Towards Exact Quantum Entropy of Black Holes

Atish Dabholkar

CNRS/University of Paris

Tata Institute, Mumbai

Cambridge

February 2010

- ***A.D. Joao Gomes, Sameer, Murthy, Ashoke Sen*** arXiv:1002.nnnn
- ***A.D. Sameer Murthy, Don Zagier***

  arXiv:1002.nnnn
- ***Miranda Cheng, A. D.*** arXiv:0809.0234,
- ***A.D., Joao Gomez, Sameer Murthy***

  arXiv:0802.0761
- ***A. D., Davide Gaiotto,Suresh Nampuri***

  hep-th/0702150; hep-th/0612011; 0603066

References

- **Dijkgraaf, Verlinde, Verlinde;**
- **Cardoso, de Wit, Kappelli , Mohaupt**
- **Kawai;**
- **Gaiotto, Strominger, Xi, Yin**
- **Sen**
- **David, Jatkar, Sen; Banerjee,Srivastava**
- **Cheng and Verlinde**

Black Hole Entropy & Counting

- For a class of black holes within string theory, we now have a satisfactory statistical understanding of black hole entropy in terms of microstate counting in accordance with Boltzmann Relation.

$$S_{BH} = \log(d)$$

Finite Size Effects

- This agreement is checked mostly in the thermodynamic limit of large size of the horizon and under certain assumptions.
- Given such a remarkable but approximate agreement it is natural to ask what is this an approximation to? Is there an exact quantity that we can define on both sides which can be systematically expanded?

Questions

- How to compute finite size effects?
- What is the choice of the ensemble?
- Can we define exact quantum entropy?
- Can we compute exact degeneracy?
- Is Index the same as Degeneracy? Why?
- Can we compute the corrections in a systematic expansion including both perturbative and nonperturbative effects?

- Even posing some of these questions correctly involves many important conceptual issues. Explicit computations involve new tools and new symmetries.

- Finite size effects depend on quantum corrections to two-derivative Einstein-Hilbert-Maxwell actions and are thus very important probes of microstructure of quatum gravity. One would like to explore them in a systematic way.

Thermodynamics	Statististical Mech
Black Hole (charge Q)	Brane (charge Q)
Bekenstein-Hawking	Strominger-Vafa
Wald + Nonlocal	Exact Index
AdS_2	CFT_2
Z_{AdS}	Z_{CFT}
Macroscopic	Microscopic

Objective

- Various partial computations, arguments, and confusions exist in the literature.
- Our objective will be to make *reliable and explicit computations on microscopic and macroscopoic sides including finite size effects and compare them*. We will obtain a number of exact results (for BPS black holes) on both sides in perfect agreement with each other.

(1) Which Ensemble?

- Various thermodynamic ensembles are equivalent only in the thermodynamic limit. Even to talk about finite size effects on both sides, we need to determine what ensemble to use. Microcanonical? Grand Canonical? Mixed? What quantities to compare?
- *Use holographic partition function which is narturally in* ***microcanonica****l ensemble.*

(2) Why Index = Degeneracy?

- Usually exact counting only possible for topolgical quantities such as indices. But Entropy equals logartithm of absolute degenearcy. In general,

$$\mathbf{n_B - n_F \neq n_B + n_F}$$

- *However, for black hole horizons it **is** true.*

(3) Can we compute? *Microscopic side*

- Can we compute exact quantum degeneracies microscopicallly?
- For N=4 dyons this problem has essentially been solved. We now have the exact counting formula for dyons in ***all*** duality orbits at ***all*** points in moduli space.
- *Partition functions are Siegel forms*

(3) Can we compute? *Macroscopic side*

- Taking into account effects of all higher derivatives terms etc. may be possible in principle, but seems practically impossible. How do we proceed?
- We will use a nonrenormalization theorem that follows from a combination of anomaly inflow and supersymmetry to determine *exact* Wald entropy in a certain limit.

(4) Wall-crossing

- Often the index includes contributions from multi-centered black hole bound states which fall apart upon crossing walls in the moduli space: *wall-crossing phenomenon.*
- Degeneracy jumps upon crossing the walls. How to compute it everywhere?
- *Degeneracy changes but partition function does not!*

5) Poles and Walls

- Degeneracies given by Fourier coefficients of partition functions that are *meromorphic* Siegel modular forms.
- Fourier contour depend on moduli in a precise way are inequivalent because of the poles.
- Poles correspond to walls and residue gives the jump in the degeneracy.

6) Modular Symmetry

- Given these exact answers, how to compute the large charge asymptotics systematically including all subleading corrections?
- Modular symmetry is very useful and can lead to Hardy-Ramanujan-Rademacher exapansion. Wall-crossing seems to lead to loss of modularity.
- *Mock modular forms*

7) Borcherds Symmetry

- The partition functions of these dyons show hints of a huge symmetry – Borcherds-Kac-Moody superalgebra.
- *Satisfies Weyl-Kac denominator identity of in nontrivial ways. All root multiplicities can can be determined.*
- Physics interpretation partially understood -- Weyl group governs wall-crossings.

- In this talk, I will address questions 1 to 3 for black holes that preseve eight or more supersymmetries presenting the results along with some details.
- I will summarize only the results for topics 4-7 in the context of for black holes in N=4 compactifications.

Wald Entropy and beyond

- Wald entropy can incorporate the corrections to Bekenstein-Hawking entropy from all higher-derivative ***local*** terms in the effective action. But 1PI quantum effective actions include ***nonanalytic*** and ***nonlocal*** terms. How to incorporate these effects systematically?
- These terms are in many cases essential for duality invariance of entropy.

Quantum Entropy Function

- One can define the Euclidean string partition function using holography to generalize the notion of Bekenstein-Hawking-Wald entropy including all local and nonlocal corrections.
- In the large charge limit, ignoring nonlocal terms it reduces to Wald entropy. **SEN**

$\mathbf{AdS_2/CFT_1}$

- A proper definition becomes possible in the context of Holography. For a black hole, we are led to $\mathbf{AdS_2/CFT_1}$
- In two dimensions Coulomb potential grows at the boundary instead of falling. This leads to a different boundary condition for the bulk gauge fields.

$$\log(r) \quad \text{vs} \quad r^{-d+1}$$

Choice of Ensemble

- Holography for 2d Euclidean AdS implies that the comparison is most natural in the ***microcanonical*** ensemble.
- By contrast, in higher-dimensional AdS one normally fixes the constant mode of the gauge field at the boundary which corresponds to fixing chemical potential implyig ***grand-canonical*** ensemble.

$$\mathbf{Z_{AdS_2}(Q) = Z_{CFT_1}(Q)}$$

- When we have only local effective action one can show (Sen)

$$\mathbf{Z_{AdS_2}(Q)} \sim \exp[\mathbf{S_{Wald}(Q)}]$$

- By definition, in microcanonical ensemble

$$\mathbf{Z_{CFT_1}(Q) := d(Q)}$$

- This gives the precise formulation of the comparison we want to make. Can we compute both sides and compare them?

Index = Degeneracy?

- In many cases this assumption is true for the leading entropy but unclear at subleading order. No theoretical rationale.
- A number of puzzles at subleading order.
- Five-dimensional examples where this is not true. One can sometimes define a modified Index, but when and why?
- No index for BPS black holes in AdS5.

Basic Argument

- If we have at least four unbroken supersymmetries then together with the SU(1, 1) symmetry of $\mathbf{AdS_2}$, closure of algebra implies an SU(2) symmetry.
- SU(1,1|2) superalgebra at the horizon.
- Microstates associated with this horizon are thus invariant under this SU(2) symmetry.

Sen

$$(-\mathbf{1})^{\mathbf{F}} := \exp\left[\mathbf{2}\pi\mathbf{iJ}\right] = \mathbf{1}$$

- One can use the Cartan generator of this SU(2) to define fermion number
- Because the black hole horizon is invariant, all microstates take the same value for the fermion number.
- This argument does not work in situations such as the one-sixteenth BPS black holes in AdS5 because less than four susys.

Spacetime Index

- Thus Boltzman relation gives a way compute using this argument the ***index*** of the horizon degrees of freedom.
- One can now put together the contribution from multiple horizons and hairs.
- `Hair' degrees of freedom are localized outside the horizon.
- Spacetime index counts all.

Definition of the index

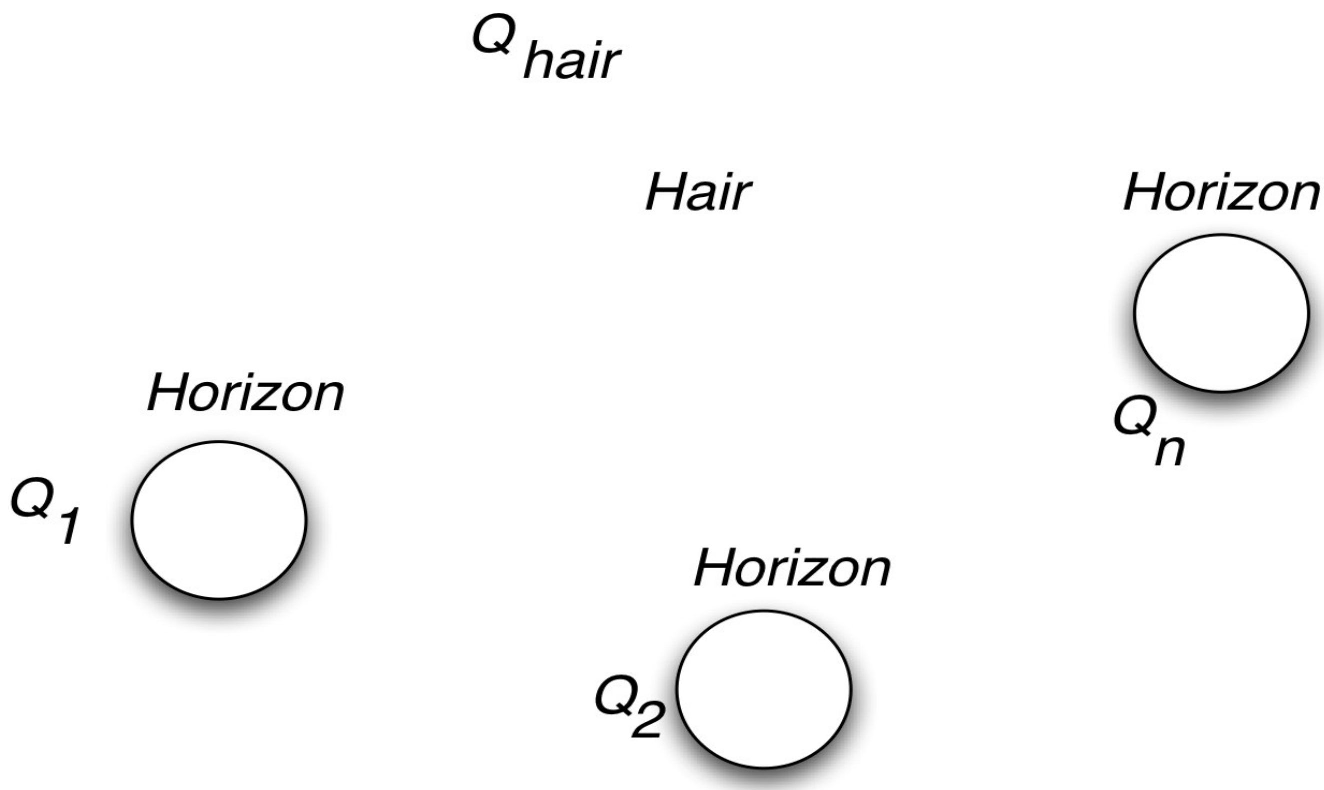
- Spacetime index is a helicity supertrace. In four dimensions if the black hole breaks 4n supersymmetries, there is standard index for counting these dyons

$$B_{2n} = \frac{1}{(2n)!} Tr\left[(-1)^F (2h)^{2n}\right] = \frac{1}{(2n)!} Tr\left[e^{2\pi i h}\,(2h)^{2n}\right]$$

where h is spin in rest frame. It is a natural index for an asymptotic observer.

Macroscopic Spacetime index

-



- Putting together hair and horizon contributions we get

$$B_{2n} = \frac{1}{(2n)!} \left[Tr_{hor}(-1)^{2h_{hor}}\right] \left[Tr_{hair}(-1)^{2h_{hair}}(2h_{hair})^{2n}\right]$$

- If the only hair are from unbroken susy,

$$Tr_{hair}(-1)^{2h_{hair}}(2h_{hair})^{2n} = (-1)^n(2n)!$$

- Spacetime index thus equals degeneracy:

$$B_{2n} = (-1)^n Tr_{hor}(-1)^{2h_{hor}} = (-1)^n d_{hor}$$

Explicit tests

- We would like to test these formal arguments in concrete examples where finite effects are visible and calculable.
- We need to be able compute both macro and micro answers exactly.
- We will consider systems that preserve at least eight supersymmetries in four and five dimensions with and without spin.

Five Dimensions

- Consider Type-IIB on $\mathbf{K3} \times \mathbf{S^1}$.
- We consider the Strominger-Vafa system with ***Q1*** D1-branes, ***Q5*** D5-branes wrapping the circle and ***n*** units of momentum along the circle.
- But we will consider limits when *only one charge becomes large*, other remaining finite to explore finite charge effects.

Cardy Limits and $\mathbf{AdS_3}$

- If the charge that is becoming large can be viewed as momentum around a circle in some duality frame, then that circle becomes large in this limit.
- We have larger geometric symmetries of $\mathbf{AdS_3}$ an instead of $\mathbf{AdS_2 \times S^1}$
- This corresponds to the Cardy limit in the dual CFT2.

Different limits in 5d

- ***Q1, Q5*** fixed ***n*** large,
  Type-II Cardy limit
- ***Q5, n*** fixed ***Q1*** large,
  Heterotic Cardy limit.

We will now compute both macroscopic and microscopic sides using combination of anomaly, susy, scaling, and exact computations in the microscopic theory.

Macroscopic computation

- In the limit when the circle is large, we have full SO(2, 2) symmetries of $\mathbf{AdS_3}$
- In this case, the exact wald entropy is completely captured by the left-moving central charge $\mathbf{c_L}$ of the boundary CFT2 and the momentum $\mathbf{n}$

$$\mathbf{S_{Wald} = 2\pi\sqrt{\frac{1}{6}c_L\, n}}$$

Anomaly Inflow

- The central charge is related to conformal anomaly in the boundary.
- An anomaly on the boundary CFT reflects lack of gauge invariance and lack of current conservation. The total theory is of course gauge invariant. So there must be an inflow of anomaly onto the boundary to account for this nonconservation.

Chern-Simons terms

- Chern-Simons terms in the bulk are not gauge invariant in the presence of boundaries. Noninvariance of CS terms is precisely the amount of anomaly that `inflows' onto the boundary so that there is net current conservation.
- Together with (0, 4) supersymmetry this is enough to determine the central charges.

Central Charges from CS terms

- Coefficient of Lorentz Chern-Simons determines $\mathbf{c_L - c_R}$
- Coefficient of gauge Chern-Simons term determines $\mathbf{k_R}$
- Supersymmetry determines

$$\mathbf{c_R = 6k_R}$$

Harvey, Minasian, Moore; Kraus, Larsen

Nonrenormalization

- Basically, all correlators of the stress-tensor multiplet in the boundary are determined from the Virasoro and current algebra OPEs which depend on the various central charges c_R, c_L, k_R
- Since stress-tensor correlators determine the S-matrix completely, it must be always possible to do field redefinitions to get rid of all terms except the CS terms.

Determination of the CS terms

- Our task is thus reduced to finding the coefficients of CS terms in $\mathbf{AdS_3}$ near horizon geometry but this typically can receive quantum corrections and can depend on details of compactification.
- We will first relate them to quantities at asymptotic infinities that are easier to determine using scaling and knowledge of 10d action.

Horizon +Hair

- If we can determine the hair degrees of freedom, then the inflow of anomaly from the hair can be evaluated. Using current conservation we can relate the horizon CS terms to asymptotic CS terms.
- It is then sufficient to determine the CS terms relevant for the asymptotic observer responsible for the anomaly inflow onto the brane configuration.

Scaling Argument

$$\mathbf{S^{(l)}_{BH}(\lambda^2 q, p, \lambda Q) = \lambda^{2-2l} S_{BH}(q, p, Q)}$$

- Here q and p are electric and magnetic NS-NS charges, and Q are the RR charges. Follows from scaling properties of effective action at *l* loops.
- This allows us to determine at what loop corrections can arise.

Example: Type-II Cardy limit

- Central charge is quantized so corrections can only be powers of charges and not inverse charges.
- Leading supergravity answer gives

$$\mathbf{c_L = 6Q_1Q_5}$$

- Only possibilities are corrections linear in $\mathbf{Q_1}$ and $\mathbf{Q_5}$ or a constant shift.

- Linear terms will lead to corrections that scale as $\mathbf{1/Q_1}$ or $\mathbf{1/Q_5}$. Since these charges are from RR sector, by scaling such corrections can arise only at half-loop which is not possible.
- Constant term will give correction at one-loop scaling as $\mathbf{1/Q_1 Q_5}$. But cannot involve any 3-forms since it is independent of the charge. No such purely gravitiational CS type terms at one-loop. Hence, we have $\mathbf{c_L = 6 Q_1 Q_5}$ *exactly*.

Example: Heterotic Cardy Limit

- Similar arugments in Heterotic frame, imply only corretion linear in $\mathbf{Q_5}$ is allowed at one-loop in Type-IIA frame which comes from $\int \mathbf{B \wedge X_8}$ term.

- Exact central charge is then

$$\mathbf{c_L = 6Q_5 n + 18Q_5}$$

Microscopic Computations

- In the heterotic duality frame, we have NS5-branes so we don't know the microcopic CFT. We cannot simply use the Strominger-Vafa CFT to read of the central charge because taking *n* finite and *Q1* large is the *anti-Cardy* limit for this theory. We need to be able work out the asymptotics in efficient ways.

Finite Charges

- Fortunately we can use the knowledge of the exact partition function and a clever trick *Castro & Murthy; Banerjee*
- Use 4d-5d lift to relate 5d black holes at the center of Taub-NUT to 4d black holes.
- Use S-duality in 4d to convert electric to magnetic. Flips n and Q1 and converts anti-Cardy $c \gg L_0$ to Cardy $L_0 \gg c$

Four dimensions

- Type IIB on $\mathbf{K3 \times T^2}$
- We can add an additional KK-monopole charge K to the system
- Now, if we take n large and other charges finite, the appropriate duality frame is the M-theory frame and n becomes the momentum along this circle. We have to study AdS3/Cardy limit of this system.

MSW string

- In M-theory we have an M5-brane wrapping a $\mathbf{P} \times \mathbf{S^1}$ where $\mathbf{P}$ is a 4-cycle in $\mathbf{K3} \times \mathbf{T^2}$ giving an effective Maldacena-Strominger-Witten string wrapping the circle.
- The central charges of the effective CFT can be determined from index theory following MSW.

Microscopic counting in MSW

- For $\mathbf{K3} \times \mathbf{T^2}$, there are some subtleties because of the odd cycles which are not present in a general Calabi-Yau but one can determine the central charges.
- Naive application gives wrong results!
- The relevant quantity turns out to be not the central chage but an effective central charge that determines growth of an index.

Degneracy = Index !

- Five-dimensional Black Holes with spin.

$\mathcal{M}$	Limit	$\log d^{macro}$	$\log d^{micro}_{ind}$	$\log d^{micro}_{deg}$
$K3$	Type-II Cardy	$\pi\sqrt{4Q_1Q_5\left(n-\frac{J^2}{4Q_1Q_5}\right)}$	$\pi\sqrt{4Q_1Q_5\left(n-\frac{J^2}{4Q_1Q_5}\right)}$	$\pi\sqrt{4\left(Q_1Q_5+2\right)n}$
$K3$	Heterotic Cardy	$\pi\sqrt{4Q_5(n+3)\left(Q_1-\frac{J^2}{Q_5(n-1)}\right)}$	$\pi\sqrt{4Q_5(n+3)\left(Q_1-\frac{J^2}{Q_5(n-1)}\right)}$	–
T^4	Type-II Cardy	$\pi\sqrt{4Q_1Q_5\left(n-\frac{J^2}{4Q_1Q_5}\right)}$	$\pi\sqrt{4Q_1Q_5\left(n-\frac{J^2}{4Q_1Q_5}\right)}$	$\pi\sqrt{4\left(Q_1Q_5+2\right)n}$

- Once again, when there is disagreement, macroscopic degeneracy agrees with the micoscopic index and not with degeneracy

Degeneracy = Index !

- Four dimensional black holes

$\mathcal{M}$	$\log d^{macro}$	$\log d^{micro}_{ind}$	$\log d^{micro}_{deg}$
$K3$	$\pi\sqrt{4(Q_1Q_5K+4K)n}$	$\pi\sqrt{4(Q_1Q_5K+4K)n}$	$\pi\sqrt{4(Q_1Q_5K+4K+1)n}$
T^4	$\pi\sqrt{4(Q_1Q_5K)n}$	$\pi\sqrt{4(Q_1Q_5K)n}$	$\pi\sqrt{4(Q_1Q_5K+3)n}$

Macroscopics agrees with the microscopic index and not the degeneracy computed at weak coupling resolving puzzles raised by

Lambert; Cardoso, de Wit, Mohaupt

It is worth emphasizing the following points

- These results are *exact* in the limit when only one charge is large. We did not make any assumption about keeping only F-type terms etc. The choice of the ensemble was not a separate conjecture as in OSV.
- These arguments give a theoretical explanation of why in earlier computations it was justified to compare a microscopic *index* with black hole degeneracy. It is in fact necessary for subleading corrections.

Dyons in N=4

- Quarter-BPS dyons are specified by electric charge q and magnetic charge p, both vectors of the T-duality group. It is useful to classify the duality invariants. Classification of invariants of an arithmetic group is an interesting & subtle problem in number theory.
- A.D. Gaiotto, Namuri; Sen, Banerjee, Srivastava, Mandal; A.D. Gomes, Murthy

Duality Invariants

- The partition functios are classified by a single integer duality invariant

$$\mathbf{I = gcd(q \wedge p)}$$

- Degeneracies depend only on three T-duality invariant integers

$$\mathbf{(p^2, q^2, q \cdot p) = (2m, 2n, l)}$$

Exact Degeneracies

$$d(m,n,l)|_{\lambda} = \int_{C(\lambda)} \frac{p^{-m} q^{-l} y^{-l}}{\Phi_{10}(\Omega)}\, d^3\Omega$$

where $\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix}$

$$p := e^{2\pi i \sigma} \; q := e^{2\pi i \tau} \; y := e^{2\pi i z}$$

- Where $\boldsymbol{\Phi_{10}(\Omega)}$ is an explicitly known function called Igusa cusp form of wt 10

Contour, Poles, Walls

- The contour depend in a precise way on the moduli.
- The Igusa form has double zeroes and hence the partition function has an intricate structure of double poles
- Poles correspond to walls in moduli space.
- Degeneracies jumps when the contour hits a pole given by the residue at the pole.

Single-centered black holes

- Choosing the contour corresponding to the attractor moduli we can extract the degeneracies of single horizons.
- Note that the partition function has a double pole. As a consequence, the degeneracies of single-centered black holes are not coefficients of a modular form. *Modular symmetry is apparently lost.*

Modular Symmetry

- Modular symmetry is very important in these context because it relates high temperature to low temperature.

$$\mathbf{f}(\mathbf{1}/\tau) = \tau_{\mathbf{2}}^{\mathbf{k}}\,\mathbf{f}(\tau); \qquad \tau := \mathbf{2}\pi\beta$$

- Without it, we would not know how to do asymptotic expansions such as the Cardy formula more exactly Hardy-Ramanujan-Rademacher formula.

Modularity and Holography

- Modular symmetry is also important for $\mathbf{AdS_3/CFT_2}$ holography. The Euclidean $\mathbf{AdS_3}$ has a $\mathbf{T^2}$ as a boundary. The $\mathbf{SL(2,\mathbb{Z})}$ modular symmetry of the black hole partition function is a geometric symmetry that acts on the complex structure parameter of this torus.

How to restore modularity?

- Lack of modularity is of a very special type and can be restored using some very recent developments in number theory from 2005 following the work of Zwegers.
- The idea of mock modular forms goes back to Ramanjuan but a proper characterization of these forms was unclear until recently.

Mock modular form

- The lack of modularity of a mock modular form $\mathbf{f}(\tau)$ is controlled by another modular form $\mathbf{h}(\tau)$ called a `shadow' such that one can define a real analytic completion $\hat{\mathbf{f}}(\tau,\bar{\tau})$ which is modular but no longer holomorphic.

$$\hat{\mathbf{f}}(\tau,\bar{\tau}) = \mathbf{f}(\tau) + \mathbf{h}^*(\tau,\bar{\tau})$$

$$(\mathbf{2}\pi\mathbf{i}\tau_{\mathbf{2}})^{\mathbf{k}} \frac{\partial}{\partial\bar{\tau}} \hat{\mathbf{f}}(\tau,\bar{\tau}) = \bar{\mathbf{h}}(\tau)$$

Holomorphic Anomaly

- The shadow can be viewed also as a holomorphic anomaly in the completion.
- Mock modularity is closely related to meromorphy of the partition function. Which in turn is closely related to wall-crossing and non-comapctness of the microscopic CFT. This idea therefore is expected to have wider applications.

Borcherds Symmetry

- The square root of the dyon partition function has a product representation and a sum representation. It is highly nontrivial mathematical fact that the product representation exists.
- Using these representation one can show that the partition function is a square of a character of a Verma module of a Borcherds-Kac-Moody superalgebra.

Denominator Identity

- There are infinite imaginary simple roots.
- The Weyl-Kac denominator identity is satisfied and can be used to determine all root multiplicities.
- The Weyl group controls the wall-crossing. Weyl chambers correpond to chambers in the moduli space where the degeneracy is constant.