Stringy Differential Geometry, beyond Riemann

Imtak Jeon

Sogang University, Seoul

12 July. 2011

Talk is based on works

in collaboration with Jeong-Hyuck Park and Kanghoon Lee

- **Differential geometry with a projection: Application to double field theory**
  JHEP 1104:014 (2011), arXiv:1011.1324

- **Double field formulation of Yang-Mills theory**
  Phys.Lett.B701:260-264 (2011) , arXiv:1102.0419

- **Stringy differential geometry, beyond Riemann**
  arXiv: 1105.6294

Introduction

- In Riemannian geometry, the fundamental object is the metric, $g_{\mu\nu}$.
- String theory puts $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ on an equal footing.
- This may suggests the existence of a veiled unifying description of them, beyond Riemann.

Introduction

Symmetry

- guides the structure of Lagrangians and organizes the physical laws into simple forms.

- for example, in Maxwell theory,
  - U(1) gauge symmetry forbids $m^2 A_\mu A^\mu$
  - Lorentz symmetry unifies the original 4 eqs into 2.

Introduction

- Essence of Riemannian geometry: Diffeomorphism
  - $\partial_\mu \longrightarrow \nabla_\mu = \partial_\mu + \Gamma_\mu$
  - $\nabla_\lambda g_{\mu\nu} = 0 \longrightarrow \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$
  - Curvature, $[\nabla_\mu, \nabla_\nu], \longrightarrow R_g$.
- Main purpose : Generalization to a geometry for stringy theory.
- Key property in string theory : T-duality

Outline

- T-duality in string theory
- String effective action and Double Field Theory
- Stringy differential geometry as underlying geometry of DFT.
    - Local inertial frame
- Non-Abelian YM gauge field in DFT.

Closed string on a Toroidal background

- For closed string wraping around a circle, the modes expansion is

$$X = X_L + X_R \,,$$

$$X_L(\sigma^+) = \tfrac{1}{2}(x + \tilde{x}) + \tfrac{1}{2}(p + w)\sigma^+ + \cdots \,,$$

$$X_R(\sigma^-) = \tfrac{1}{2}(x - \tilde{x}) + \tfrac{1}{2}(p - w)\sigma^- + \cdots \,.$$

where

$$\sigma^\pm := \tau \pm \sigma, \qquad \begin{array}{ll} p: \text{ momentum mode}\,, & w: \text{ winding mode}\,, \\ x: \text{ center of mass}\,, & \tilde{x} \text{ is introduced}\,. \end{array}$$

- Physical condition (Virasoro constraint) implies the level matching and the on-shell condition,

$$\begin{array}{l} L_0 - \bar{L}_0 = 0 \\ L_0 + \bar{L}_0 = 2 \end{array} \quad \Longleftrightarrow \quad \begin{array}{l} N - \bar{N} - p \cdot w = 0 \\ \mathcal{M}^2 = p^2 + w^2 + 2(N + \bar{N} - 2) \end{array}$$

T-duality

- T-duality transformation is

$$X_L^\mu + X_R^\mu \longrightarrow X_L^\mu - X_R^\mu \,,$$

such that the two pairs, (x, p) and $(\tilde{x}, w)$, are exchanged by each other,

$$(x, \tilde{x}, p, w) \longrightarrow (\tilde{x}, x, w, p) \,,$$

Then, the level matching condition and the mass spectrum is invariant.

- On d-dimensional toroidal background, the T-duality is realized as $O(d, d, Z)$ transformation preserving $\mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{2d \times 2d}$.

- x^i : Conjugate to the momenta p_i

  $\tilde{x}_i$: Conjugate to the winding number w^i

  It is natural to introduce the additional coordinate $\tilde{x}_i$

- Closed string field theory treats momenta and winding modes symmetrically. So, target spacetime fields are also depend on both x and $\tilde{x}$.: $\Phi(x^i, \tilde{x}_i)$[Kugo, Zwiebach 1992]

Effective theory

- Describes a gravity : $g_{\mu\nu}$, $B_{\mu\nu}$, ϕ are on an equal footing completing the massless sector.
- Low energy effective action of them:

$$S_{\text{eff.}} = \int \mathrm{d}x^D \sqrt{-g} e^{-2\phi} \left(R_g + 4\partial_\mu \phi \partial^\mu \phi - \tfrac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \right)$$

- Diffeomorphism and one-form gauge symmetry are manifest

$$x^\mu \to x^\mu + \delta x^\mu \,, \qquad B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \,.$$

Effective theory

- Describes a gravity : $g_{\mu\nu}, B_{\mu\nu}, \phi$ are on an equal footing completing the massless sector.
- Low energy effective action of them:

$$S_{\text{eff.}} = \int \mathrm{d}x^D \sqrt{-g} e^{-2\phi} \left(R_g + 4\partial_\mu \phi \partial^\mu \phi - \tfrac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \right)$$

- Though not manifest, if there is isometry, this enjoys T-duality which mixes $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$ (Buscher)
- Include the $\tilde{x}$ dependence. $\longrightarrow$ double field theory

Double field theory (DFT)

- Hull and Zwiebach , later with Hohm constructed action which has explicit T-duality,

$$S_{\rm DFT} = \int {\rm d}y^{2D}\, e^{-2d}\Big[\ \mathcal{H}^{AB}\left(4\partial_A\partial_B d - 4\partial_A d\partial_B d + \tfrac{1}{8}\partial_A\mathcal{H}^{CD}\partial_B\mathcal{H}_{CD} - \tfrac{1}{2}\partial_A\mathcal{H}^{CD}\partial_C\mathcal{H}_{BD}\right) + 4\partial_A\mathcal{H}^{AB}\partial_B d - \partial_A\partial_B\mathcal{H}^{AB}\ \Big]\,.$$

- d is double field theory 'dilaton' given by

$$e^{-2d} = \sqrt{-g}e^{-2\phi}$$

and $\mathcal{H}_{AB}$ is $2D \times 2D$ matrix, 'generalized metric'

$$\mathcal{H}_{AB} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\kappa}B_{\kappa\sigma} \\ B_{\rho\kappa}g^{\kappa\nu} & g_{\rho\sigma} - B_{\rho\kappa}g^{\kappa\lambda}B_{\lambda\sigma} \end{pmatrix}\,.$$

- Spacetime dimension is 'formally' doubled from D to $2D$,

$$x^\mu \to y^A = (\tilde{x}_\mu, x^\nu)\,, \quad \partial_\mu \to \partial_A = (\tilde{\partial}^\mu, \partial_\nu)\,.$$

Double field theory (DFT)

- Indices, $A, B, C, \cdots$, are $2D$-dimensional vector indices, which can be lowered or raised by $O(D,D)$ metric $\mathcal{J}_{AB}$.

$$\mathcal{J}_{AB} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

- Under global $O(D,D)$ rotation, $L \in O(D,D)$,

$$\mathcal{H}_{AB}(y) \longrightarrow L_A{}^C L_B{}^D \mathcal{H}_{CD}(y') , \qquad d(y) \longrightarrow d(y') ,$$

  manifest that DFT action is invariant.

- Note that $O(D,D)$ T-duality is background independent.

Double field theory (DFT)

- By 'Level matching condition,'

  DFT action $\equiv$ D-dimensional closed string effective action

- It possesses a gauge symmetry, say *double gauge symmetry*, through 'generalized Lie derivative';

  double gauge symmetry $=$ diffeomorphism $+$ 1-form gauge symmety

Level matching constraint

- Level matching condition for the massless sector,

$$p \cdot w \equiv 0 \quad \Longleftrightarrow \quad \frac{\partial^2}{\partial \tilde{x}_\mu \partial x^\mu} = \partial_A \partial^A \equiv 0\,.$$

- Strong constraint : all the fields and the gauge parameters as well as all of their products should be annihilated by $\partial^2 = \partial_A \partial^A$

$$\partial^2 \Phi \equiv 0\,, \qquad \partial_A \Phi_1 \partial^A \Phi_2 \equiv 0 \qquad \Longleftrightarrow \quad \tilde{\partial} \equiv 0\,.$$

- DFT is restricted in the strong constraint.
- Actually meaning that theory is not truly doubled: we can impose $\tilde{\partial} \equiv 0$ by $O(D,D)$ rotation.

- DFT is reorganization of effective action: Upon the level matching constraint,

$$S_{\text{DFT}} \Longrightarrow S_{\text{eff.}} = \int \mathrm{d}x^D \sqrt{-g} e^{-2\phi} \left(R_g + 4(\partial\phi)^2 - \tfrac{1}{12} H^2 \right).$$

Double gauge symmetry

- Introducing an unifying doubled gauge parameter,

$$X^A = (\Lambda_\mu, \delta x^\nu)$$

- diffeomorphism and 1-form gauge transformation is expressed in unified fashion, upon the level matching constraint,

$$\delta_X \mathcal{H}_{AB} \equiv X^C \partial_C \mathcal{H}_{AB} + 2\partial_{[A} X_{C]} \mathcal{H}^C{}_B + 2\partial_{[B} X_{C]} \mathcal{H}_A{}^C \,,$$

$$\delta_X \left(e^{-2d}\right) \equiv \partial_A \left(X^A e^{-2d}\right) \,.$$

- In fact, these coincide with the generalized Lie derivative,

$$\delta_X \mathcal{H}_{AB} = \hat{\mathcal{L}}_X \mathcal{H}_{AB} \,, \qquad \delta_X (e^{-2d}) = \hat{\mathcal{L}}_X (e^{-2d}) = -2(\hat{\mathcal{L}}_X d) e^{-2d}.$$

Generalized Lie derivative

- Definition, Siegel, Courant, Grana ...

$$\hat{\mathcal{L}}_X T_{A_1\cdots A_n} := X^B \partial_B T_{A_1\cdots A_n} + \omega \partial_B X^B T_{A_1\cdots A_n} + \sum_{i=1}^{n} 2\partial_{[A_i} X_{B]} T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_n}.$$

- *cf.* ordinary one,

$$\mathcal{L}_X T_{A_1\cdots A_n} := X^B \partial_B T_{A_1\cdots A_n} + \omega \partial_B X^B T_{A_1\cdots A_n} + \sum_{i=1}^{n} \partial_{A_i} X_B T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_n}.$$

- Definition of tensor(density) in DFT : 'double gauge covariant' quantity

$$\delta_X T_{A_1 A_2\cdots A_n} = \hat{\mathcal{L}}_X T_{A_1 A_2\cdots A_n} .$$

- $\mathcal{H}_{AB}$: rank 2 generalized tensor, e^{-2d}: weight 1 generalized scalar,

Algebra of generalized Lie derivative

- Commutator of generalized Lie derivatives is closed, up to the level matching condition, by using **c**-bracket,

$$[\hat{\mathcal{L}}_X, \hat{\mathcal{L}}_Y] \equiv \hat{\mathcal{L}}_{[X,Y]_{\mathbf{C}}} \,,$$

where $[X, Y]_{\mathbf{C}}$ denotes C-bracket

$$[X, Y]^A_{\mathbf{C}} := X^B \partial_B Y^A - Y^B \partial_B X^A + \tfrac{1}{2} Y^B \partial^A X_B - \tfrac{1}{2} X^B \partial^A Y_B \,,$$

known to be euqivalent the Courant bracket if dropping $\tilde{\partial}$.

Diffeomorphism & one-form gauge symmetry

- Direct computation shows that

$$\hat{\mathcal{L}}_X \mathcal{H}_{AB} \equiv X^C \partial_C \mathcal{H}_{AB} + 2\partial_{[A} X_{C]} \mathcal{H}^C{}_B + 2\partial_{[B} X_{C]} \mathcal{H}_A{}^C \,,$$

$$\hat{\mathcal{L}}_X \left(e^{-2d}\right) \equiv \partial_A \left(X^A e^{-2d}\right) \,,$$

are symmetry of DFT action by Hull, Zwiebach and Hohm,

$$S_{\mathrm{DFT}} = \int \mathrm{d}y^{2D}\, e^{-2d}\, \mathcal{R}(\mathcal{H}, d) \,,$$

where 'generalized curvature' is

$$\begin{aligned} \mathcal{R}(\mathcal{H}, d) = \quad & \mathcal{H}^{AB} \left(4\partial_A \partial_B d - 4\partial_A d \partial_B d + \tfrac{1}{8}\partial_A \mathcal{H}^{CD} \partial_B \mathcal{H}_{CD} - \tfrac{1}{2}\partial_A \mathcal{H}^{CD} \partial_C \mathcal{H}_{BD}\right) \\ & + 4\partial_A \mathcal{H}^{AB} \partial_B d - \partial_A \partial_B \mathcal{H}^{AB} \,. \end{aligned}$$

- This expression may be analogous to the case of writing the scalar curvature, R_g, in terms of the metric and its derivative.

Diffeomorphism & one-form gauge symmetry

- Direct computation shows that

$$\hat{\mathcal{L}}_X \mathcal{H}_{AB} \equiv X^C \partial_C \mathcal{H}_{AB} + 2\partial_{[A} X_{C]} \mathcal{H}^C{}_B + 2\partial_{[B} X_{C]} \mathcal{H}_A{}^C \,,$$

$$\hat{\mathcal{L}}_X \left(e^{-2d}\right) \equiv \partial_A \left(X^A e^{-2d}\right) \,,$$

are symmetry of the action by Hull, Zwiebach and Hohm,

$$S_{\mathrm{DFT}} = \int \mathrm{d}y^{2D}\, e^{-2d}\, \mathcal{R}(\mathcal{H}, d) \,,$$

where

$$\begin{aligned} \mathcal{R}(\mathcal{H}, d) = \quad & \mathcal{H}^{AB} \left(4\partial_A \partial_B d - 4\partial_A d \partial_B d + \tfrac{1}{8}\partial_A \mathcal{H}^{CD} \partial_B \mathcal{H}_{CD} - \tfrac{1}{2}\partial_A \mathcal{H}^{CD} \partial_C \mathcal{H}_{BD}\right) \\ & + 4\partial_A \mathcal{H}^{AB} \partial_B d - \partial_A \partial_B \mathcal{H}^{AB} \,. \end{aligned}$$

- What is the underlying geometry?

Diffeomorphism & one-form gauge symmetry

- Direct computation shows that

$$\hat{\mathcal{L}}_X \mathcal{H}_{AB} \equiv X^C \partial_C \mathcal{H}_{AB} + 2\partial_{[A} X_{C]} \mathcal{H}^C{}_B + 2\partial_{[B} X_{C]} \mathcal{H}_A{}^C \,,$$

$$\hat{\mathcal{L}}_X \left(e^{-2d}\right) \equiv \partial_A \left(X^A e^{-2d}\right) \,,$$

are symmetry of the action by Hull, Zwiebach and Hohm,

$$S_{\mathrm{DFT}} = \int \mathrm{d}y^{2D}\, e^{-2d}\, \mathcal{R}(\mathcal{H}, d) \,,$$

where

$$\begin{aligned} \mathcal{R}(\mathcal{H}, d) = \quad & \mathcal{H}^{AB} \left(4\partial_A \partial_B d - 4\partial_A d \partial_B d + \tfrac{1}{8}\partial_A \mathcal{H}^{CD} \partial_B \mathcal{H}_{CD} - \tfrac{1}{2}\partial_A \mathcal{H}^{CD} \partial_C \mathcal{H}_{BD}\right) \\ & + 4\partial_A \mathcal{H}^{AB} \partial_B d - \partial_A \partial_B \mathcal{H}^{AB} \,. \end{aligned}$$

- connection and covariant derivative of the geometry?

Stringy differential geometry

We propose a novel differential geometry which

- **enables us to rewrite the low energy effective action as a single term in a geometrical manner,**

$$S_{\text{eff.}} = \int \mathrm{d}x^{2D}\, e^{-2d}\, \mathcal{H}^{AB} S_{AB}\,,$$

- **treats the three objects of the massless sector in a unified way,**
- **manifests not only diffeomorphism and one-form gauge symmetry but also $\mathbf{O}(D,D)$ T-duality,**

Stringy differential geometry

- Motivated by the observation that,

$$\mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}$$

is of the most general form to satisfy

$$\mathcal{H}_A{}^C \mathcal{H}_C{}^B = \delta_A{}^B \,, \qquad \mathcal{H}_{AB} = \mathcal{H}_{BA} \,,$$

and the upper left $D \times D$ block of $\mathcal{H}$ is non-degenerate,

- we focus on a symmetric *projection*,

$$P_A{}^B P_B{}^C = P_A{}^C \qquad P_{AB} = P_{BA} \,,$$

which is related to $\mathcal{H}$ by

$$P_{AB} = \tfrac{1}{2}(\mathcal{J}_{AB} + \mathcal{H}_{AB}) \,.$$

- Three basic objects:

$$\mathcal{J}_{AB} \,, \qquad P_{AB} \,, \qquad d \,.$$

Stringy differential geometry

- We postulate a “semi-covariant" derivative, ∇_A,

$$\nabla_C T_{A_1 A_2 \cdots A_n} = \partial_C T_{A_1 A_2 \cdots A_n} - \omega \Gamma^B{}_{BC} T_{A_1 A_2 \cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n} \, .$$

- In particular,

$$\nabla_C (e^{-2d}) = \partial_C e^{-2d} - \Gamma^B{}_{BC} e^{-2d} = -2(\nabla_C d) e^{-2d}$$

$$\Longrightarrow \quad \nabla_C d := \partial_C d + \tfrac{1}{2} \Gamma^B{}_{BC}$$

Stringy differential geometry

- We demand the following compatibility conditions,

$$\nabla_A \mathcal{J}_{BC} = 0\,, \qquad \nabla_A P_{BC} = 0\,, \qquad \nabla_A d = 0\,,$$

  as for the unifying description of the massless modes
  (*cf.* $\nabla_\lambda g_{\mu\nu} = 0$ in Riemannian geometry).

- Further we require,

$$\Gamma_{CAB} + \Gamma_{CBA} = 0\,, \qquad \Gamma_{ABC} + \Gamma_{CAB} + \Gamma_{BCA} = 0\,.$$

Stringy differential geometry

- Then, we may replace ∂_A by ∇_A in $\hat{\mathcal{L}}_X$ and also in $[X,Y]^A_{\mathrm{C}}$,

$$\hat{\mathcal{L}}_X T_{A_1\cdots A_n} = X^B \nabla_B T_{A_1\cdots A_n} + \omega \nabla_B X^B T_{A_1\cdots A_n} + \sum_{i=1}^n 2\nabla_{[A_i} X_{B]} T_{A_1\cdots A_{i-1}}{}^B{}_{A_{i+1}\cdots A_n} \,,$$

$$[X,Y]^A_{\mathrm{C}} = X^B \nabla_B Y^A - Y^B \nabla_B X^A + \tfrac{1}{2} Y^B \nabla^A X_B - \tfrac{1}{2} X^B \nabla^A Y_B \,.$$

- *cf.* In Riemannian geometry, torsion free condition implies

$$\mathcal{L}_X T_{\mu_1\cdots\mu_n} = X^\nu \nabla_\nu T_{\mu_1\cdots\mu_n} + \omega \nabla_\nu X^\nu T_{\mu_1\cdots\mu_n} + \sum_{i=1}^n \nabla_{\mu_i} X^\nu T_{\mu_1\cdots\mu_{i-1}\nu\mu_{i+1}\cdots\mu_n} \,,$$

$$[X,Y]^\mu = X^\nu \nabla_\nu Y^\mu - Y^\nu \nabla_\nu X^\mu \,.$$

Stringy differential geometry

- Explicitly, the connection is

$$\Gamma_{CAB} = 2(P\partial_C P\bar{P})_{[AB]} + 2\big(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E} - P_{[A}{}^{D}P_{B]}{}^{E}\big)\partial_D P_{EC}$$

$$-\tfrac{4}{D-1}\big(\bar{P}_{C[A}\bar{P}_{B]}{}^{D} + P_{C[A}P_{B]}{}^{D}\big)\big(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\big)\,.$$

where $\bar{P}$ is the complementary projection,

$$\bar{P} = \tfrac{1}{2}(1-\mathcal{H})\,.$$

and P and $\bar{P}$ are orthogonal,

$$P\bar{P} = \bar{P}P = 0\,.$$

- Is this derviativ ∇_A double gauge covariant?

Stringy differential geometry

- For usefulness, we set

$$\mathcal{P}_{CAB}{}^{DEF} := P_C{}^D P_{[A}{}^{[E} P_{B]}{}^{F]} + \tfrac{2}{D-1} P_{C[A} P_{B]}{}^{[E} P^{F]D} \,,$$

$$\bar{\mathcal{P}}_{CAB}{}^{DEF} := \bar{P}_C{}^D \bar{P}_{[A}{}^{[E} \bar{P}_{B]}{}^{F]} + \tfrac{2}{D-1} \bar{P}_{C[A} \bar{P}_{B]}{}^{[E} \bar{P}^{F]D} \,,$$

  which satisfy

$$\mathcal{P}_{CABDEF} = \mathcal{P}_{DEFCAB} = \mathcal{P}_{C[AB]D[EF]} \,,$$

$$\mathcal{P}_{CAB}{}^{DEF} \mathcal{P}_{DEF}{}^{GHI} = \mathcal{P}_{CAB}{}^{GHI} \,,$$

$$\mathcal{P}^A{}_{ABDEF} = 0 \,, \quad P^{AB} \mathcal{P}_{ABCDEF} = 0 \,, \quad \textit{etc.}$$

- The connection belongs to the kernel of these rank six-projectors,

$$\mathcal{P}_{CAB}{}^{DEF} \Gamma_{DEF} = 0 \,, \qquad \bar{\mathcal{P}}_{CAB}{}^{DEF} \Gamma_{DEF} = 0 \,.$$

Stringy differential geometry

- Under *double-gauge transform*, $\delta_X P_{AB} = \hat{\mathcal{L}}_X P_{AB}$ and $\delta_X d = \hat{\mathcal{L}}_X d$, the diffeomorphism and the one-form gague tranform,
  we obtain

$$(\delta_X - \hat{\mathcal{L}}_X)\Gamma_{CAB} \equiv 2\big[(\mathcal{P}+\bar{\mathcal{P}})_{CAB}{}^{FDE} - \delta_C^{\ F}\delta_A^{\ D}\delta_B^{\ E}\big]\partial_F\partial_{[D}X_{E]}\ ,$$

  and

$$(\delta_X - \hat{\mathcal{L}}_X)\nabla_C T_{A_1\cdots A_n} \equiv \sum_i 2(\mathcal{P}+\bar{\mathcal{P}})_{CA_i}{}^{BFDE}\partial_F\partial_{[D}X_{E]}T_{A_1\cdots B\cdots A_n}\ .$$

- Hence, these are not double-gauge covariant,

$$\delta_X \ \neq\ \hat{\mathcal{L}}_X\ .$$

Stringy differential geometry

- However, the characteristic property of our derivative, ∇_A, is that, combined with the projections, it can generate various $\mathbf{O}(D,D)$ and double-gauge covariant quantities:

$$P_C{}^D \bar{P}_{A_1}{}^{B_1} \bar{P}_{A_2}{}^{B_2} \cdots \bar{P}_{A_n}{}^{B_n} \nabla_D T_{B_1 B_2 \cdots B_n} \ ,$$

$$\bar{P}_C{}^D P_{A_1}{}^{B_1} P_{A_2}{}^{B_2} \cdots P_{A_n}{}^{B_n} \nabla_D T_{B_1 B_2 \cdots B_n} \ ,$$

$$P^{AB} \nabla_A T_B \ , \qquad \bar{P}^{AB} \nabla_A T_B \ ,$$

$$P^{AB} \bar{P}_{C_1}{}^{D_1} \cdots \bar{P}_{C_n}{}^{D_n} \nabla_A \nabla_B T_{D_1 \cdots D_n} \ ,$$

$$\bar{P}^{AB} P_{C_1}{}^{D_1} \cdots P_{C_n}{}^{D_n} \nabla_A \nabla_B T_{D_1 \cdots D_n} \ .$$

- This suggests us to call ∇_A as *semi-covariant derivative* .

Curvature

- The usual curvature,

$$R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^{E} \Gamma_{BED} - \Gamma_{BC}{}^{E} \Gamma_{AED} \,,$$

satisfying

$$[\nabla_A, \nabla_B] T_{C_1 C_2 \cdots C_n} = -\Gamma_{DAB} \nabla^D T_{C_1 C_2 \cdots C_n} + \sum_{i=1}^{n} R_{C_i DAB}\, T_{C_1 \cdots C_{i-1}}{}^{D}{}_{C_{i+1} \cdots C_n} \,,$$

is NOT double-gauge covariant,

$$\delta_X R_{ABCD} \neq \hat{\mathcal{L}}_X R_{ABCD} \,.$$

It satisfy $R_{ABCD} = R_{[AB][CD]}$, but

$$R_{ABCD} \neq R_{CDAB} \,.$$

Generalized curvature

- Instead, we define, as for a key quantity in our formalism,

$$S_{ABCD} := \tfrac{1}{2}\left(\mathcal{R}_{ABCD} + \mathcal{R}_{CDAB} - \Gamma^{E}{}_{AB}\Gamma_{ECD}\right) .$$

- This can be read off from the commutaor,

$$P_I{}^A \bar{P}_J{}^B [\nabla_A, \nabla_B] T_C \equiv 2 P_I{}^A \bar{P}_J{}^B S_{CDAB} T^D .$$

- It can be shown, by brute force computation, to satisfy
  - just like the Riemann curvature,

$$S_{ABCD} = \tfrac{1}{2}(S_{[AB][CD]} + S_{[CD][AB]}) \equiv S_{\{ABCD\}} , \qquad S_{A[BCD]} = 0 ,$$

  - and further

$$P_I{}^A P_J{}^B \bar{P}_K{}^C \bar{P}_L{}^D S_{ABCD} \equiv 0 , \qquad P_I{}^A \bar{P}_J{}^B P_K{}^C \bar{P}_L{}^D S_{ABCD} \equiv 0 , \qquad etc.$$

Generalized curvature

- S_{ABCD} is not double-gauge covariant.
- Under the double-gauge transformations, we get

$$(\delta_X - \hat{\mathcal{L}}_X)S_{ABCD} \equiv 4\nabla_{\{A}\left[(\mathcal{P}+\bar{\mathcal{P}})_{BCD\}}{}^{EFG}\partial_E\partial_{[F}X_{G]}\,.\right]\,.$$

- Nevertheless, contracting indices we can obtain covariant quantities.

Covariant curvature

By setting

$$S_{AB} = S_{BA} := S^{C}{}_{ACB} \,,$$

which turns out to be traceless,

$$S^{A}{}_{A} \equiv 0 \,,$$

and contracting with projection operators, we get

- Double-gauge covariant rank two-tensor,

$$P_{I}{}^{A} \bar{P}_{J}{}^{B} S_{AB} \qquad : \text{generalized curvature tensor}$$

- Double-gauge covariant scalar,

$$\mathcal{H}^{AB} S_{AB} \qquad : \text{generalized curvature scalar}$$

Reproduction of DFT

- Natural DFT action is proposed by

$$S_{\rm DFT} = \int {\rm d}y^{2D}\, e^{-2d}\, \mathcal{H}^{AB} S_{AB}\,,$$

- In fact, the covariant scalar constitutes the effective action as

$$\mathcal{H}^{AB} S_{AB} \equiv R_g + 4\Box\phi - 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu}\,.$$

- It also agrees with Hull, Zwiebach and Hohm,

$$\begin{aligned}\mathcal{H}^{AB} S_{AB} \equiv \quad & \mathcal{H}^{AB}\left(4\partial_A\partial_B d - 4\partial_A d\partial_B d + \tfrac{1}{8}\partial_A\mathcal{H}^{CD}\partial_B\mathcal{H}_{CD} - \tfrac{1}{2}\partial_A\mathcal{H}^{CD}\partial_C\mathcal{H}_{BD}\right)\\ & +4\partial_A\mathcal{H}^{AB}\partial_B d - \partial_A\partial_B\mathcal{H}^{AB}\,.\end{aligned}$$

Deriving Equations of motion

- It is easy to rederive the equation of motion.
- Under arbitrary infinitesimal transformations of the dilaton and the projection, we get

$$\delta S_{\text{eff.}} \equiv \int \mathrm{d}y^{2D}\, 2e^{-2d} \left(\delta P^{AB} S_{AB} - \delta d\, \mathcal{H}^{AB} S_{AB} + \delta S_{AB} \right) .$$

- The third therm is total derivative as

$$\delta S_{ABCD} = \nabla_{[A} \delta \Gamma_{B]CD} + \nabla_{[C} \delta \Gamma_{D]AB} \,, \qquad \nabla_A d = 0 \,,$$

where explicitly

$$\begin{aligned} \delta \Gamma_{CAB} = \; & 2P_{[A}^{\;\;D} \bar{P}_{B]}^{\;\;E} \nabla_C \delta P_{DE} + 2(\bar{P}_{[A}^{\;\;D} \bar{P}_{B]}^{\;\;E} - P_{[A}^{\;\;D} P_{B]}^{\;\;E}) \nabla_D \delta P_{EC} \\ & - \tfrac{4}{D-1} (\bar{P}_{C[A} \bar{P}_{B]}^{\;\;D} + P_{C[A} P_{B]}^{\;\;D})(\partial_D \delta d + P_{E[G} \nabla^G \delta P^E{}_{D]}) \\ & - \Gamma_{FDE}\, \delta(\mathcal{P} + \bar{\mathcal{P}})_{CAB}{}^{FDE} \,, \end{aligned}$$

Deriving Equations of motion

- It is easy to rederive the equation of motion.
- Under arbitrary infinitesimal transformations of the dilaton and the projection, we get

$$\delta S_{\text{eff.}} \equiv \int \mathrm{d}y^{2D}\, 2e^{-2d} \left(\delta P^{AB} S_{AB} - \delta d\, \mathcal{H}^{AB} S_{AB} \right) .$$

- from the relation

$$\delta P = P \delta P \bar{P} + \bar{P} \delta P P \,,$$

the equations of motions are easily obtained:

$$P_{(I}{}^{A} \bar{P}_{J)}{}^{B} S_{AB} = 0 \,, \qquad \mathcal{H}^{AB} S_{AB} = 0 \,.$$

Local inertial frame and Double-vielbein

- $\mathcal{J}_{AB}$ and $\mathcal{H}_{AB}$ can be simultaneously diagonalized,

$$\mathcal{J} = \begin{pmatrix} V & \bar{V} \end{pmatrix} \begin{pmatrix} \eta^{-1} & 0 \\ 0 & -\bar{\eta} \end{pmatrix} \begin{pmatrix} V & \bar{V} \end{pmatrix}^t ,$$

$$\mathcal{H} = \begin{pmatrix} V & \bar{V} \end{pmatrix} \begin{pmatrix} \eta^{-1} & 0 \\ 0 & \bar{\eta} \end{pmatrix} \begin{pmatrix} V & \bar{V} \end{pmatrix}^t .$$

  Here η and $\bar{\eta}$ are two copies of the D-dimensional Minkowskian metric. Both V and $\bar{V}$ are $2D \times D$ matrices which we name '*double-vielbein*'.

- They must satisfy

$$\begin{array}{lll} V = PV\,, & V\eta^{-1}V^t = P\,, & V^t\mathcal{J}V = \eta\,, \quad V^t\mathcal{J}\bar{V} = 0\,, \\ \bar{V} = \bar{P}\bar{V}\,, & \bar{V}\bar{\eta}\,\bar{V}^t = -\bar{P}\,, & \bar{V}^t\mathcal{J}\bar{V} = -\bar{\eta}^{-1}\,. \end{array}$$

- There are two copies of independent vielbein(Siegel, Tseytlin)

Local inertial frame and Double-vielbein

- Our dobule-vielbein is of the general form,

$$V_{Am} = \frac{1}{\sqrt{2}} \begin{pmatrix} e_m{}^{\mu} \\ \\ (B+e)_{\nu m} \end{pmatrix}, \quad \bar{V}_A{}^{\bar{n}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{e}^{\bar{n}\mu} \\ \\ (\bar{B}-\bar{e})_{\nu}{}^{\bar{n}} \end{pmatrix},$$

where $B_{\nu m} = B_{\nu\lambda} e^{\lambda}{}_m$ and $\bar{B}_{\nu}{}^{\bar{n}} = B_{\nu\lambda}\bar{e}^{\lambda\bar{n}}$.

  - Here, $e_{\mu}{}^{m}$ and $\bar{e}_{\nu}{}^{\bar{n}}$ are two copies of the D-dimensional vielbein corresponding to the same spacetime metric,

$$e_{\mu}{}^{m} e_{\nu m} = \bar{e}_{\mu}{}^{\bar{n}} \bar{e}_{\nu\bar{n}} = g_{\mu\nu} \, .$$

  - We may identify $(B+e)_{\mu}{}^{m}$ and $(\bar{B}-\bar{e})_{\nu}{}^{\bar{n}}$ as two copies of the vielbein for the winding mode coordinate, $\tilde{x}_{\mu}$, since

$$(B+e)_{\mu}{}^{m}(B+e)_{\nu m} = (\bar{B}-\bar{e})_{\mu}{}^{\bar{n}}(\bar{B}-\bar{e})_{\nu\bar{n}} = (g - Bg^{-1}B)_{\mu\nu} \, .$$

Local inertial frame and Double vielbein

- Internal symmetry group is

$$\mathbf{SO}(1,D-1)\times\overline{\mathbf{SO}}(D-1,1)\,,$$

- Taking single diagonal local Lorentz group $\mathbf{SO}(1,D-1)$ or $\overline{\mathbf{SO}}(D-1,1)$ by gauge fixing corresponds to

$$V_{Am}=\frac{1}{\sqrt{2}}\begin{pmatrix} e_m{}^{\mu} \\ (B+e)_{\nu m}\end{pmatrix},\quad \bar{U}_A{}^m:=\frac{1}{\sqrt{2}}\begin{pmatrix} (e^{-1})^{m\mu} \\ (B-e)_{\nu}{}^{m}\end{pmatrix},$$

or

$$U_{A\bar{m}}=\frac{1}{\sqrt{2}}\begin{pmatrix} \bar{e}_m{}^{\mu} \\ (B+\bar{e})_{\nu m}\end{pmatrix},\quad \bar{V}_A{}^{\bar{m}}:=\frac{1}{\sqrt{2}}\begin{pmatrix} (\bar{e}^{-1})^{\bar{m}\mu} \\ (B-\bar{e})_{\nu}{}^{\bar{m}}\end{pmatrix}.$$

where we define “twins" of the double-vielbein, by exchanging $e_{\mu m}$ and $\bar{e}_{\mu\bar{m}}$ in V and $\bar{V}$.

- While V and $\bar{V}$ are $\mathbf{O}(D,D)$ covariant, the twins are not $\mathbf{O}(D,D)$ covariant.

Double spin connection

- Gauging each diagonal local Lorentz symmetry, we get *doubled spin connection*

$$\Omega_{Amn} = \bar{P}_A{}^B V_{Cm} \nabla_B V^C{}_n - P_A{}^B \bar{U}_{Cm} \nabla_B \bar{U}^C{}_n \,.$$

or

$$\bar{\Omega}_{A\bar{m}\bar{n}} \quad = \bar{P}_A{}^B U_{C\bar{m}} \nabla_B U^C{}_{\bar{n}} - P_A{}^B \bar{V}_{C\bar{m}} \nabla_B \bar{V}^C{}_{\bar{n}} \,.$$

- Upon the level matching constraint, they are expressed in terms of D-dimensional notation,

$$\Omega_A \equiv \begin{pmatrix} -\frac{1}{2} H^\mu \\ \\ \omega_\nu - \frac{1}{2} B_{\nu\rho} H^\rho \end{pmatrix} , \quad \bar{\Omega}_A \equiv \begin{pmatrix} -\frac{1}{2} \bar{H}^\mu \\ \\ \bar{\omega}_\nu - \frac{1}{2} B_{\nu\rho} \bar{H}^\rho \end{pmatrix} ,$$

Pull back to D-dimensional theory

- Double-vielbein can pull back the chiral and the anti-chiral $2D$ indices to the more familiar D-dimensional ones

- We pull back the double-gauge covariant rank two-tensor to obtain,

$$S_{AB}V^{A}{}_{m}\bar{V}^{B}{}_{\bar{n}} \equiv R_{m\bar{n}} + 2D_m D_{\bar{n}}\phi - \tfrac{1}{4}H_{m\mu\nu}H_{\bar{n}}{}^{\mu\nu} + (\partial^{\lambda}\phi)H_{\lambda m\bar{n}} - \tfrac{1}{2}\nabla^{\lambda}H_{\lambda m\bar{n}}\,.$$

- As expected, its symmetric and the anti-symmetric parts correspond to the equations of motion of the effective action for $g_{\mu\nu}$ and $B_{\mu\nu}$ respectively.

- Pullback of various covariant quantities

$$V^A{}_l \mathcal{D}_A T_{\bar{k}_1 \bar{k}_2 \cdots \bar{k}_n} \equiv \frac{1}{\sqrt{2}} \hat{D}_l T_{\bar{k}_1 \bar{k}_2 \cdots \bar{k}_n} \ ,$$

$$\bar{V}^A{}_{\bar{l}} \mathcal{D}_A T_{k_1 k_2 \cdots k_n} \equiv \frac{1}{\sqrt{2}} \hat{D}_{\bar{l}} T_{k_1 k_2 \cdots k_n} \ ,$$

$$P^{AB} \mathcal{D}_A T_{B \bar{k}_1 \cdots \bar{k}_n} \equiv \frac{1}{\sqrt{2}} \hat{D}^l T_{l \bar{k}_1 \cdots \bar{k}_n} - \sqrt{2} \hat{D}^l \phi \, T_{l \bar{k}_1 \cdots \bar{k}_n} \ ,$$

$$\bar{P}^{AB} \mathcal{D}_A T_{B k_1 \cdots k_n} \equiv -\frac{1}{\sqrt{2}} \hat{D}^{\bar{l}} T_{\bar{l} k_1 \cdots k_n} + \sqrt{2} \hat{D}^{\bar{l}} \phi \, T_{\bar{l} k_1 \cdots k_n} \ ,$$

$$P^{AB} \mathcal{D}_A \mathcal{D}_B T_{\bar{k}_1 \cdots \bar{k}_n} \equiv \frac{1}{2} \hat{D}^\mu \hat{D}_\mu T_{\bar{k}_1 \cdots \bar{k}_n} - \hat{D}^\mu \phi \, \hat{D}_\mu T_{\bar{k}_1 \cdots \bar{k}_n} \ ,$$

$$\bar{P}^{AB} \mathcal{D}_A \mathcal{D}_B T_{k_1 \cdots k_n} \equiv -\frac{1}{2} \hat{D}^\mu \hat{D}_\mu T_{k_1 \cdots k_n} + \hat{D}^\mu \phi \, \hat{D}_\mu T_{k_1 \cdots k_n} \ ,$$

where we put, as for D-dimensional tensors,

$$\begin{aligned}
T_{k_1 k_2 \cdots k_n} &= T_{A_1 A_2 \cdots A_n} V^{A_1}{}_{k_1} V^{A_2}{}_{k_2} \cdots V^{A_n}{}_{k_n} \ , \\
T_{\bar{k}_1 \bar{k}_2 \cdots \bar{k}_n} &= T_{A_1 A_2 \cdots A_n} \bar{V}^{A_1}{}_{\bar{k}_1} \bar{V}^{A_2}{}_{\bar{k}_2} \cdots \bar{V}^{A_n}{}_{\bar{k}_n} \ , \\
T_{\bar{l} k_1 \cdots k_n} &= T_{B A_1 \cdots A_n} \bar{V}^B{}_{\bar{l}} V^{A_1}{}_{k_1} \cdots V^{A_n}{}_{k_n} \ , \\
T_{l \bar{k}_1 \cdots \bar{k}_n} &= T_{B A_1 \cdots A_n} V^B{}_l \bar{V}^{A_1}{}_{\bar{k}_1} \cdots \bar{V}^{A_n}{}_{\bar{k}_n} \ ,
\end{aligned}$$

which are $\mathbf{O}(D,D)$ singlets, and we set, for $\hat{D}_m = (e^{-1})_m{}^\mu \hat{D}_\mu$ and $\hat{D}_{\bar{n}} = (\bar{e}^{-1})_{\bar{n}}{}^\mu \hat{D}_\mu$,

$$\hat{D}_\mu := \nabla_\mu + (\omega_\mu + \tfrac{1}{2} H_\mu) + (\bar{\omega}_\mu - \tfrac{1}{2} \bar{H}_\mu) .$$

Symmetry structure

- Symmetry structure for the double field theory
  - $\mathbf{O}(D,D)$ 'T-duality'
  - Double-gauge symmetry $\begin{cases} \text{Diffeomorphism} \\ \text{One-form gauge symmetry for } B_{\mu\nu} \end{cases}$
- Commutator between $so(D,D)$ and generalized Lie derivative does not generate any symmetry of double field theory.

$$[\delta_h, \hat{\mathcal{L}}_X] = \hat{\mathcal{L}}_Y \,, \quad \partial^C Y^A \partial_C T_{B_1 B_2 \cdots B_n} \neq 0 \,.$$

- $O(D,D)$ transformation rotates the entire hyperplane on which DFT lives.

Application to Yang-Mills

- Symmetry structure for the double field theory
  - $\mathbf{O}(D,D)$ 'T-duality'
  - Double-gauge symmetry $\begin{cases} \text{Diffeomorphism} \\ \text{One-form gauge symmetry for } B_{\mu\nu} \end{cases}$
  - two copies of local Lorentz symmetry

Application to Yang-Mills

- Symmetry structure for the double field theory
  - $\mathbf{O}(D,D)$ 'T-duality'
  - Double-gauge symmetry $\begin{cases} \text{Diffeomorphism} \\ \text{One-form gauge symmetry for } B_{\mu\nu} \end{cases}$
  - Yang-Mills gauge symmetry

- Apply the formalism to couple the non-Abelian Yang-Mills gauge field to the DFT action.

Application to Yang-Mills

- We postulate a vector potential, $\mathcal{V}_A$, which
  - is $\mathbf{O}(D,D)$ and double-gauge covariant,
  - and transforms under non-Abelian gauge group, $\mathbf{g} \in \mathbf{G}$,

$$\mathcal{V}_A \quad \longrightarrow \quad \mathbf{g}\mathcal{V}_A\mathbf{g}^{-1} - i(\partial_A\mathbf{g})\mathbf{g}^{-1} \, .$$

Application to Yang-Mills

- The usual field strength,

$$F_{AB} = \partial_A \mathcal{V}_B - \partial_B \mathcal{V}_A - i\left[\mathcal{V}_A, \mathcal{V}_B\right] \ ,$$

is YM gauge covariant, but it is NOT double-gauge covariant,

$$\delta_X F_{AB} \neq \hat{\mathcal{L}}_X F_{AB} \ .$$

Application to Yang-Mills

- Instead, we consider with the semi-covariant derivative,

$$\mathcal{F}_{AB} := \nabla_A \mathcal{V}_B - \nabla_B \mathcal{V}_A - i\,[\mathcal{V}_A, \mathcal{V}_B] = F_{AB} - \Gamma^C{}_{AB}\mathcal{V}_C\,.$$

- While this is neither YM gauge nor double-gauge covariant,

$$\mathcal{F}_{AB} \quad \longrightarrow \quad \mathbf{g}\mathcal{F}_{AB}\mathbf{g}^{-1} + i\Gamma^C{}_{AB}(\partial_C \mathbf{g})\mathbf{g}^{-1}\,,$$

$$\delta_X \mathcal{F}_{AB} \neq \hat{\mathcal{L}}_X \mathcal{F}_{AB}\,,$$

- if projected properly, it can be covariant up to level matching condtion,

$$P_A{}^C \bar{P}_B{}^D \mathcal{F}_{CD} \longrightarrow P_A{}^C \bar{P}_B{}^D \mathbf{g}\mathcal{F}_{CD}\mathbf{g}^{-1}\,,$$

$$\delta_X(P_A{}^C \bar{P}_B{}^D \mathcal{F}_{CD}) = \hat{\mathcal{L}}_X(P_A{}^C \bar{P}_B{}^D \mathcal{F}_{CD})\,.$$

Application to Yang-Mills

$P_A{}^C \bar{P}_B{}^D \mathcal{F}_{CD}$ is DFT field strength which is fully covariant with respect to

- $\mathbf{O}(D, D)$ T-duality
- Gauge symmetry
  - Double gauge = Diffeomorphism + one form gauge symmetry
  - Yang-Mills gauge

Yang-Mills action

- Our double field formulation of Yang-Mills action is

$$S_{\mathrm{YM}} = g_{\mathrm{YM}}^{-2} \int \mathrm{d}y^{2D}\, e^{-2d}\, \mathrm{Tr}\Big(P^{AB} \bar{P}^{CD} \mathcal{F}_{AC} \mathcal{F}_{BD} \Big) \ ,$$

- Manifestly this action is invariant under $O(D,D)$ T-duality, double-gauge and Yang-Mills gauge transformation.

- Corresponding D-dimensional action of the Double field YM action?

Yang-Mills in components

- Decompose the vector potential into chiral and anti-chiral ones,

$$\mathcal{V}_A = V_A^+ + V_A^- \,,$$

$$V_A^+ = P_A{}^B \mathcal{V}_B \,, \qquad V_A^- = \bar{P}_A{}^B \mathcal{V}_B \,.$$

- Their general forms are

$$V_A^+ = \tfrac{1}{2} \begin{pmatrix} A^{+\lambda} \\ \\ (g+B)_{\mu\nu} A^{+\nu} \end{pmatrix} , \qquad V_A^- = \tfrac{1}{2} \begin{pmatrix} -A^{-\lambda} \\ \\ (g-B)_{\mu\nu} A^{-\nu} \end{pmatrix} .$$

Yang-Mills in components

- With the field redefinition,

$$A_\mu := \frac{1}{2}(A_\mu^+ + A_\mu^-)\,, \qquad \phi_\mu := \frac{1}{2}(A_\mu^+ - A_\mu^-)\,,$$

  we get a general form of double guage field

$$\mathcal{V}_A = \begin{pmatrix} \phi^\lambda \\ A_\mu + B_{\mu\nu}\phi^\nu \end{pmatrix}.$$

- A_μ and ϕ_ν will be YM gauge connection and YM gauge covariant one-form respectively.

Yang-Mills in components

- Turning off the $\tilde{x}$-dependence reduces the action to

$$S_{\rm YM} \equiv g_{\rm YM}^{-2} \int {\rm d}x^D \, \sqrt{-g} e^{-2\phi} \, {\rm Tr}\Big(-\tfrac{1}{4} \hat{f}^{\mu\nu} \hat{f}_{\mu\nu} \Big) \ ,$$

where

$$\hat{f}_{\mu\nu} := f_{\mu\nu} - D_\mu \phi_\nu - D_\nu \phi_\mu + i \, [\phi_\mu, \phi_\nu] + H_{\mu\nu\lambda} \phi^\lambda \ ,$$

and

$$\begin{aligned} {\rm Tr}\Big(\hat{f}_{\mu\nu} \hat{f}^{\mu\nu} \Big) = \ & {\rm Tr}\Big(f_{\mu\nu} f^{\mu\nu} + 2 D_\mu \phi_\nu D^\mu \phi^\nu + 2 D_\mu \phi_\nu D^\nu \phi^\mu - [\phi_\mu, \phi_\nu][\phi^\mu, \phi^\nu] \\ & + 2i f_{\mu\nu} [\phi^\mu, \phi^\nu] + 2 \left(f^{\mu\nu} + i [\phi^\mu, \phi^\nu] \right) H_{\mu\nu\sigma} \phi^\sigma + H_{\mu\nu\sigma} H^{\mu\nu}{}_\tau \phi^\sigma \phi^\tau \ \Big) \ . \end{aligned}$$

- For T-duality, we need YM gauge covariant 1-form field.
- Similar to topologically twisted Yang-Mills, but differs in detail.
- Curved D-branes are known to convert adjoint scalars into one-form,

$$\phi^a \to \phi_\mu, \qquad \text{Bershadsky}$$

More on fully covariant quantities

- Even power of the field strength, $\hat{\mathcal{F}}^{AB} := P_A{}^C \bar{P}_B{}^D \mathcal{F}_{CD}$,

$$\mathrm{Tr}\left(\hat{\mathcal{F}}^{A_1B_1}\hat{\mathcal{F}}_{A_2B_1}\hat{\mathcal{F}}^{A_2B_2}\hat{\mathcal{F}}_{A_3B_2}\cdots\hat{\mathcal{F}}^{A_nB_n}\hat{\mathcal{F}}_{A_1B_n}\right) .$$

- For the Abelian group, DBI type action

$$\det\left(\eta_{AB} + \kappa\,\hat{\mathcal{F}}_{AC}\hat{\mathcal{F}}_B{}^C\right) = \det\left(\eta_{AB} + \kappa\,\hat{\mathcal{F}}_{CA}\hat{\mathcal{F}}^C{}_B\right) ,$$

  κ is a constant.
  No square root is necessary since this is a scalar, not a density.

Concluding remarks

- $\mathbf{O}(D,D)$ T-duality, diffeomorphism, one-form gauge symmetry fixes the low energy effective action,

$$S_{\text{eff.}} = \int \mathrm{d}x^D \, e^{-2d} \, \mathcal{H}^{AB} S_{AB} \, .$$

- Non-Abelian Yang-Mills field is incorporated in DFT formulation.

Concluding remarks

- Yet, string theory interpretation of the YM theory is not clear.
- D-brane in DFT Albertsson, Dai, Kao, Lin
- Fermion in DFT Coimbra, Strickland-Constable, Waldram
- Extension to RR fields Hull, Kwak, Zwiebach
- Application to 'doubled sigma model' and generalization to $\mathcal{M}$-theory are of interest Hull, Berman, Perry, ... etc...

- Concluding:
  Perhaps, our formalism may provide some clue to a new framework for string theory, beyond Riemann.

Concluding remarks

- Yet, string theory interpretation of the YM theory is not clear.
- D-brane in DFT Albertsson, Dai, Kao, Lin
- Fermion in DFT Coimbra, Strickland-Constable, Waldram
- Extension to RR fields Hull, Kwak, Zwiebach
- Application to 'doubled sigma model' and generalization to $\mathcal{M}$-theory are of interest Hull, Berman, Perry, ... etc...

- Concluding:
  Perhaps, our formalism may provide some clue to a new framework for string theory, beyond Riemann.

Thank you.