FROM WEAK TO STRONG COUPLING IN ABJM THEORY

Marcos Mariño
University of Geneva

Based on [M.M.-Putrov, 0912.1458]
[Drukker-M.M.-Putrov, 1007.1453]
[in progress]

Two well-known virtues of large *N* string/gauge theory dualities:

- The *large radius limit* of string theory is dual to the *strong coupling regime* in the gauge theory

$$\frac{R}{\ell_s} \gg 1 \leftrightarrow \lambda \gg 1$$

- The *genus expansion* of the string theory can be in principle mapped to the *1/N* expansion of the gauge theory

These virtues have their counterparts:

- It is hard to test the duality, since one has to do calculations at strong 't Hooft coupling in the gauge theory. More ambitiously, one would like to have results *interpolating between weak and strong coupling*

- It is hard to obtain information beyond the planar limit, even in the gauge theory side.

In this talk I will report on some recent progress on these problems in ABJM theory and its string dual.

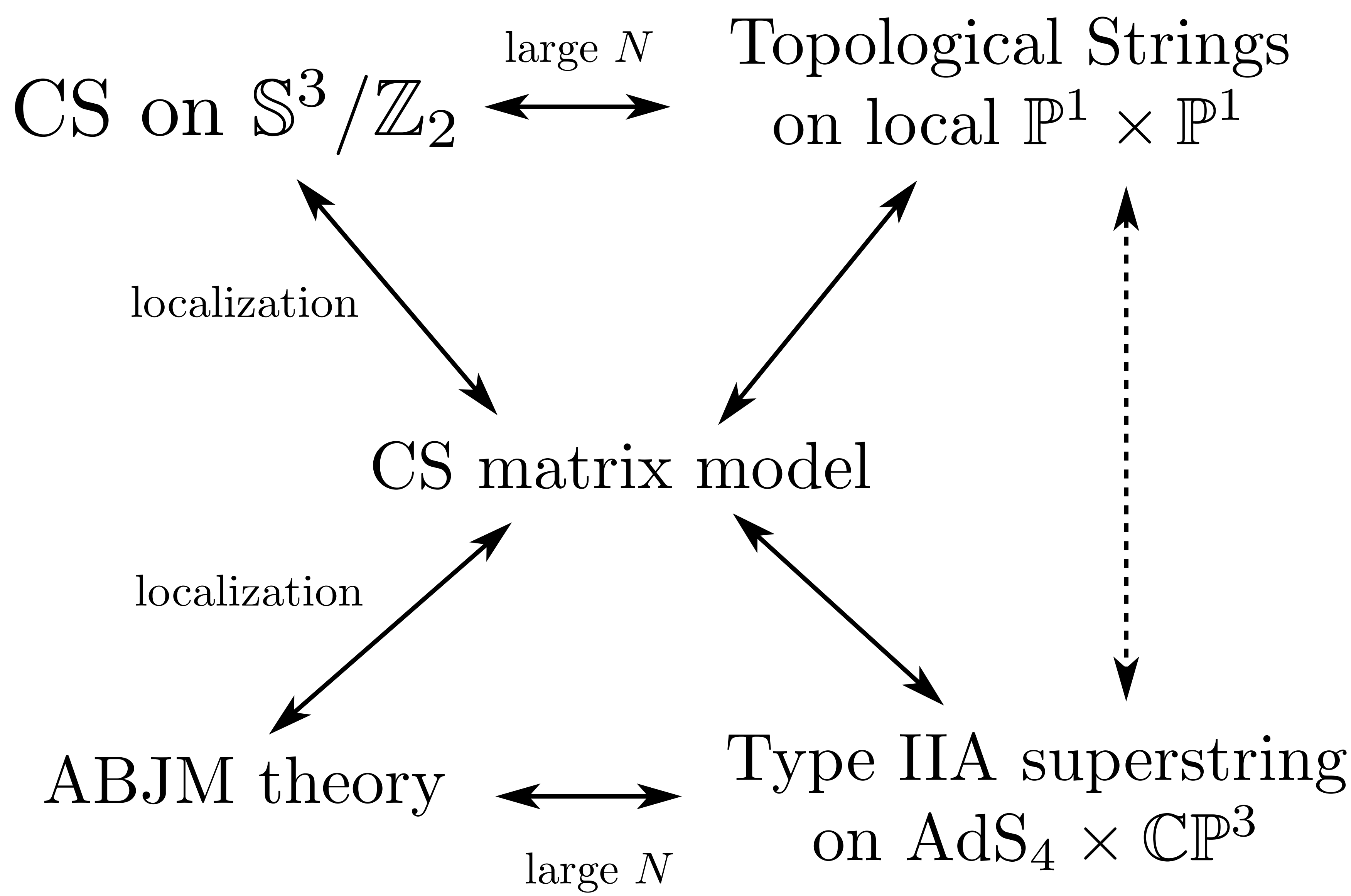
In particular, I will present *exact results (interpolating functions)* for the planar 1/2 BPS Wilson loop vev and for the planar free energy on the thee-sphere.

The strong coupling limit is in perfect agreement with the AdS dual, and in particular provides the first quantitative test of the $N^{3/2}$ behaviour of the M2 brane theory

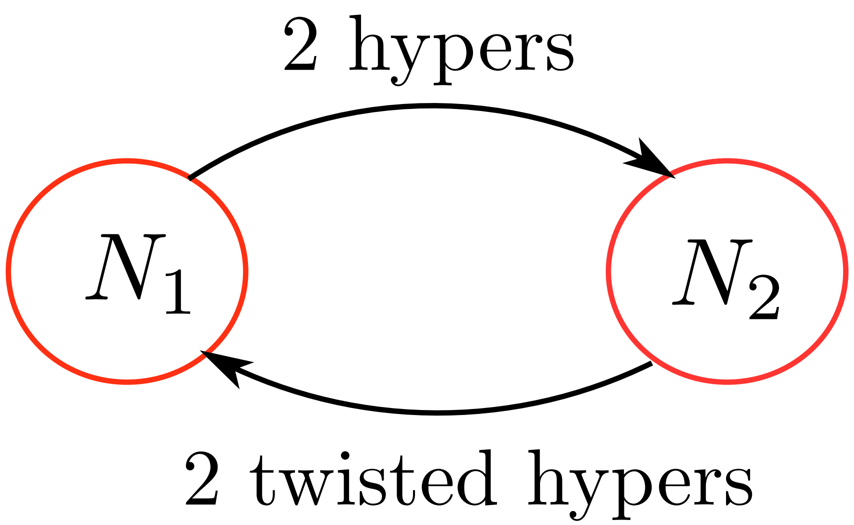
Moreover, I will show that it is possible to calculate explicitly the free energy *for all genera* (very much like in non-critical string theory).

This makes possible to address some *nonperturbative aspects of the genus expansion* in a quantitative way (large order behavior, Borel summability, spacetime instantons...)

We will rely on the following "chain of dualities", which relates a sector of ABJM theory to a topological gauge/string theory via a matrix model:



A B J M theory



$$U(N_1)_k \times U(N_2)_{-k}$$

CS theories + 4 hypers C in the bifundamental; related to supergroup $U(N_1|N_2)$ theory via [Gaiotto-Witten]

two 't Hooft couplings $\lambda_i = \frac{N_i}{k}$

This is a 3d SCFT which (conjecturally) describes $\min(N_1, N_2)$ M2 branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity, with $|N_1 - N_2|$ fractional branes

Note: "ABJM slice" refers to $\lambda_1 = \lambda_2 = \lambda$

Gravity dual

M-theory on $\mathrm{AdS}_4 \times \mathbb{S}^7/\mathbb{Z}_k$ $\xrightarrow{\text{Hopf reduction}}$ type IIA theory/$\mathrm{AdS}_4 \times \mathbb{P}^3$

$$\mathrm{d}s^2 = \frac{R^2}{4\ell_s^2}\left(\mathrm{d}s^2_{\mathrm{AdS}_4} + 4\mathrm{d}s^2_{\mathbb{CP}^3}\right)$$

Gauge/gravity dictionary:

$$\left(\frac{R}{\ell_s}\right)^2 = \left(32\pi^2\hat{\lambda}\right)^{1/2}$$

$$g_{\mathrm{st}} = \frac{1}{k}\left(32\pi^2\hat{\lambda}\right)^{1/4}$$

$$B = \lambda_1 - \lambda_2 + \frac{1}{2}$$

Warning!
shifted charges

$$\hat{\lambda} = \lambda_1 - \frac{1}{2}\left(B^2 - \frac{1}{4}\right) - \frac{1}{24}$$

[Bergman-Hirano, Aharony et al.]

Wilson loops

Circular 1/6 BPS Wilson loops: they involve only one of the gauge connections, but they know about the other node through the bifundamentals

$$W_R^{1/6} = \mathrm{Tr}_R \mathrm{P} \exp\left[\mathrm{i}\int \left(A_1 \cdot \dot{x} + |\dot{x}| C\overline{C}\right)\right]$$

1/2 BPS Wilson loops constructed by [Drukker-Trancanelli]. They exploit the hidden supergroup structure, and they are symmetric in the two nodes

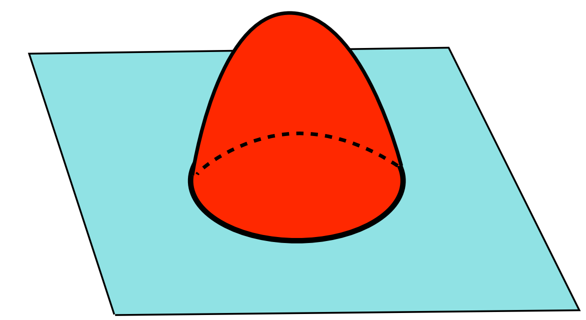
$$W_{\mathcal{R}}^{1/2} = \mathrm{sTr}_{\mathcal{R}} \mathrm{P} \exp\left[\mathrm{i}\int \begin{pmatrix} A_1 \cdot \dot{x} + \cdots & \\ & -A_2 \cdot \dot{x} + \cdots \end{pmatrix}\right]$$

rep $U(N_1|N_2)$ circle $U(N_1)$ connection $U(N_2)$ connection

Two string/gravity predictions

1) 1/2 BPS Wilson loop from fundamental string

$$\langle W_{\square}^{1/2} \rangle_{\text{planar}} \sim \mathrm{e}^{\pi\sqrt{2\hat\lambda}}$$



2) The planar free energy of the Euclidean theory on $\mathbb{S}^3$ should be given by the (regularized) Euclidean Einstein-Hilbert action on AdS4

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \sinh^2(\rho)\,\mathrm{d}\Omega^2_{\mathbb{S}^3},$$

$$-F(N,k) \approx S_{\mathrm{AdS}_4} = \frac{\pi}{2G_N} = \frac{\pi\sqrt{2}}{3} k^2 \hat\lambda^{3/2}, \qquad \hat\lambda \gg 1,\ g_{\mathrm{st}} \ll 1$$

[Emparan-Johnson-Myers] using *universal counterterms*

Nonzero and probing the 3/2 growth!

Exact interpolation from a matrix model

Similar story: 1/2 BPS Wilson loop in N=4 SYM. The string prediction at strong coupling can be derived in the gauge theory from an exact interpolating function [Ericksson-Semenoff-Zarembo, Drukker-Gross]

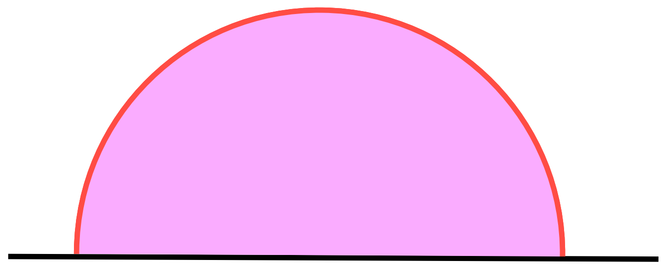
$$\frac{1}{N}\langle W_{\Box}\rangle_{\rm planar} = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \begin{cases} \sim \mathrm{e}^{\sqrt{\lambda}} & \lambda \gg 1 \\ 1 + \dfrac{\lambda}{8} + \cdots & \lambda \ll 1 \end{cases}$$

$$\lambda = g_{\rm YM}^2 N$$

Rationale: the path integral calculating of the vev of the Wilson loop reduces to a *Gaussian matrix model*

$$\langle W_R\rangle = \frac{1}{Z}\int \mathrm{d}M\, \mathrm{e}^{-\frac{2N}{\lambda}\mathrm{Tr}\, M^2}\,\mathrm{Tr}_R \mathrm{e}^M$$

This is the *simplest matrix model*, and the planar density of eigenvalues is the famous Wigner semicircle distribution



$$\rho(z) = \frac{2}{\pi\lambda}\sqrt{\lambda - z^2}$$

$$\frac{1}{N}\langle W_{\square}\rangle_{\mathrm{planar}} = \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \rho(z)\mathrm{e}^{z}\mathrm{d}z$$

One can also compute *1/N* corrections systematically

This conjecture was finally proved by using *localization techniques* [Pestun].

Reduction to a matrix model in ABJM

Localization techniques were extended to the ABJM theory in a beautiful paper by [Kapustin-Willett-Yaakov]. The partition function on $\mathbb{S}^3$ is given by the following matrix integral:

contribution CS gauge fields

$$Z_{\mathrm{ABJM}}(N_1, N_2, g_{\mathrm{top}})$$

$$= \frac{1}{N_1! N_2!} \int \prod_{i=1}^{N_1} \frac{\mathrm{d}\mu_i}{2\pi} \prod_{j=1}^{N_2} \frac{\mathrm{d}\nu_j}{2\pi} \frac{\prod_{i<j} \left(2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right)^2 \left(2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right)^2}{\prod_{i,j}\left(2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right)^2} \mathrm{e}^{-\frac{1}{2g_{\mathrm{top}}}\left(\sum_i \mu_i^2 - \sum_j \nu_j^2\right)}$$

contribution 4 hypers

$$g_{\mathrm{top}} = \frac{2\pi \mathrm{i}}{k}$$

We "just" need the planar solution, but *exact* in the 't Hooft parameters, in order to go to strong coupling

Relation to Chern-Simons matrix models

Shortcut: relate this to *CS matrix models*

[M.M. building on Lawrence-Rozansky] [AKMV, Halmagyi-Yasnov]

U(N) (pure!) CS theory on $\mathbb{S}^3$:

$$Z_{\mathbb{S}^3}(N, g_{\rm top}) = \frac{1}{N!}\int \prod_{i=1}^{N} \frac{\mathrm{d}\mu_i}{2\pi} \prod_{i<j} \left(2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right)^2 \mathrm{e}^{-\frac{1}{2g_{\rm top}}\sum_i \mu_i^2}$$

can be rederived with SUSY localization [Kapustin et al.]

U(N) (pure!) CS theory on L(2,1)= $\mathbb{S}^3/\mathbb{Z}_2$

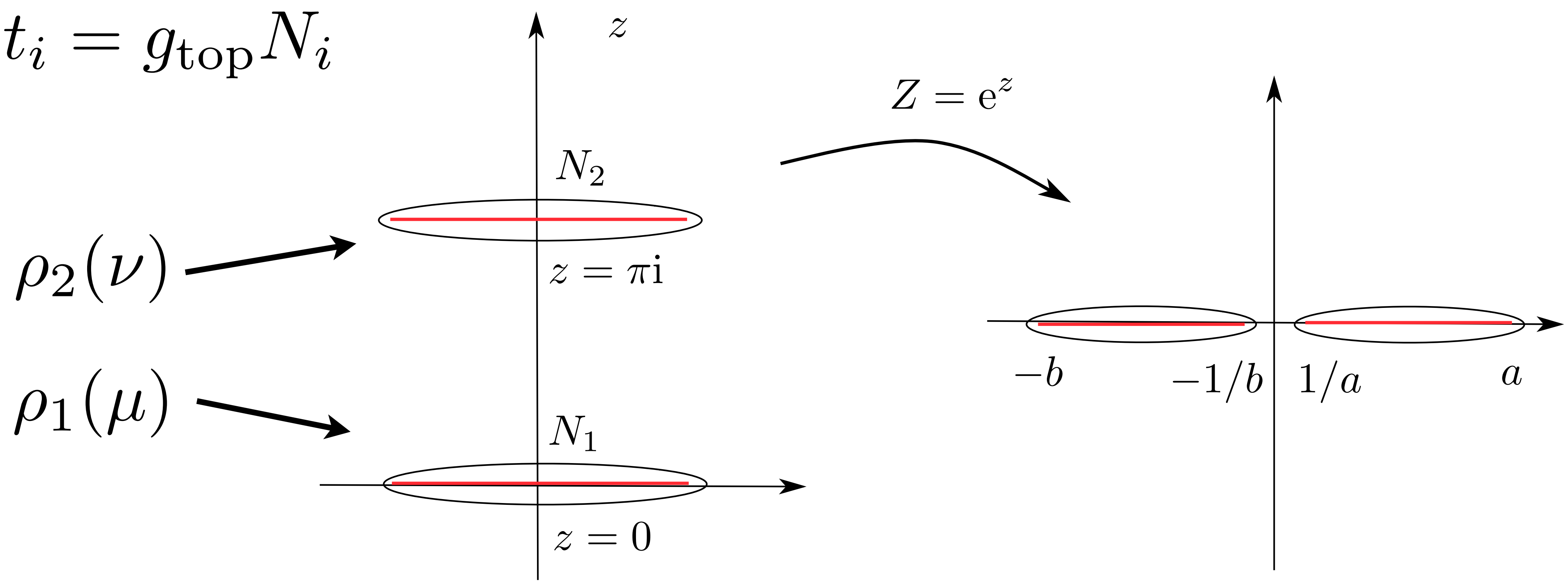
$$Z_{L(2,1)}(N, g_{\rm top}) = \sum_{N_1+N_2=N} Z_{L(2,1)}(N_1, N_2, g_{\rm top})$$

sum over flat connections

$$Z_{L(2,1)}(N_1, N_2, g_{\rm top}) = \frac{1}{N_1! N_2!} \int \prod_{i=1}^{N_1} \frac{\mathrm{d}\mu_i}{2\pi} \prod_{j=1}^{N_2} \frac{\mathrm{d}\nu_j}{2\pi} \prod_{i<j} \left(2\sinh\left(\frac{\mu_i - \mu_j}{2}\right)\right)^2 \left(2\sinh\left(\frac{\nu_i - \nu_j}{2}\right)\right)^2$$
$$\times \quad \prod_{i,j} \left(2\cosh\left(\frac{\mu_i - \nu_j}{2}\right)\right)^2 \mathrm{e}^{-\frac{1}{2g_{\rm top}}\left(\sum_i \mu_i^2 + \sum_j \nu_j^2\right)}$$

This is a *two-cut* model with two 't Hooft parameters

$$t_i = g_{\rm top} N_i$$



Superficially similar to the matrix model describing ABJM...

1/N expansion $$F(N_1, N_2, g_{\rm top}) = \sum_{g\geq 0} g_{\rm top}^{2g-2} F_g(t_1, t_2)$$

Fact: The ABJM MM is the *supermatrix version* of the *L(2,1)* MM.

They are related by $t_2 \longrightarrow -t_2$ [M.M.-Putrov]

i.e. $$t_1 = 2\pi \mathrm{i}\lambda_1, \quad t_2 = -2\pi \mathrm{i}\lambda_2$$

CS matrix model ABJM theory

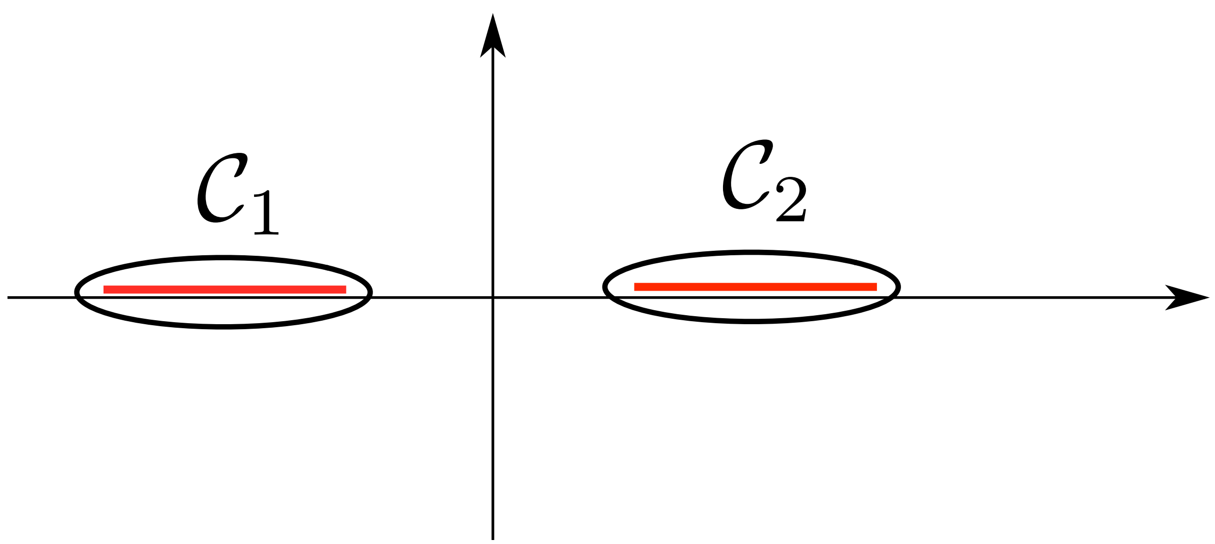
$$\boxed{\text{The } 1/N \text{ expansion of the lens space matrix model gives the } 1/N \text{ expansion of the ABJM free energy on the three-sphere}}$$

Planar solution: matrix model approach

The planar solution of the CS lens space matrix model has been known for some time [AKMV, Halmagyi-Yasnov]. The solution is elegantly encoded in a *resolvent* or *spectral curve*

$$\omega_0(z) = 2\log\left(\frac{e^{-t/2}}{2}\left[\sqrt{(Z+b)(Z+1/b)} - \sqrt{(Z-a)(Z-1/a)}\right]\right)$$

$$t = t_1 + t_2$$



discontinuity across the cuts=densities

$$\rho_k(z) = -\frac{t}{t_k}\frac{1}{2\pi \mathrm{i}}\left(\omega_0(z+\mathrm{i}\epsilon) - \omega_0(z-\mathrm{i}\epsilon)\right)$$

All the planar information is given by *period integrals* of the resolvent

$$t_i = \frac{1}{4\pi \mathrm{i}} \oint_{\mathcal{C}_i} \omega_0(z) dz, \quad i = 1, 2$$

$$\frac{\partial F_0}{\partial t_1} - \frac{\partial F_0}{\partial t_2} - \pi \mathrm{i} t = - \int_{-1/b}^{1/a} \omega_0(z) dz$$

We have to understand what are the weak and the strong coupling limit in terms of the geometry of the curve (i.e. the location of the cuts). In ABJM theory we also want the 't Hooft parameters to be imaginary

We first consider the ABJM slice. It turns out that all quantities are naturally expressed in terms of a real variable κ, closely related to the positions of the cuts

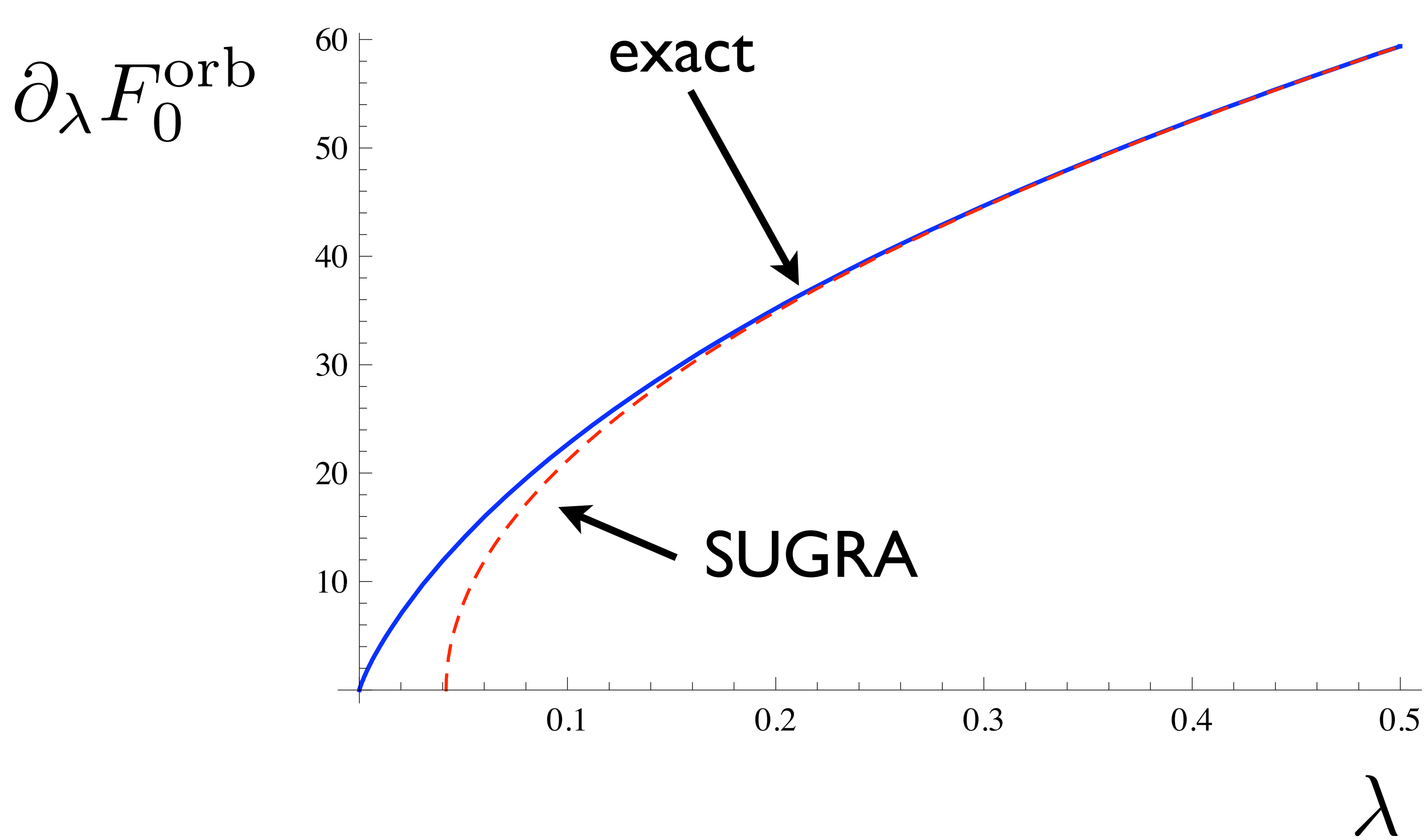
weak coupling	strong coupling
$\kappa = 0$	$\kappa = \infty$
$a, b \sim 1$	$a \sim \mathrm{i}\kappa,\ b \sim -\mathrm{i}\kappa$
$\lambda \sim \kappa$	$\lambda \sim \log^2(\kappa)$
$\partial_\lambda F_0 \sim \kappa \log \kappa$	$\partial_\lambda F_0 \sim \log \kappa$

$$\boxed{F_0(\lambda) \sim \lambda^{3/2}, \quad \lambda \gg 1}$$

We can in fact write very explicit interpolating functions:

$$\lambda(\kappa) = \frac{\kappa}{8\pi}\ {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\frac{\kappa^2}{16}\right)$$

$$\partial_\lambda F_0^{\rm orb}(\lambda) = \frac{\kappa}{4} G^{2,3}_{3,3}\left(\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 0, & 0, & -\frac{1}{2} \end{matrix} \middle| -\frac{\kappa^2}{16}\right) + 4\pi^3 \mathrm{i}\lambda$$



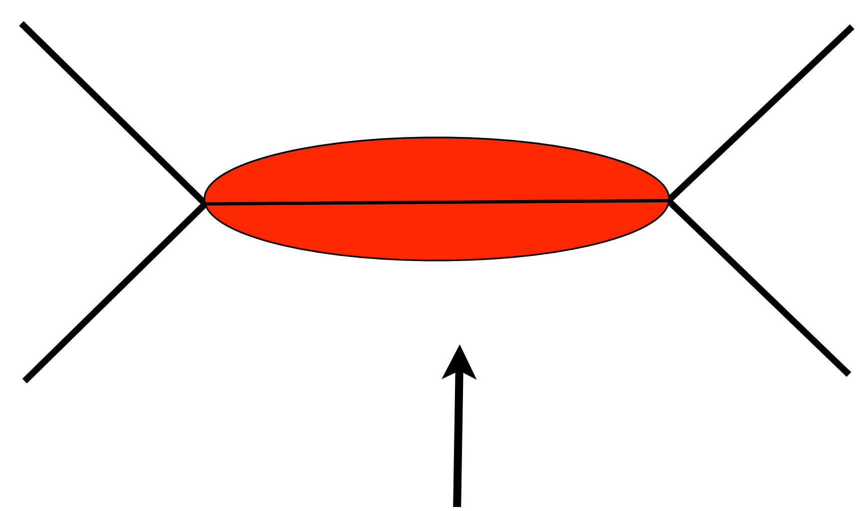
Planar limit from topological strings

Pure CS theory on *L(p,1)* has a large *N* topological string dual [AKMV]. The genus g free energies (for a fixed, generic flat connection) are equal to the genus g free energies of topological string theory on a *toric CY manifold*

$$F_g^{\mathrm{CS}}(t_i = g_s N_i) = F_g^{\mathrm{TS}}(t_i = \mathrm{moduli})$$

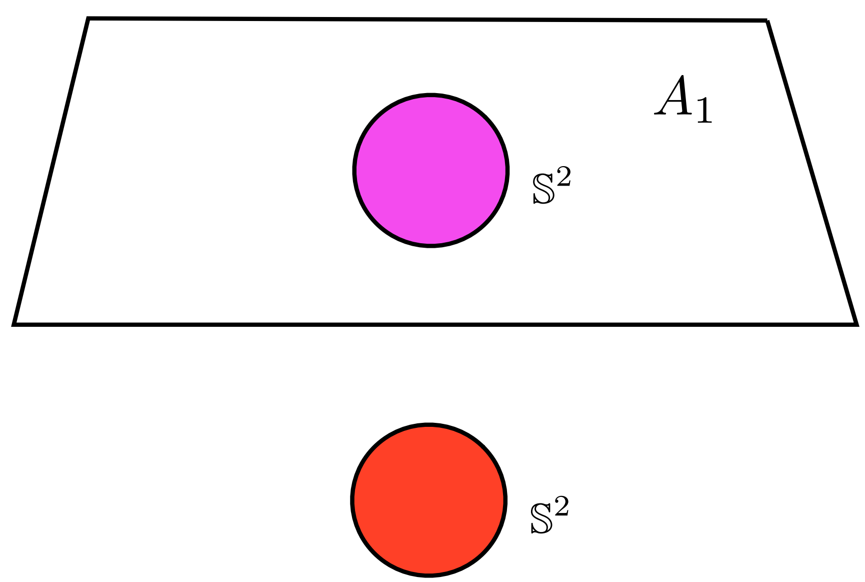
For *p=1* (i.e. *M*=$\mathbb{S}^3$) this is the original Gopakumar-Vafa large *N* duality. The toric CY is the *resolved conifold*

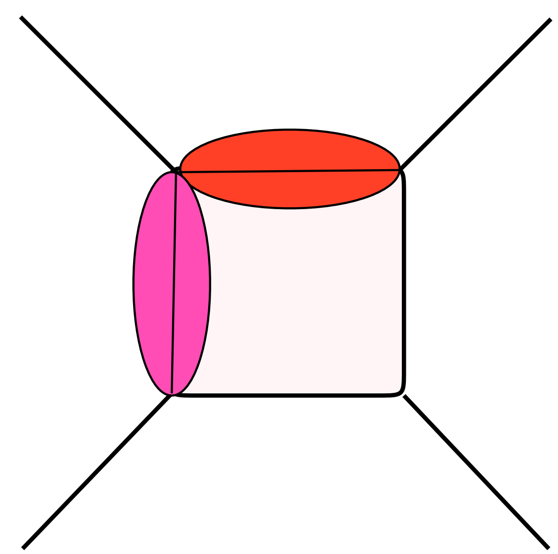
$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{S}^2$$



(single) 't Hooft parameter= (complexified) area of two-sphere

For p=2 (*i.e.* the lens space L(2,1)) the CY target is local $\mathbb{P}^1 \times \mathbb{P}^1$. It has two complexified Kahler moduli measuring the sizes of the two-spheres





Topological Strings
on local $\mathbb{P}^1 \times \mathbb{P}^1$

CS matrix model

ABJM theory

$$F_g^{\mathrm{ABJM}}(\lambda_1, \lambda_2) = F_g^{\mathrm{CS}}(2\pi \mathrm{i}\lambda_1, -2\pi \mathrm{i}\lambda_2) = F_g^{\mathrm{TS}}(2\pi \mathrm{i}\lambda_1, -2\pi \mathrm{i}\lambda_2)$$

relation between
matrix models

topological large
N duality

In particular, *the planar free energy in ABJM is just the prepotential of the topological string!* (a standard calculation in mirror symmetry)

B field and worldsheet instantons

We can now add the B-field. At strong coupling we have

$$\lambda_1(\kappa, B) = \frac{1}{2}\left(B^2 - \frac{1}{4}\right) + \frac{1}{24} + \frac{\log^2\kappa}{2\pi^2} + f\left(\frac{1}{\kappa^2}, \cos(2\pi B)\right)$$

we reproduce the shifts!

series

$$\boxed{F_0(\hat\lambda, B) = \frac{4\pi^3\sqrt{2}}{3}\hat\lambda^{3/2} - \pi^3 \mathrm{i}(\lambda_1^2 - \lambda_2^2) + \mathcal{O}\left(\mathrm{e}^{-2\pi\sqrt{2\hat\lambda}\pm 2\pi\mathrm{i}B}\right)}$$

after multiplying by $g_{\rm top}^{-2}$ it matches the AdS4 calculation !

phase of the partition function

calculable series of *worldsheet instantons* on $\mathbb{CP}^1 \subset \mathbb{CP}^3$

[Sorokin et al.]

Back to Wilson loops

$$\langle W_{\square}^{1/2} \rangle = \mathrm{e}^{\pi \mathrm{i} B} \frac{\kappa(\hat{\lambda}, B)}{2} \approx \underbrace{\mathrm{e}^{\pi\sqrt{2\hat{\lambda}}}}_{\text{AdS prediction}} \left(1 + \mathcal{O}\left(\underbrace{\mathrm{e}^{-2\pi\sqrt{2\hat{\lambda}} \pm 2\pi \mathrm{i} B}}_{\text{worldsheet instanton corrections}}\right)\right)$$

B=0: [M.M.-Putrov]

This corresponds to a (topological) disk string amplitude in the topological string picture

One can refine this computation to obtain vevs for 1/6 BPS Wilson loops [M.M.-Putrov] and for 1/2 BPS "giant" Wilson loops [Drukker-M.M.-Putrov]

Beyond the planar approximation

It turns out that one can compute the full *1/N* expansion of the free energy in a systematic (and efficient!) way, at least in the ABJM slice

The higher genus free energies in topological string theory can be obtained from the BCOV holomorphic anomaly equations. Schematically,

$$\partial_{\bar{t}} F_g(t,\bar{t}) = \text{functional of } F_{g'<g}(t,\bar{t})$$

Direct integration [Yau,Klemm+Huang-M.M.-...] : formulate them in terms of *modular forms* and impose boundary conditions at special points in moduli space. In local CYs they are fully integrable

$$F_2 = \frac{1}{432bd^2}\left(-\frac{5}{3}E_2^3 + 3bE_2^2 - 2E_4E_2\right) + \frac{16b^3 + 15db^2 + 21d^2b + 2d^3}{12960bd^2}$$

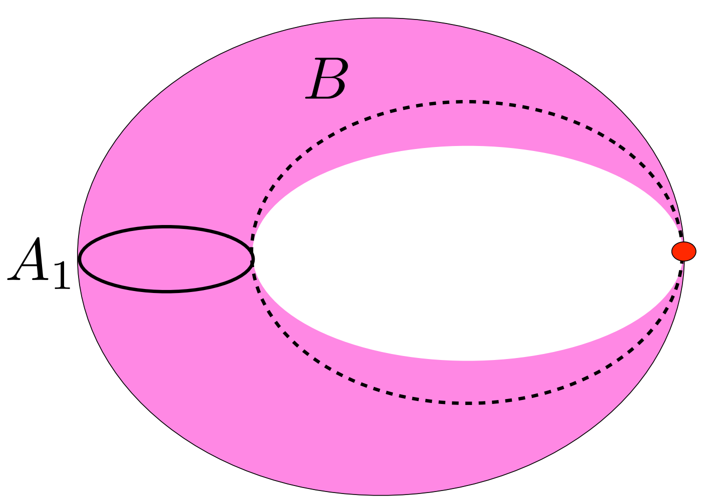
Upgrading the matrix models of non-critical strings: we have an integrable structure encoding a $1/N$ matrix model expansion, similar to the Painleve-type nonlinear ODEs

We can now address some nonperturbative issues *in the string coupling constant* by looking at the large genus behavior

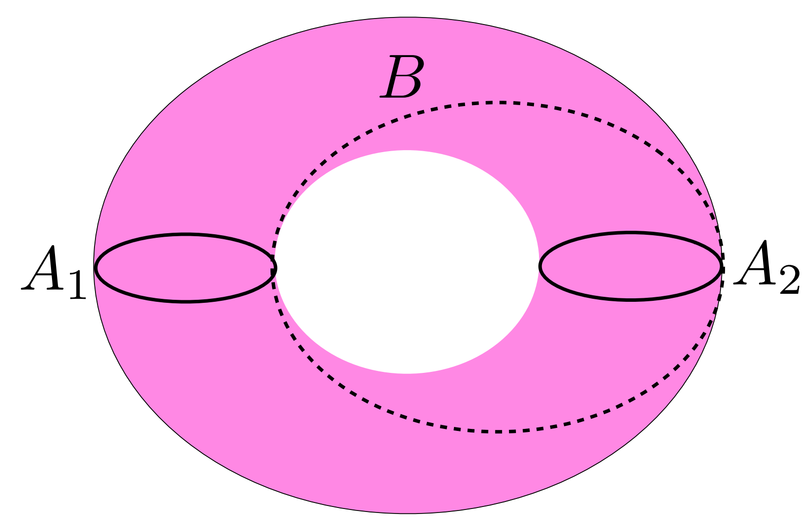
$$F_g(\lambda) \sim (2g)!(A_{\rm st}(\lambda))^{-2g}, \quad \lambda > \frac{1}{2}$$ [cf. Shenker]

$$A_{\rm st}(\lambda) \propto \frac{1}{\pi}\partial_\lambda F_0(\lambda) + \pi^2 {\rm i}$$ (complex) eigenvalue tunneling

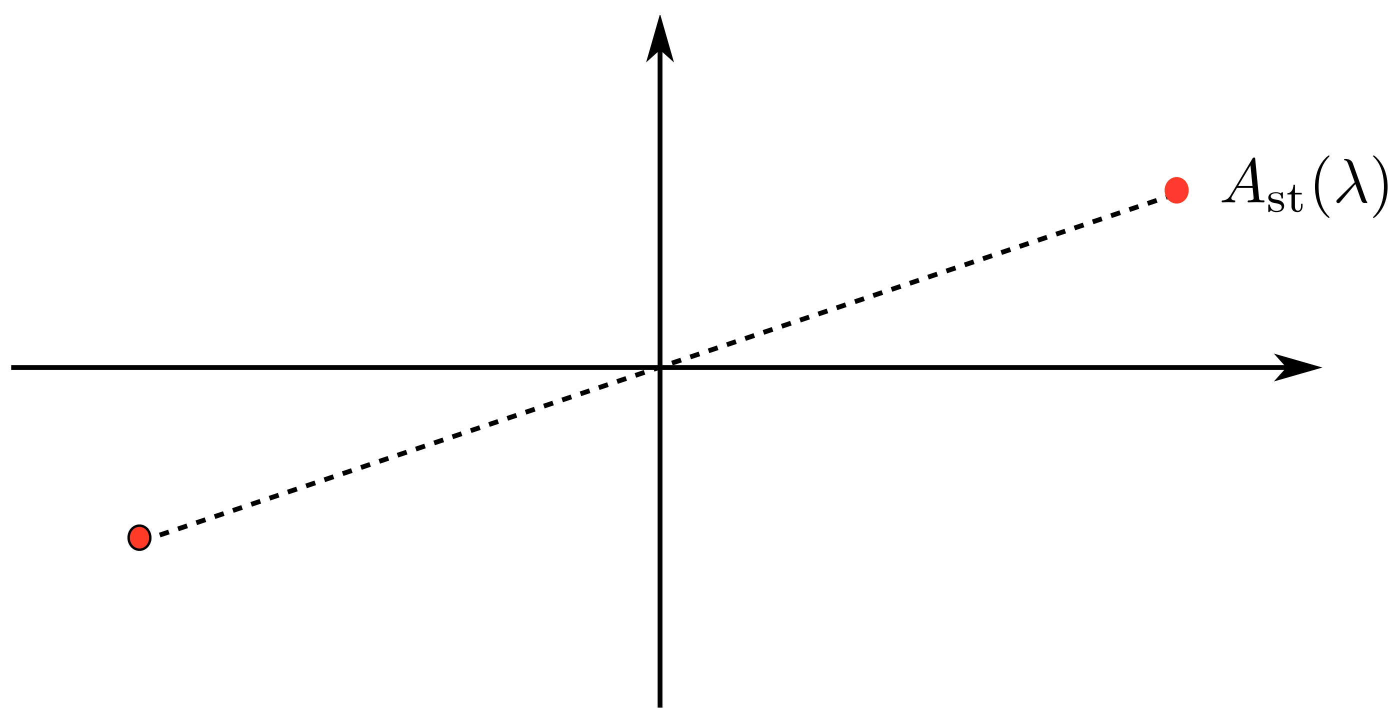
one-cut: [Shenker, David, Seiberg-Shih, M.M.-Schiappa-Weiss]



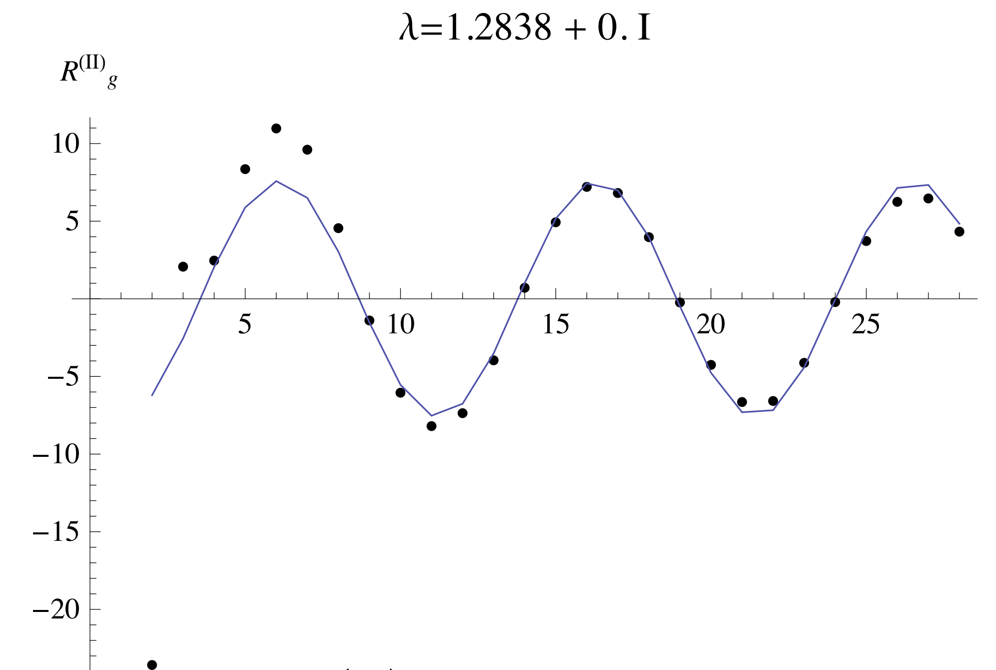
two-cuts: [Klemm-M.M.-Rauch]



Complex instantons: superstring perturbation theory on AdS4xCP3 is Borel summable for all nonzero ‘t Hooft coupling/radius!



Borel plane of the string coupling constant



$$\frac{F_g(\lambda)}{(2g)!|A_{\rm st}(\lambda)|^{-2g}} \sim \cos(2g\theta_A(\lambda)+\delta)$$

At strong coupling we find: $$A_{\rm st} \approx \frac{R^3}{4}\left(1+\frac{2\pi \mathrm{i}}{R^2}\right)$$

Euclidean D2 brane wrapping $\mathbb{RP}^3$?

Borel summability *invisible* in SUGRA -a stringy effect!

Conifold singularity and analytic continuation

In the ABJM slice, the conifold locus takes place at *imaginary* 't Hooft coupling and there is a double-scaling limit giving the *c=1* string:

$$F_g \sim \frac{B_{2g}}{2g(2g-2)} \left(\frac{\lambda - \lambda_c}{\log(\lambda - \lambda_c)} \right)^{2-2g} \qquad \lambda_c = -\frac{2\mathrm{iK}}{\pi^2}$$

In this regime (with imaginary CS coupling) the genus expansion is *no longer* Borel summable (real instantons)

All this seems to give a concrete realization of the scenario advocated for Polyakov to go to de Sitter space

Adding matter

We can deform ABJM by adding Nf matter fields in the (anti)fundamental [Gaiotto-Jafferis]. The resulting theory has N=3 SUSY and its M-theory dual involves a tri-Sasakian manifold

$$F_{\mathcal{N}=3}(\mathbb{S}^3) = -N^2 \frac{2\pi}{3\sqrt{2\lambda}} \frac{1+N_f/k}{\sqrt{1+N_f/(2k)}}$$

quenched approximation

$N_f \ll N$

$$= -N^2 \frac{2\pi}{3\sqrt{2\lambda}} - \frac{\pi}{4} N_f N \sqrt{2\lambda} + \mathcal{O}(N_f^2)$$

[Couso-M.M.-Putrov]

$$\int_{-A}^{A} \mathrm{d}\mu \, \rho_1(\mu) \log\left[2\cosh\frac{\mu}{2}\right]$$

strong coupling limit of planar correlator!

Unquenched/Veneziano limit (arbitrary Nf): explicit solution available, but harder to analyze (no CY picture!)

Conclusions and open problems

- We have used matrix models/topological strings to derive important aspects of ABJM theory at strong coupling. It is of course possible to analyze related 3d SCFTs with the same tools (matter, massive type IIA...)

- Concrete predictions for worldsheet instanton corrections, which should be better understood. Direct calculation in type IIA? Localization?

- Is there an *a priori* reason for the connection with topological strings?

- Nonperturbative effects controlling genus expansion: identify them in both the gauge theory (large *N* instantons?) and in the superstring theory (wrapped D-branes?)

Appendix: Supermatrix models

Hermitian supermatrix

$$\Phi = \begin{pmatrix} A & \Psi \\ \Psi^\dagger & C \end{pmatrix}$$

A, C Hermitian, Grassmann even

Ψ complex, Grassmann odd

$$Z_{\mathrm{s}}(N_1|N_2) = \int \mathcal{D}\Phi\, e^{-\frac{1}{g_s}\mathrm{Str}V(\Phi)}$$

[Yost, Alvarez-Gaume-Mañes, Dijkgraaf-Vafa, ...]

Assume the eigenvalues are *real* (physical supermatrix model):

$$Z_{\mathrm{s}}(N_1|N_2) = \int \prod_{i=1}^{N_1} d\mu_i \prod_{j=1}^{N_2} d\nu_j \frac{\prod_{i<j}\left(\mu_i-\mu_j\right)^2\left(\nu_i-\nu_j\right)^2}{\prod_{i,j}\left(\mu_i-\nu_j\right)^2} e^{-\frac{1}{g_s}\left(\sum_i V(\mu_i)-\sum_j V(\nu_j)\right)}$$

compare to

$$Z_{\mathrm{b}}(N_1,N_2) = \int \prod_{i=1}^{N_1} d\mu_i \prod_{j=1}^{N_2} d\nu_j \prod_{i<j}\left(\mu_i-\mu_j\right)^2\left(\nu_i-\nu_j\right)^2 \prod_{i,j}\left(\mu_i-\nu_j\right)^2 e^{-\frac{1}{g}\left(\sum_i V(\mu_i)+\sum_j V(\nu_j)\right)}$$