum_{a=1}^{G} \frac{k_a}{4\pi} \int \mathrm{Tr} \left(A_a \wedge \mathrm{d}A_a + \frac{2}{3} A_a \wedge A_a \wedge A_a - \bar{\chi}_a \chi_a + 2 D_a \sigma_a \right) ,$$

where k_a are called the **CS levels**. Gauge fields are non-dynamical.

- The matter term is

$$S_{\rm matter} = \int \mathrm{d}^3 x \ \mathrm{d}^4 \theta \sum_{\Phi_{ab}} \mathrm{Tr} \left(\Phi_{ab}^{\dagger} e^{-V_a} \Phi_{ab} e^{V_b} \right) .$$

- The superpotential term is

$$S_{\rm potential} = \int \mathrm{d}^3 x \ \mathrm{d}^2 \theta \, W(\Phi_{ab}) + \mathrm{c.c.} \ .$$

What Is Special in $2+1$ dimensions?

- The Yang–Mills coupling has mass dimension $1/2$ in $(2+1)$ dimensions
  - All theories are strongly coupled in the IR
- The CS levels k_a are integer valued

  (so that the path integral is invariant under large gauge transformation)
  - **Non-renormalisable theorem (NRT):** Each k_a is not renormalised beyond a possible finite 1-loop shift [Witten '99]
- The action are classically marginal (k_a have mass dimension 0)
- NRT $\Rightarrow$ The action is also quantum mechanically exactly marginal

  (Any quantum correction is irrelevant in the IR or can be absorbed by field redef.) [Gaiotto-Yin '07]
- The theory is conformally invariant at the quantum level

The Mesonic Moduli Space

- The vacuum equations:
  - F-terms: $\partial_{X_{ab}} W = 0$
  - 1st D-terms: $\sum_{b=1}^{G} X_{ab} X_{ab}^{\dagger} - \sum_{c=1}^{G} X_{ca}^{\dagger} X_{ca} + [X_{aa}, X_{aa}^{\dagger}] = 4k_a \sigma_a$
  - 2nd D-terms: $\sigma_a X_{ab} - X_{ab} \sigma_b = 0$.
  - Note that the fields X_{ab}, σ_a are matrices, and no summation convention.
- Space of solutions of these eqns are called the mesonic moduli space, $\mathcal{M}^{\text{mes}}$.
- The F-terms and the LHS of the 1st D-terms are familiar in 3+1 dimensions
- The RHS of 1st D-terms and 2nd D-terms are new in 2+1 dimensions.

Quiver Gauge Theories

What is a quiver gauge theory?

- It is a gauge theory associated with a directed graph with nodes and arrows.
  - Each node represents each factor in the gauge group $\mathcal{G}$.
  - Each arrow going from a node a to a different node b represents a field X_{ab} in the bifundamental rep. $(\mathbf{N}, \overline{\mathbf{N}})$ of $U(N)_a \times U(N)_b$.
  - Each loop on a node a represents a field ϕ_a in the adjoint rep. of $U(N)_a$.
  - **Drawback:** A quiver diagram does NOT fix the superpotential

- For a $(2+1)$d CS quiver theory, need to assign the CS levels k_a to each node.

Abelian CS Quiver Theories

- Take $N = 1$. Gauge group $\mathcal{G} = U(1)^G$.
- The fields X_{ab}, σ_a are just **complex numbers**.
- The **vacuum equations** do the following things:
  - Set all σ_a to a single field, say σ. It is a real field.
  - Impose the following condition on the CS levels: $\sum_a k_a = 0$.
- Define the CS coefficient: $k \equiv \gcd(\{k_a\})$.

Moduli Space of a CS Quiver Theory

Let's consider first the abelian case $N = 1$.

- Solving the vacuum equations in 2 steps:
  1. Solving F-terms. The space of solutions of F-terms is the Master space, $\mathcal{F}^\flat$.
  2. Further solving D-terms: Modding out $\mathcal{F}^\flat$ by **the gauge symmetry**.
- Among the original gauge symmetry $U(1)^G$, one is a diagonal $U(1)$; it does not couple to matter fields $\quad\rightarrow\quad$ We are left with $U(1)^{G-1}$.
- Up to this point, the process is the same for a (3+1)d theory living on a D3-brane probing CY_3

Moduli Space of a CS Quiver Theory

- 1st D-terms: $$\sum_{b=1}^{G} X_{ab}X_{ab}^{\dagger} - \sum_{c=1}^{G} X_{ca}^{\dagger}X_{ca} + [X_{aa}, X_{aa}^{\dagger}] = 4k_a\sigma$$
  - The CS levels induce FI-like terms: $4k_a\sigma$.
  - This gives a fibration of CY_3 over $\mathbb{R}$ $\Rightarrow$ Total space is CY_4
- The mesonic moduli space $\mathcal{M}^{\mathrm{mes}}$ is a CY_4.
- Remaining D-terms gauge redundancy: $U(1)^{G-2}$ (baryonic directions)
- Therefore, the mesonic moduli space can be written as

$$\mathcal{M}^{\mathrm{mes}}_{N=1,k} = \left(\mathcal{F}^{\flat}//U(1)^{G-2}\right)/\mathbb{Z}_k$$

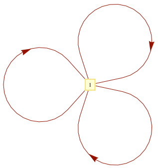
- For higher N, the moduli space is

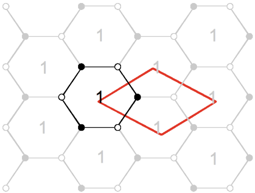
$$\mathcal{M}^{\mathrm{mes}}_{N,k} = \mathrm{Sym}^{N}\left(\mathcal{M}^{\mathrm{mes}}_{N=1,k}\right)$$

Part III: Brane Tilings

What is known in 3+1 dimensions?

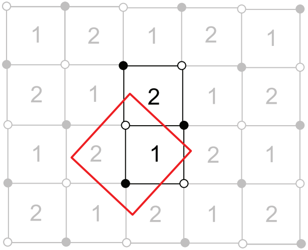
- SCFTs on D3-branes probing CY_3 are best described in terms of brane tilings [Hanany *et al.* from '05]
- The gravity dual of each theory is on the $\mathrm{AdS}_5 \times Y_5$ background (Y_5 being a 5 dimensional Sasaki-Einstein manifold)
- **Example:** The $\mathcal{N} = 4$ Super Yang-Mills (Y_5 is a 5-sphere S^5)





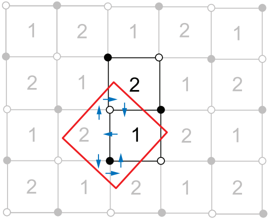
Tiling-Quiver Dictionary

- **Example:** The $\mathcal{N} = 1$ conifold theory [Klebanov-Witten '98]



- $2n$ **sided face** $=$ $U(N)$ gauge group with nN flavours
- **Edge** $=$ A chiral field charged under the two gauge group corresponding to the faces it separates
- D **valent node** $=$ A D-th order interaction term in superpotential

Comments on Brane Tilings



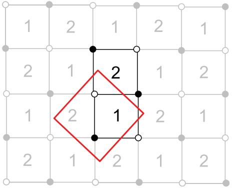
- **Graph is bipartite:** Nodes alternate between clockwise (white) and anticlockwise (black) orientations of arrows.
- Black (white) nodes connected to white (black) only
- Odd sided faces are forbidden by anomaly cancellation condition
- White (black) nodes give $+$ $(-)$ sign in the superpotential

  Conifold theory: $W = \mathrm{Tr}(X^1_{12}X^1_{21}X^2_{12}X^2_{21} - X^1_{12}X^2_{21}X^2_{12}X^1_{21})$

Brane Tilings for $(2+1)$d Theories

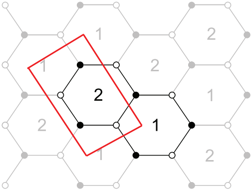
- Assign a CS level to each gauge group (node in quiver & face in the tiling). Rules above still work!
- Each brane tiling (with specified CS levels) defines a unique Lagrangian for an $\mathcal{N} = 2$ CS theory (4 supercharges) in 2+1 dimensions.
- The tiling has an interpretation of a network of D4-branes and NS5-brane ending on the NS5-brane in Type IIA. (Imamura & Kimura '08)
- Largest known family of SCFTs in $(2+1)$ dimensions!

Example: The ABJM Theory [Aharony, Bergman, Jafferis, Maldacena '08]



- Gauge group: $U(N) \times U(N)$. CS levels: $(k, -k)$.
- Superpotential: $W = \mathrm{Tr}(X^1_{12} X^1_{21} X^2_{12} X^2_{21} - X^1_{12} X^2_{21} X^2_{12} X^1_{21})$.
- The case of $N = 1$ and $k = 1$: $W = 0$
  - The F-terms admit any complex solutions of X^i_{12}, X^i_{21} $(i = 1, 2)$
  - The Master space is $\mathcal{F}^\flat = \mathbb{C}^4$
  - The mesonic moduli space is $\mathcal{M}^{\mathrm{mes}}_{N=1} = \mathcal{F}^\flat // U(1)^{G-2} = \mathbb{C}^4$
  - The moduli space generated by X^i_{12}, X^i_{21} (each has scaling dimension $1/2$)
  - These are free scalar fields

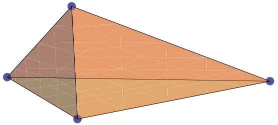
Example: A Conifold $(\mathcal{C}) \times \mathbb{C}$ Theory



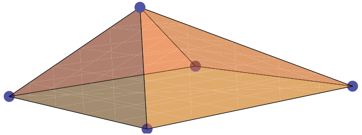
- Gauge group: $U(1) \times U(1)$. CS levels: $(1, -1)$.
- Superpotential: $W = \text{Tr}\left(\phi_1(X^2_{12}X^1_{21} - X^1_{12}X^2_{21}) + \phi_2(X^2_{21}X^1_{12} - X^1_{21}X^2_{12})\right)$
- The $\mathbb{C}$ is parametrised by $\phi_1 = \phi_2$, and the $\mathcal{C}$ is generated by X^i_{12}, X^i_{21}.
- Non-trivial scaling dimensions: $1/2$ for ϕ's and $3/4$ for X's (by symmetry argument)
- These values agree with a computation on the gravity dual (volume minimisation of SE^7). This is a (weak) test of AdS/CFT.

Toric Structures

- The moduli space of $N=1$ theories admits a toric structure, due to the $U(1)$ quotients in $\mathcal{M}^{\text{mes}}_{N=1,k=1} = \mathcal{F}^\flat // U(1)^{G-2}$
- The toric data of the moduli space are collected in the toric diagram, which is unique up to a $GL(3,\mathbb{Z})$ transformation
- There is a prescription (called the forward algorithm) in going from brane tilings to toric diagrams



The toric diagram of $\mathbb{C}^4$



The toric diagram of $\mathcal{C} \times \mathbb{C}$

Part IV: Toric Phases

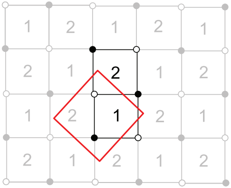
Toric Duality

- There are some models which have different brane tilings, but have the same mesonic moduli space in the IR.
- These models are said to be toric dual to each other. Each of these models is referred to as toric phase.
- In $(3+1)$d, toric duality is understood to be Seiberg duality (Feng, Hanany, He, Uranga; Beasley, Plesser '01). This is however not clear in $(2+1)$d.
- The following quantities are matched between toric phases:
  - Mesonic moduli spaces & toric diagrams
  - Chiral operators & partition functions (Hilbert series)
  - Global symmetries
  - Scaling dimensions (R-charges) of chiral operators

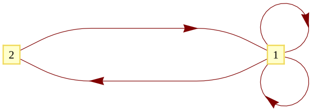
Phases of The $\mathbb{C}^4$ Theory

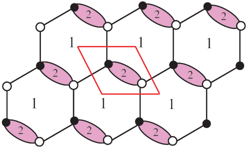
- Phase I: The ABJM model ($k_1 = -k_2 = 1$)

  Note: In $(3+1)$d, these two pictures correspond to the conifold theory.



- Phase II: The Hanany-Vegh-Zaffaroni (HVZ) model ($k_1 = -k_2 = 1$)





The toric diagram of $\mathbb{C}^4$

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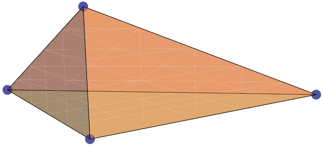
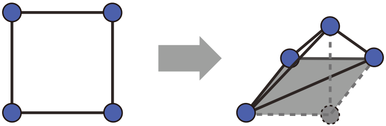


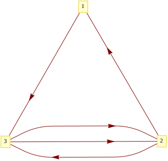
The lift of a point in toric diagram due to CS levels $(1, -1)$

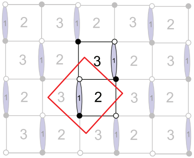


The $(3+1)$d conifold theory The $(2+1)$d ABJM model

Phases of The Conifold $(\mathcal{C}) \times \mathbb{C}$ Theory

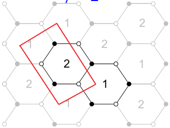
- Phase I: $k_1 = -k_2 = 1, k_3 = 0$



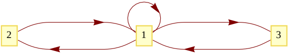


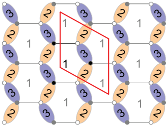
- Phase II: $k_1 = -k_2 = 1$

  Note: In $(3+1)$d, these two pictures correspond to the $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ theory.



- Phase III: $k_1 = 0, k_2 = -k_3 = 1$





The toric diagram of $\mathcal{C} \times \mathbb{C}$

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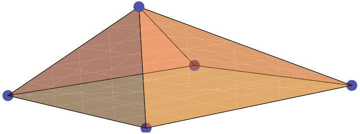

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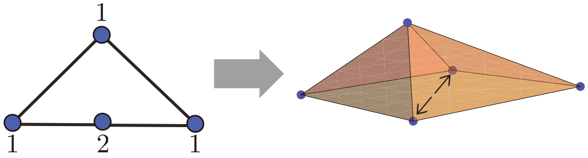
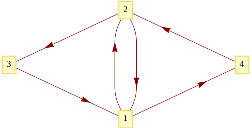


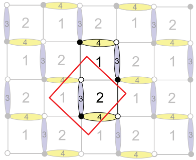
The $(3+1)$d $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ theory

The $(2+1)$d $\mathcal{C} \times \mathbb{C}$ theory

Phases of The D_3 Theory

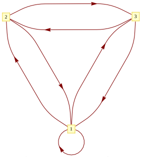
- Phase I: $k_1 = k_2 = -k_3 = -k_4 = 1$

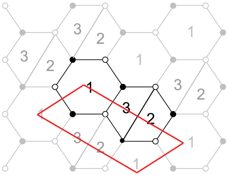




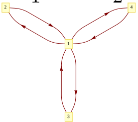
- Phase II: $k_1 = -k_2 = 1, k_3 = 0$

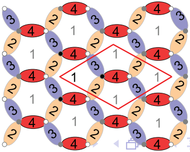
  Note: In $(3+1)$d, these are of the SPP theory.





- Phase III: $k_1 = -k_2 = k_3 = -k_4 = 1$





The toric diagram of D_3

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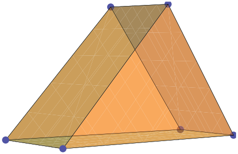

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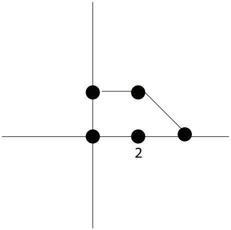
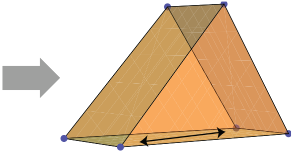




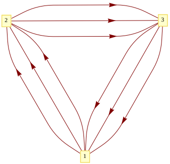
The $(3+1)$d SPP theory The $(2+1)$d D_3 theory

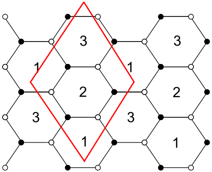
Part V: Fano 3-folds

What are Fano surfaces?

- Fano n-folds are n dim complex manifolds admitting positive curvatures
- Fano 2-folds are $\mathbb{P}^1 \times \mathbb{P}^1$ and the del Pezzo surfaces dP_n (which are $\mathbb{P}^2$ blown-up at $0 \leq n \leq 8$ points). Only $\mathbb{P}^1 \times \mathbb{P}^1$ and $dP_{n=0,1,2,3}$ are toric.
- There are precisely 18 different smooth toric Fano 3-folds (Batyrev '82). Their toric diagrams are known (http://malham.kent.ac.uk/grdb/FanoForm.php).
- Study theories on M2-branes probing a cone over Fano 3-folds
- Problem: Translate toric data to brane tilings

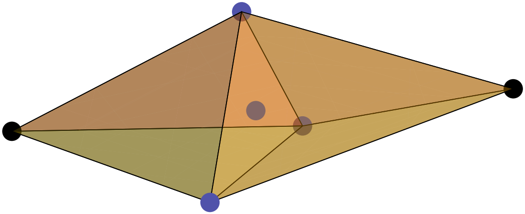
The $M^{1,1,1}$ theory





- Gauge group: $U(1) \times U(1) \times U(1)$. The CS levels: $\vec{k} = (1, 1, -2)$
- The mesonic global symmetry is $\mathfrak{G} = SU(3) \times SU(2) \times U(1)_R$
- The scaling dimensions of quiver fields $X^i_{12}, X^i_{23}, X^i_{31}$ are $7/9, 7/9, 4/9$.
- The operators are in the rep $(3n, 0; 2n)_{2n}$ of $\mathfrak{G}$. This can be computed directly from the field theory side (using Hilbert series) and confirms the known KK spectrum.

The $M^{1,1,1}$ theory from a cone over $\mathbb{P}^2 \times \mathbb{P}^1$

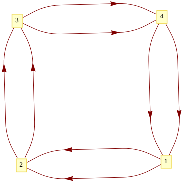


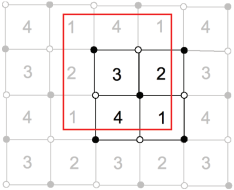
The toric diagram of the $M^{1,1,1}$ theory ($\mathbb{P}^2 \times \mathbb{P}^1$)

- The 4 blue points form the toric diagram of $\mathbb{P}^2$
- The 2 black points together with the blue internal point form the toric diagram of $\mathbb{P}^1$

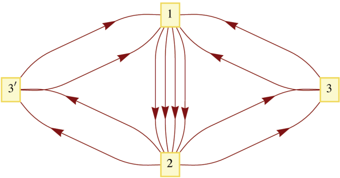
The $Q^{1,1,1}/\mathbb{Z}_2$ theory

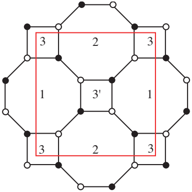
- Phase I: $k_1 = -k_2 = -k_3 = k_4 = 1$



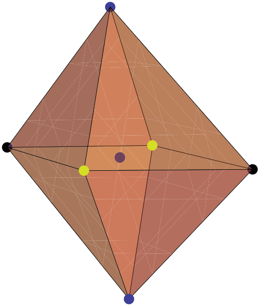


- Phase II: $k_1 = k_2 = -k_3 = -k_{3'} = 1$





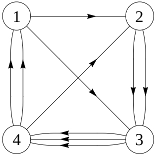
The $Q^{1,1,1}/\mathbb{Z}_2$ theory from a cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

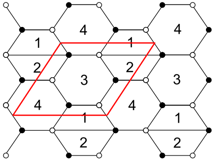


The toric diagram of the $Q^{1,1,1}/\mathbb{Z}_2$ theory ($\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$).

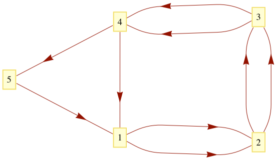
- The mesonic global symmetry is $SU(2)^3 \times U(1)_R$
- The mesonic operators are in the rep $(2n; 2n; 2n)_{2n}$ of $SU(2)^3 \times U(1)_R$.

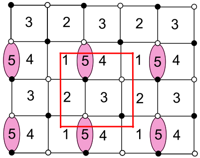
- The $dP_1 \times \mathbb{P}^1$ theory, $\vec{k} = (1, 1, -1, -1)$





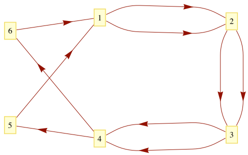
- The $dP_2 \times \mathbb{P}^1$ theory, $\vec{k} = (1, 1, -1, 0, -1)$

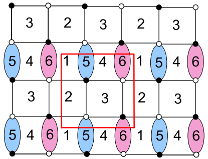




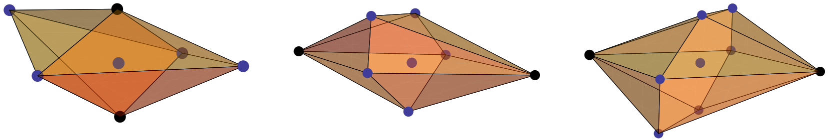
The $dP_n \times \mathbb{P}^1$ theories (continued)

- The $dP_3 \times \mathbb{P}^1$ theory, $\vec{k} = (0, 0, 0, 0, -1, 1)$





- The toric diagrams of (i) $dP_1 \times \mathbb{P}^1$, (ii) $dP_2 \times \mathbb{P}^1$, (iii) $dP_3 \times \mathbb{P}^1$



Conclusions

- All theories described are conjectured to live on the worldvolume of M2-branes probing the CY_4, which is also the mesonic moduli space
- Infinite families of SCFTs
- A variety of scaling dimensions
- Toric duality