Stringy Differential Geometry, beyond Riemann

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- **Differential geometry with a projection: Application to double field theory**
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- **Double field formulation of Yang-Mills theory**

- **Stringy differential geometry, beyond Riemann**
  arXiv: 1105.6294
Introduction

• In Riemannian geometry, the fundamental object is the metric, $g_{\mu\nu}$.

• String theory puts $g_{\mu\nu}$, $B_{\mu\nu}$ and $\phi$ on an equal footing.

• This may suggests the existence of a veiled unifying description of them, beyond Riemann.
Introduction

Symmetry

- guides the structure of Lagrangians and organizes the physical laws into simple forms.

- for example, in Maxwell theory,
  - U(1) gauge symmetry forbids $m^2A_\mu A^\mu$
  - Lorentz symmetry unifies the original 4 eqs into 2.
• Essence of Riemannian geometry: Diffeomorphism
  • $\partial_\mu \longrightarrow \nabla_\mu = \partial_\mu + \Gamma_\mu$
  • $\nabla_\lambda g_{\mu\nu} = 0 \longrightarrow \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$
  • Curvature, $[\nabla_\mu, \nabla_\nu], \longrightarrow R_g$.
• Main purpose: Generalization to a geometry for stringy theory.
• Key property in string theory: T-duality
Outline

- T-duality in string theory
- String effective action and Double Field Theory
- *Stringy differential geometry* as underlying geometry of DFT.
  - Local inertial frame
- Non-Abelian YM gauge field in DFT.
Closed string on a Toroidal background

- For closed string wrapping around a circle, the modes expansion is

\[ X = X_L + X_R , \]

\[ X_L(\sigma^+) = \frac{1}{2}(x + \tilde{x}) + \frac{1}{2}(p + w)\sigma^+ + \cdots , \]

\[ X_R(\sigma^-) = \frac{1}{2}(x - \tilde{x}) + \frac{1}{2}(p - w)\sigma^- + \cdots . \]

where

\[ \sigma^\pm := \tau \pm \sigma , \quad p : \text{momentum mode} , \quad w : \text{winding mode} , \]

\[ x : \text{center of mass} , \quad \tilde{x} \text{ is introduced} . \]

- Physical condition (Virasoro constraint) implies the level matching and the on-shell condition,

\[ L_0 - \bar{L}_0 = 0 \quad \iff \quad N - \bar{N} - p \cdot w = 0 \]

\[ L_0 + \bar{L}_0 = 2 \quad \iff \quad \mathcal{M}^2 = p^2 + w^2 + 2(N + \bar{N} - 2) \]
T-duality

- T-duality transformation is

\[ X_L^\mu + X_R^\mu \longrightarrow X_L^\mu - X_R^\mu , \]

such that the two pairs, \((x, p)\) and \((\tilde{x}, w)\), are exchanged by each other,

\[ (x, \tilde{x}, p, w) \longrightarrow (\tilde{x}, x, w, p), \]

Then, the level matching condition and the mass spectrum is invariant.

- On \(d\)-dimensional toroidal background, the T-duality is realized as \(O(d, d, Z)\) transformation preserving \(\mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{2d \times 2d} \).
• $x^i$: Conjugate to the momenta $p_i$

$\tilde{x}_i$: Conjugate to the winding number $w^i$

It is natural to introduce the additional coordinate $\tilde{x}_i$

• Closed string field theory treats momenta and winding modes symmetrically. So, target spacetime fields are also depend on both $x$ and $\tilde{x}$: $\Phi(x^i, \tilde{x}_i)$ [Kugo, Zwiebach 1992]
Effective theory

- Describes a gravity: $g_{\mu\nu}, B_{\mu\nu}, \phi$ are on an equal footing completing the massless sector.
- Low energy effective action of them:

$$S_{\text{eff.}} = \int dx^D \sqrt{-g} e^{-2\phi} \left( R_g + 4 \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \right)$$

- Diffeomorphism and one-form gauge symmetry are manifest

$$x^\mu \rightarrow x^\mu + \delta x^\mu, \quad B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu.$$
Effective theory

• Describes a gravity: $g_{\mu\nu}, B_{\mu\nu}, \phi$ are on an equal footing completing the massless sector.

• Low energy effective action of them:

$$S_{\text{eff.}} = \int \! d^D x \sqrt{-g} e^{-2\phi} \left( R_g + 4 \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \right)$$

• Though not manifest, if there is isometry, this enjoys T-duality which mixes \{ $g_{\mu\nu}, B_{\mu\nu}, \phi$ \} (Buscher)

• Include the $\tilde{x}$ dependence. $\rightarrow$ double field theory
Double field theory (DFT)

- Hull and Zwiebach, later with Hohm constructed action which has explicit T-duality,

\[ S_{\text{DFT}} = \int dy^{2D} e^{-2d} \left[ \mathcal{H}^{AB} \left( 4\partial_A \partial_B d - 4\partial_A d \partial_B d + \frac{1}{8} \partial_A \mathcal{H}^{CD} \partial_B \mathcal{H}_{CD} \right) 
- \frac{1}{2} \partial_A \mathcal{H}^{CD} \partial_C \mathcal{H}_{BD} \right] + 4\partial_A \mathcal{H}^{AB} \partial_B d - \partial_A \partial_B \mathcal{H}^{AB} \right]. \]

- \( d \) is double field theory 'dilaton' given by

\[ e^{-2d} = \sqrt{-g} e^{-2\phi} \]

and \( \mathcal{H}_{AB} \) is \( 2D \times 2D \) matrix, 'generalized metric'

\[ \mathcal{H}_{AB} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\kappa} B_{\kappa\sigma} \\ B_{\rho\kappa} g^{\kappa\nu} & g_{\rho\sigma} - B_{\rho\kappa} g^{\kappa\lambda} B_{\lambda\sigma} \end{pmatrix}. \]

- Spacetime dimension is 'formally' doubled from \( D \) to \( 2D \),

\[ x^\mu \rightarrow y^A = (\tilde{x}_\mu, x^\nu), \quad \partial_\mu \rightarrow \partial_A = (\tilde{\partial}^\mu, \partial_\nu). \]
Double field theory (DFT)

• Indices, $A, B, C, \cdots$, are $2D$-dimensional vector indices, which can be lowered or raised by $O(D, D)$ metric $J_{AB}$.

\[ J_{AB} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \]

• Under global $O(D, D)$ rotation, $L \in O(D, D)$,

\[ \mathcal{H}_{AB}(y) \longrightarrow L_A^C L_B^D \mathcal{H}_{CD}(y') , \quad d(y) \longrightarrow d(y') , \]

manifest that DFT action is invariant.

• Note that $O(D, D)$ T-duality is background independent.
Double field theory (DFT)

- By ’Level matching condition,’

  DFT action \( \equiv D \)-dimensional closed string effective action

- It possesses a gauge symmetry, say *double gauge symmetry*, through ’generalized Lie derivative’;

  double gauge symmetry = diffeomorphism + 1-form gauge symmetry
Level matching constraint

• Level matching condition for the massless sector,

\[ p \cdot w \equiv 0 \iff \frac{\partial^2}{\partial \tilde{x}_\mu \partial x^\mu} = \partial_A \partial^A \equiv 0. \]

• Strong constraint: all the fields and the gauge parameters as well as all of their products should be annihilated by \( \partial^2 = \partial_A \partial^A \)

\[ \partial^2 \Phi \equiv 0, \quad \partial_A \Phi_1 \partial^A \Phi_2 \equiv 0 \iff \tilde{\partial} \equiv 0. \]

• DFT is restricted in the strong constraint.

• Actually meaning that theory is not truly doubled: we can impose \( \tilde{\partial} \equiv 0 \) by \( O(D, D) \) rotation.

• DFT is reorganization of effective action: Upon the level matching constraint,

\[ S_{\text{DFT}} \implies S_{\text{eff.}} = \int dx^D \sqrt{-g} e^{-2\phi} \left( R_g + 4(\partial \phi)^2 - \frac{1}{12} H^2 \right). \]
Double gauge symmetry

• Introducing an unifying doubled gauge parameter,

\[ X^A = (\Lambda_\mu, \delta x^\nu) \]

• Diffeomorphism and 1-form gauge transformation is expressed in unified fashion, upon the level matching constraint,

\[ \delta_X \mathcal{H}_{AB} \equiv X^C \partial_C \mathcal{H}_{AB} + 2\partial_{[A} X_C] \mathcal{H}^C_{B} + 2\partial_{[B} X_C] \mathcal{H}^C_A , \]

\[ \delta_X (e^{-2d}) \equiv \partial_A (X^A e^{-2d}) . \]

• In fact, these coincide with the generalized Lie derivative,

\[ \delta_X \mathcal{H}_{AB} = \hat{\mathcal{L}}_X \mathcal{H}_{AB} , \quad \delta_X (e^{-2d}) = \hat{\mathcal{L}}_X (e^{-2d}) = -2(\hat{\mathcal{L}}_X d)e^{-2d} . \]
Generalized Lie derivative

- Definition, Siegel, Courant, Grana ...

\[ \hat{\mathcal{L}}_X T_{A_1 \ldots A_n} := X^B \partial_B T_{A_1 \ldots A_n} + \omega \partial_B X^B T_{A_1 \ldots A_n} + \sum_{i=1}^n 2\partial_{[A_i} X_{B]} T_{A_1 \ldots A_{i-1} B A_{i+1} \ldots A_n}. \]

- cf. ordinary one,

\[ \mathcal{L}_X T_{A_1 \ldots A_n} := X^B \partial_B T_{A_1 \ldots A_n} + \omega \partial_B X^B T_{A_1 \ldots A_n} + \sum_{i=1}^n \partial_{A_i} X_B T_{A_1 \ldots A_{i-1} B A_{i+1} \ldots A_n}. \]

- Definition of tensor(density) in DFT: 'double gauge covariant' quantity

\[ \delta_X T_{A_1 A_2 \ldots A_n} = \hat{\mathcal{L}}_X T_{A_1 A_2 \ldots A_n}. \]

- \( \mathcal{H}_{AB} \): rank 2 generalized tensor, \( e^{-2d} \): weight 1 generalized scalar,
Commutator of generalized Lie derivatives is closed, up to the level matching condition, by using \( c \)-bracket,

\[
[\hat{\mathcal{L}}_X, \hat{\mathcal{L}}_Y] \equiv \hat{\mathcal{L}}_{[X,Y]_C},
\]

where \([X, Y]_C\) denotes \( C \)-bracket

\[
[X, Y]_C^A := X^B \partial_B Y^A - Y^B \partial_B X^A + \frac{1}{2} Y^B \partial^A X_B - \frac{1}{2} X^B \partial^A Y_B,
\]

known to be equivalent the Courant bracket if dropping \( \tilde{\partial} \).
Diffeomorphism & one-form gauge symmetry

- Direct computation shows that
  \[ \hat{L}_X \mathcal{H}_{AB} \equiv X^C \partial_C \mathcal{H}_{AB} + 2 \partial_{[A} X_{C]} \mathcal{H}_{B}^C + 2 \partial_{[B} X_{C]} \mathcal{H}_{A}^C , \]
  \[ \hat{L}_X (e^{-2d}) \equiv \partial_A (X^A e^{-2d}) , \]
  are symmetry of DFT action by Hull, Zwiebach and Hohm,
  \[ S_{\text{DFT}} = \int \text{d}y^{2D} \ e^{-2d} \ \mathcal{R}(\mathcal{H}, d) , \]
  where 'generalized curvature' is
  \[ \mathcal{R}(\mathcal{H}, d) = \mathcal{H}^{AB} \left( 4 \partial_A \partial_B d - 4 \partial_A d \partial_B d + \frac{1}{8} \partial_A \mathcal{H}^{CD} \partial_B \mathcal{H}_{CD} - \frac{1}{2} \partial_A \mathcal{H}^{CD} \partial_C \mathcal{H}_{BD} \right) \]
  \[ + 4 \partial_A \mathcal{H}^{AB} \partial_B d - \partial_A \partial_B \mathcal{H}^{AB} . \]

- This expression may be analogous to the case of writing the scalar curvature, \( R_g \), in terms of the metric and its derivative.
Diffeomorphism & one-form gauge symmetry

- Direct computation shows that

\[ \hat{L}_X \mathcal{H}_{AB} \equiv X^C \partial_C \mathcal{H}_{AB} + 2 \partial_{[A} X_{C]} \mathcal{H}^C_B + 2 \partial_{[B} X_{C]} \mathcal{H}^C_A , \]

\[ \hat{L}_X (e^{-2d}) \equiv \partial_A (X^A e^{-2d}) , \]

are symmetry of the action by Hull, Zwiebach and Hohm,

\[ S_{DFT} = \int dy^{2D} e^{-2d} \mathcal{R} (\mathcal{H}, d) , \]

where

\[ \mathcal{R} (\mathcal{H}, d) = \mathcal{H}^{AB} (4 \partial_A \partial_B d - 4 \partial_A d \partial_B d + \frac{1}{8} \partial_A \mathcal{H}^{CD} \partial_B \mathcal{H}_{CD} - \frac{1}{2} \partial_A \mathcal{H}^{CD} \partial_C \mathcal{H}_{BD} ) + 4 \partial_A \mathcal{H}^{AB} \partial_B d - \partial_A \partial_B \mathcal{H}^{AB} . \]

- What is the underlying geometry?
Diffeomorphism & one-form gauge symmetry

• Direct computation shows that

\[ \hat{L}_X H_{AB} \equiv X^C \partial_C H_{AB} + 2\partial_{[A} X_{C]} H^C_B + 2\partial_{[B} X_{C]} H_A^C, \]

\[ \hat{L}_X (e^{-2d}) \equiv \partial_A (X^A e^{-2d}), \]

are symmetry of the action by Hull, Zwiebach and Hohm,

\[ S_{\text{DFT}} = \int dy^{2D} e^{-2d} R(H, d), \]

where

\[ R(H, d) = H^{AB} (4\partial_A \partial_B d - 4\partial_A d \partial_B d + \frac{1}{8} \partial_A H^{CD} \partial_B H_{CD} - \frac{1}{2} \partial_A H^{CD} \partial_C H_{BD}) \]

\[ + 4\partial_A H^{AB} \partial_B d - \partial_A \partial_B H^{AB}. \]

• connection and covariant derivative of the geometry?
Stringy differential geometry

We propose a novel differential geometry which

- enables us to rewrite the low energy effective action as a single term in a geometrical manner,

\[ S_{\text{eff.}} = \int dx^{2D} e^{-2d} \mathcal{H}^{AB} S_{AB} , \]

- treats the three objects of the massless sector in a unified way,

- manifests not only diffeomorphism and one-form gauge symmetry but also \( O(D, D) \) T-duality,
Stringy differential geometry

• Motivated by the observation that,

\[ \mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix} \]

is of the most general form to satisfy

\[ \mathcal{H}_A^C \mathcal{H}_C^B = \delta_A^B, \quad \mathcal{H}_{AB} = \mathcal{H}_{BA}, \]

and the upper left \( D \times D \) block of \( \mathcal{H} \) is non-degenerate,

• we focus on a symmetric projection,

\[ P_A^B P_B^C = P_A^C, \quad P_{AB} = P_{BA}, \]

which is related to \( \mathcal{H} \) by

\[ P_{AB} = \frac{1}{2} (\mathcal{J}_{AB} + \mathcal{H}_{AB}). \]

• Three basic objects:

\[ \mathcal{J}_{AB}, \quad P_{AB}, \quad d. \]
Stringy differential geometry

- We postulate a “semi-covariant” derivative, $\nabla_A$,

$$\nabla_C T_{A_1A_2\ldots A_n} = \partial_C T_{A_1A_2\ldots A_n} - \omega \Gamma^B_{BC} T_{A_1A_2\ldots A_n} + \sum_{i=1}^n \Gamma_{CA_i}^B T_{A_1\ldots A_{i-1}BA_{i+1}\ldots A_n}.$$

- In particular,

$$\nabla_C (e^{-2d}) = \partial_C e^{-2d} - \Gamma^B_{BC} e^{-2d} = -2(\nabla_C d)e^{-2d}$$

$$\implies \nabla_C d := \partial_C d + \frac{1}{2} \Gamma^B_{BC}$$
Stringy differential geometry

- We demand the following compatibility conditions,

\[ \nabla_A J_{BC} = 0, \quad \nabla_A P_{BC} = 0, \quad \nabla_A d = 0, \]

as for the unifying description of the massless modes
(cf. \( \nabla_\lambda g_{\mu\nu} = 0 \) in Riemannian geometry).

- Further we require,

\[ \Gamma_{CAB} + \Gamma_{CBA} = 0, \quad \Gamma_{ABC} + \Gamma_{CAB} + \Gamma_{BCA} = 0. \]
Stringy differential geometry

- Then, we may replace $\partial_A$ by $\nabla_A$ in $\hat{\mathcal{L}}_X$ and also in $[X, Y]^A_C$,

$$
\hat{\mathcal{L}}_X T_{A_1 \cdots A_n} = X^B \nabla_B T_{A_1 \cdots A_n} + \omega \nabla_B X^B T_{A_1 \cdots A_n} + \sum_{i=1}^n 2 \nabla_{[A_i X_B]} T_{A_1 \cdots A_{i-1} B \ A_{i+1} \cdots A_n} ,
$$

$$
[X, Y]^A_C = X^B \nabla_B Y^A - Y^B \nabla_B X^A + \frac{1}{2} Y^B \nabla^A X_B - \frac{1}{2} X^B \nabla^A Y_B .
$$

- cf. In Riemannian geometry, torsion free condition implies

$$
\mathcal{L}_X T_{\mu_1 \cdots \mu_n} = X^\nu \nabla_\nu T_{\mu_1 \cdots \mu_n} + \omega \nabla_\nu X^\nu T_{\mu_1 \cdots \mu_n} + \sum_{i=1}^n \nabla_{\mu_i} X^\nu T_{\mu_1 \cdots \mu_{i-1} \nu \mu_{i+1} \cdots \mu_n} ,
$$

$$
[X, Y]^\mu = X^\nu \nabla_\nu Y^\mu - Y^\nu \nabla_\nu X^\mu .
$$
Stringy differential geometry

• Explicitly, the connection is

\[ \Gamma_{CAB} = 2(P \partial_C P \bar{P})_{[AB]} + 2(\bar{P} [A \partial_D \bar{P}^E] - P [A \partial_D P^E]) \partial_D P_{EC} \]

\[ - \frac{4}{D-1} (\bar{P} C_{[A} \bar{P}^B] + P_{C[A} P_{B]}^D) (\partial_D d + (P \partial^E P \bar{P})_{[ED]}). \]

where \( \bar{P} \) is the complementary projection,

\[ \bar{P} = \frac{1}{2} (1 - \mathcal{H}) . \]

and \( P \) and \( \bar{P} \) are orthogonal,

\[ PP = \bar{P} \bar{P} = 0 . \]

• Is this derivative \( \nabla_A \) double gauge covariant?
• For usefulness, we set

\[ \mathcal{P}_{CAB}^{DEF} := P_C^D P_{[E} P_{B]}^F + \frac{2}{D-1} P_{C[A} P_B^{[E} P^{F]} D, \]

\[ \bar{\mathcal{P}}_{CAB}^{DEF} := \bar{P}_C^D \bar{P}_{[A} \bar{P}_{B]}^F + \frac{2}{D-1} \bar{P}_{C[A} \bar{P}_B^{[E} \bar{P}^{F]} D, \]

which satisfy

\[ \mathcal{P}_{CABDEF} = \mathcal{P}_{DEFCAB} = \mathcal{P}_{C[AB]D[EF]}, \]

\[ \mathcal{P}_{CAB}^{DEF} \mathcal{P}_{DEF}^{GHI} = \mathcal{P}_{CAB}^{GHI}, \]

\[ \mathcal{P}^{A}_{ABDEF} = 0, \quad \mathcal{P}^{AB} \mathcal{P}_{ABCDEF} = 0, \quad \text{etc.} \]

• The connection belongs to the kernel of these rank six-projectors,

\[ \mathcal{P}_{CAB}^{DEF} \Gamma_{DEF} = 0, \quad \bar{\mathcal{P}}_{CAB}^{DEF} \Gamma_{DEF} = 0. \]
Stringy differential geometry

• Under double-gauge transform, $\delta_X P_{AB} = \hat{\mathcal{L}}_X P_{AB}$ and $\delta_X d = \hat{\mathcal{L}}_X d$, the diffeomorphism and the one-form gauge transform, we obtain

\[
(\delta_X - \hat{\mathcal{L}}_X) \Gamma_{CAB} \equiv 2 \left[(\mathcal{P} + \bar{\mathcal{P}})_{CAB}^{FDE} - \delta_C^F \delta_A^D \delta_B^E \right] \partial_F \partial_{[D} X_{E]} ,
\]

and

\[
(\delta_X - \hat{\mathcal{L}}_X) \nabla_C T_{A_1 \cdots A_n} \equiv \sum_i 2(\mathcal{P} + \bar{\mathcal{P}})_{CA_i}^{BFDE} \partial_F \partial_{[D} X_{E]} T_{A_1 \cdots B \cdots A_n} .
\]

• Hence, these are not double-gauge covariant,

$$\delta_X \neq \hat{\mathcal{L}}_X .$$
However, the characteristic property of our derivative, $\nabla_A$, is that, combined with the projections, it can generate various $O(D, D)$ and double-gauge covariant quantities:

\[
P_C^D \bar{P}_{A_1}^{B_1} \bar{P}_{A_2}^{B_2} \cdots \bar{P}_{A_n}^{B_n} \nabla_D T_{B_1 B_2 \cdots B_n},
\]

\[
\bar{P}_C^D P_{A_1}^{B_1} P_{A_2}^{B_2} \cdots P_{A_n}^{B_n} \nabla_D T_{B_1 B_2 \cdots B_n},
\]

\[
P^{AB} \nabla_A T_B, \quad \bar{P}^{AB} \nabla_A T_B,
\]

\[
P^{AB} \bar{P}_{C_1}^{D_1} \cdots \bar{P}_{C_n}^{D_n} \nabla_A \nabla_B T_{D_1 \cdots D_n},
\]

\[
\bar{P}^{AB} P_{C_1}^{D_1} \cdots P_{C_n}^{D_n} \nabla_A \nabla_B T_{D_1 \cdots D_n}.
\]

This suggests us to call $\nabla_A$ as \textit{semi-covariant derivative}.
Curvature

- The usual curvature,

\[ R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}^E \Gamma_{BED} - \Gamma_{BC}^E \Gamma_{AED}, \]

satisfying

\[ [\nabla_A, \nabla_B] T_{C_1 C_2 \ldots C_n} = -\Gamma_{DAB} \nabla^D T_{C_1 C_2 \ldots C_n} + \sum_{i=1}^{n} R_{C_i DAB} T_{C_1 \ldots C_{i-1}^D C_{i+1} \ldots C_n}, \]

is NOT double-gauge covariant,

\[ \delta_X R_{ABCD} \neq \hat{\mathcal{L}}_X R_{ABCD}. \]

It satisfy \( R_{ABCD} = R_{[AB][CD]} \), but

\[ R_{ABCD} \neq R_{CDAB}. \]
Generalized curvature

• Instead, we define, as for a key quantity in our formalism,

\[ S_{ABCD} := \frac{1}{2} \left( \mathcal{R}_{ABCD} + \mathcal{R}_{CDAB} - \Gamma^E_{AB} \Gamma_{ECD} \right) . \]

• This can be read off from the commutaor,

\[ P_I^A \bar{P}_J^B [\nabla_A, \nabla_B] T_C \equiv 2 P_I^A \bar{P}_J^B S_{CDAB} T^D . \]

• It can be shown, by brute force computation, to satisfy
  • just like the Riemann curvature,

\[ S_{ABCD} = \frac{1}{2} (S_{[AB][CD]} + S_{[CD][AB]} ) \equiv S_{\{ABCD\}} , \quad S_{A[BCD]} = 0 , \]

• and further

\[ P_I^A P_J^B \bar{P}_K^C \bar{P}_L^D S_{ABCD} \equiv 0 , \quad P_I^A \bar{P}_J^B P_K^C \bar{P}_L^D S_{ABCD} \equiv 0 , \quad etc. \]
Generalized curvature

- $S_{ABCD}$ is not double-gauge covariant.
- Under the double-gauge transformations, we get

$$ (\delta_X - \hat{L}_X)S_{ABCD} \equiv 4 \nabla_{\{A} (\mathcal{P} + \bar{\mathcal{P}})_{\{BCD}^{EFG} \partial_E \partial_{[F}X_{G]} \} .$$

- Nevertheless, contracting indices we can obtain covariant quantities.
Covariant curvature

By setting

\[ S_{AB} = S_{BA} = S^{C}_{ACB} \]

which turns out to be traceless,

\[ S^A_A \equiv 0 \]

and contracting with projection operators, we get

- Double-gauge covariant rank two-tensor,

  \[ P_I^A \bar{P}_J^B S_{AB} \quad : \text{generalized curvature tensor} \]

- Double-gauge covariant scalar,

  \[ \mathcal{H}^{AB} S_{AB} \quad : \text{generalized curvature scalar} \]
Reproduction of DFT

- Natural DFT action is proposed by

\[
S_{\text{DFT}} = \int dy^2 e^{-2d} \mathcal{H}^{AB} S_{AB},
\]

- In fact, the covariant scalar constitutes the effective action as

\[
\mathcal{H}^{AB} S_{AB} \equiv R_g + 4\Box \phi - 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\lambda \mu \nu} H^{\lambda \mu \nu}.
\]

- It also agrees with Hull, Zwiebach and Hohm,

\[
\mathcal{H}^{AB} S_{AB} \equiv \mathcal{H}^{AB} \left(4\partial_A \partial_B d - 4\partial_A d \partial_B d + \frac{1}{8} \partial_A \mathcal{H}^{CD} \partial_B \mathcal{H}_{CD} - \frac{1}{2} \partial_A \mathcal{H}^{CD} \partial_C \mathcal{H}_{BD}\right) + 4\partial_A \mathcal{H}^{AB} \partial_B d - \partial_A \partial_B \mathcal{H}^{AB}.
\]
Deriving Equations of motion

• It is easy to rederive the equation of motion.

• Under arbitrary infinitesimal transformations of the dilaton and the projection, we get

\[ \delta S_{\text{eff.}} \equiv \int \text{d}y^{2D} 2e^{-2d} \left( \delta P^{AB} S_{AB} - \delta d \mathcal{H}^{AB} S_{AB} + \delta S_{AB} \right). \]

• The third term is total derivative as

\[ \delta S_{ABCD} = \nabla_{[A} \delta \Gamma_{B]CD} + \nabla_{[C} \delta \Gamma_{D]AB} , \quad \nabla_A d = 0 , \]

where explicitly

\[ \delta \Gamma_{CAB} = 2P_{[A}^D \bar{P}_{B]}^E \nabla_C \delta P_{DE} + 2(\bar{P}_{[A}^D \bar{P}_{B]}^E - P_{[A}^D P_{B]}^E) \nabla_D \delta P_{EC} \]

\[ - \frac{4}{D-1} (\bar{P}_{C[A}^D \bar{P}_{B]}^D + P_{C[A}^D P_{B]}^D)(\partial_D \delta d + P_{E[G} \nabla^G \delta P_{D]}^E) \]

\[ - \Gamma_{FDE} \delta (\mathcal{P} + \bar{\mathcal{P}})_{CAB}^{FDE} , \]
Deriving Equations of motion

- It is easy to rederive the equation of motion.
- Under arbitrary infinitesimal transformations of the dilaton and the projection, we get

\[ \delta S_{\text{eff.}} \equiv \int \, dy^{2D} \, 2e^{-2d} \left( \delta P^{AB} S_{AB} - \delta d \mathcal{H}^{AB} S_{AB} \right). \]

- from the relation

\[ \delta P = P \delta P \bar{P} + \bar{P} \delta P P, \]

the equations of motions are easily obtained:

\[ P (^A \bar{P} _J)^B S_{AB} = 0, \quad \mathcal{H}^{AB} S_{AB} = 0. \]
Local inertial frame and Double-vielbein

• $\mathcal{J}_{AB}$ and $\mathcal{H}_{AB}$ can be simultaneously diagonalized,

$$
\mathcal{J} = ( V \quad \bar{V} ) \left( \begin{array}{cc} \eta^{-1} & 0 \\ 0 & -\bar{\eta} \end{array} \right) ( V \quad \bar{V} )^t ,
$$

$$
\mathcal{H} = ( V \quad \bar{V} ) \left( \begin{array}{cc} \eta^{-1} & 0 \\ 0 & \bar{\eta} \end{array} \right) ( V \quad \bar{V} )^t .
$$

Here $\eta$ and $\bar{\eta}$ are two copies of the $D$-dimensional Minkowskian metric. Both $V$ and $\bar{V}$ are $2D\times D$ matrices which we name ‘double-vielbein’.

• They must satisfy

$$
V = P V , \quad V \eta^{-1} V^t = P , \quad V^t \mathcal{J} V = \eta , \quad V^t \mathcal{J} \bar{V} = 0 ,
$$

$$
\bar{V} = \bar{P} \bar{V} , \quad \bar{V} \bar{\eta} \bar{V}^t = -\bar{P} , \quad \bar{V}^t \mathcal{J} \bar{V} = -\bar{\eta}^{-1} .
$$

• There are two copies of independent vielbein(Siegel, Tseytlin )
Local inertial frame and Double-vielbein

- Our double-vielbein is of the general form,

\[ V_A^m = \frac{1}{\sqrt{2}} \begin{pmatrix} e^\mu_m \\ (B + e)_{\nu m} \end{pmatrix}, \quad \bar{V}^\bar{\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{e}^\bar{\nu} \mu \\ (\bar{B} - \bar{e})_{\nu \bar{\nu}} \end{pmatrix}, \]

where \( B_{\nu m} = B_{\nu \lambda} e^\lambda_m \) and \( \bar{B}_{\nu \bar{\nu}} = B_{\nu \lambda} \bar{e}^\lambda_{\bar{\nu}} \).

- Here, \( e^m_\mu \) and \( \bar{e}^\bar{\nu} \) are two copies of the \( D \)-dimensional vielbein corresponding to the same spacetime metric,

\[ e^m_\mu e_{\nu m} = \bar{e}^\bar{\nu} \bar{e}_{\nu \bar{\nu}} = g_{\mu \nu}. \]

- We may identify \( (B + e)_m^\mu \) and \( (\bar{B} - \bar{e})_{\nu \bar{\nu}} \) as two copies of the vielbein for the winding mode coordinate, \( \tilde{x}_\mu \), since

\[ (B + e)_m^\mu (B + e)_{\nu m} = (\bar{B} - \bar{e})_{\mu \bar{\nu}} (\bar{B} - \bar{e})_{\nu \bar{\nu}} = (g - B g^{-1} B)_{\mu \nu}. \]
Local inertial frame and Double vielbein

• Internal symmetry group is

\[ \text{SO}(1, D-1) \times \overline{\text{SO}}(D-1, 1) , \]

• Taking single diagonal local Lorentz group \( \text{SO}(1, D-1) \) or \( \overline{\text{SO}}(D-1, 1) \) by gauge fixing corresponds to

\[
V_{Am} = \frac{1}{\sqrt{2}} \begin{pmatrix} e_{m\mu}^\nu \\ (B + e)_{\nu m} \end{pmatrix}, \quad \bar{U}_{A}^{m} := \frac{1}{\sqrt{2}} \begin{pmatrix} (e^{-1})_{m\mu}^\nu \\ (B - e)_{\nu m} \end{pmatrix},
\]

or

\[
U_{A\bar{m}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{e}_{m\mu}^\nu \\ (B + \bar{e})_{\nu m} \end{pmatrix}, \quad \bar{V}_{A\bar{m}} := \frac{1}{\sqrt{2}} \begin{pmatrix} (\bar{e}^{-1})_{m\mu}^\nu \\ (B - \bar{e})_{\nu m} \end{pmatrix}.
\]

where we define “twins” of the double-vielbein, by exchanging \( e_{\mu m} \) and \( \bar{e}_{\mu \bar{m}} \) in \( V \) and \( \bar{V} \).

• While \( V \) and \( \bar{V} \) are \( \text{O}(D, D) \) covariant, the twins are not \( \text{O}(D, D) \) covariant.
Double spin connection

- Gauging each diagonal local Lorentz symmetry, we get *doubled spin connection*

  \[ \Omega_{Amn} = \bar{P}_A^B V_{Cm} \nabla_B V^C_n - P_A^B \bar{U}_{Cm} \nabla_B \bar{U}^C_n . \]

  or

  \[ \bar{\Omega}_{A\bar{m}\bar{n}} = \bar{P}_A^B U_{C\bar{m}} \nabla_B U^C_{\bar{n}} - P_A^B \bar{V}_{C\bar{m}} \nabla_B \bar{V}^C_{\bar{n}} . \]

- Upon the level matching constraint, they are expressed in terms of \(D\)-dimensional notation,

  \[ \Omega_A \equiv \begin{pmatrix} -\frac{1}{2} H^\mu \\ \omega_\nu - \frac{1}{2} B_{\nu\rho} H^\rho \end{pmatrix}, \quad \bar{\Omega}_A \equiv \begin{pmatrix} -\frac{1}{2} \bar{H}^\mu \\ \bar{\omega}_\nu - \frac{1}{2} B_{\nu\rho} \bar{H}^\rho \end{pmatrix}, \]
Pull back to D-dimensional theory

- Double-vielbein can pull back the chiral and the anti-chiral $2D$ indices to the more familiar $D$-dimensional ones

- We pull back the double-gauge covariant rank two-tensor to obtain,

$$S_{AB} V^A_m \bar{V}^B_{\bar{n}} \equiv R_{m\bar{n}} + 2 D_m D_{\bar{n}} \phi - \frac{1}{4} H_{m\mu\nu} H_{\bar{n}\mu\nu} + (\partial^\lambda \phi) H_{\lambda m\bar{n}} - \frac{1}{2} \nabla^\lambda H_{\lambda m\bar{n}}.$$

- As expected, its symmetric and the anti-symmetric parts correspond to the equations of motion of the effective action for $g_{\mu\nu}$ and $B_{\mu\nu}$ respectively.
Pullback of various covariant quantities

\[ V^A l \bar{D}_A T_{k_1 \bar{k}_2 \ldots \bar{k}_n} \equiv \frac{1}{\sqrt{2}} \hat{D}_l T_{k_1 \bar{k}_2 \ldots \bar{k}_n} , \]

\[ \bar{V}^A l \bar{D}_A T_{k_1 k_2 \ldots k_n} \equiv \frac{1}{\sqrt{2}} \hat{D}_l T_{k_1 k_2 \ldots k_n} , \]

\[ P^{AB} D_A T^{Bk_1 \ldots k_n} \equiv \frac{1}{\sqrt{2}} \hat{D}' T^{l \bar{k}_1 \ldots \bar{k}_n} - \sqrt{2} \hat{D}' \phi T_{l \bar{k}_1 \ldots \bar{k}_n} , \]

\[ \bar{P}^{AB} D_A T_{Bk_1 \ldots k_n} \equiv - \frac{1}{\sqrt{2}} \hat{D}' \bar{T}_{l \bar{k}_1 \ldots \bar{k}_n} + \sqrt{2} \hat{D}' \bar{T}_{l k_1 \ldots k_n} , \]

\[ P^{AB} D_A D_B T_{k_1 \ldots k_n} \equiv \frac{1}{2} \hat{D}^\mu \hat{D}_\mu T_{k_1 \ldots k_n} - \hat{D}^\mu \phi \hat{D}_\mu T_{k_1 \ldots k_n} , \]

\[ \bar{P}^{AB} D_A D_B T_{k_1 \ldots k_n} \equiv - \frac{1}{2} \hat{D}^\mu \hat{D}_\mu T_{k_1 \ldots k_n} + \hat{D}^\mu \phi \hat{D}_\mu T_{k_1 \ldots k_n} , \]

where we put, as for \( D \)-dimensional tensors,

\[ T_{k_1 k_2 \ldots k_n} = T_{A_1 A_2 \ldots A_n} V^{A_1}_{A_k} V^{A_2}_{A_{\bar{k}_2}} \ldots V^{A_n}_{A_{\bar{k}_n}} , \]

\[ T^{\bar{k}_1 \bar{k}_2 \ldots \bar{k}_n} = T_{A_1 A_2 \ldots A_n} \bar{V}^{A_1}_{A_{\bar{k}_1}} \bar{V}^{A_2}_{A_{\bar{k}_2}} \ldots \bar{V}^{A_n}_{A_{\bar{k}_n}} , \]

\[ T_{l \bar{k}_1 \ldots \bar{k}_n} = T_{B A_1 \ldots A_n} \bar{V}^{B}_{l A_{\bar{k}_1}} \ldots \bar{V}^{A_n}_{A_{\bar{k}_n}} , \]

\[ T_{l \bar{k}_1 \ldots k_n} = T_{B A_1 \ldots A_n} V^B_{l A_{\bar{k}_1}} \ldots \bar{V}^{A_n}_{A_{\bar{k}_n}} , \]

which are \( O(D, D) \) singlets, and we set, for \( \hat{D}_m = (e^{-1})_m^\mu \hat{D}_\mu \) and \( \hat{D}_{\bar{n}} = (\bar{e}^{-1})_{\bar{n}}^\mu \hat{D}_\mu \),

\[ \hat{D}_\mu := \nabla_\mu + (\omega_\mu + \frac{1}{2} H_\mu) + (\bar{\omega}_\mu - \frac{1}{2} \bar{H}_\mu) . \]
Symmetry structure

• Symmetry structure for the double field theory
  
  • $O(D, D)$ ’T-duality’
  
  • Double-gauge symmetry
    
    \[
    \left\{ \begin{array}{l}
    \text{Diffeomorphism} \\
    \text{One-form gauge symmetry for } B_{\mu \nu}
    \end{array} \right.
    \]

• Commutator between $so(D, D)$ and generalized Lie derivative does not generate any symmetry of double field theory.

\[
[\delta_h, \hat{\mathcal{L}}_X] = \hat{\mathcal{L}}_Y, \quad \partial^C Y^A \partial_C T_{B_1 B_2 \cdots B_n} \neq 0.
\]

• $O(D, D)$ transformation rotates the entire hyperplane on which DFT lives.
Application to Yang-Mills

- Symmetry structure for the double field theory
  - $O(D, D)$ 'T-duality'
  - Double-gauge symmetry
    - Diffeomorphism
    - One-form gauge symmetry for $B_{\mu\nu}$
  - two copies of local Lorentz symmetry
Application to Yang-Mills

- Symmetry structure for the double field theory
  - $O(D, D)$ 'T-duality'
  - Double-gauge symmetry \{ Diffeomorphism
    One-form gauge symmetry for $B_{\mu \nu}$ \}
  - Yang-Mills gauge symmetry

- Apply the formalism to couple the non-Abelian Yang-Mills gauge field to the DFT action.
Application to Yang-Mills

- We postulate a vector potential, $\mathcal{V}_A$, which
  - is $O(D, D)$ and double-gauge covariant,
  - and transforms under non-Abelian gauge group, $g \in G$,

$$\mathcal{V}_A \longrightarrow g\mathcal{V}_Ag^{-1} - i(Ag)g^{-1}.$$
• The usual field strength,

\[ F_{AB} = \partial_A \nu_B - \partial_B \nu_A - i [\nu_A, \nu_B] , \]

is YM gauge covariant, but it is NOT double-gauge covariant,

\[ \delta_X F_{AB} \neq \hat{\mathcal{L}}_X F_{AB} . \]
Application to Yang-Mills

- Instead, we consider with the semi-covariant derivative,

\[ \mathcal{F}_{AB} := \nabla_A \nabla_B - \nabla_B \nabla_A - i [\nabla_A, \nabla_B] = F_{AB} - \Gamma^{C}_{AB} \nabla_C. \]

- While this is neither YM gauge nor double-gauge covariant,

\[ \mathcal{F}_{AB} \rightarrow g \mathcal{F}_{AB} g^{-1} + i \Gamma^{C}_{AB} (\partial_C g) g^{-1}, \]

\[ \delta_X \mathcal{F}_{AB} \neq \hat{\mathcal{L}}_X \mathcal{F}_{AB}, \]

- if projected properly, it can be covariant up to level matching condition,

\[ P^C_A \bar{P}^D_B \mathcal{F}_{CD} \rightarrow P^C_A \bar{P}^D_B g \mathcal{F}_{CD} g^{-1}, \]

\[ \delta_X (P^C_A \bar{P}^D_B \mathcal{F}_{CD}) = \hat{\mathcal{L}}_X (P^C_A \bar{P}^D_B \mathcal{F}_{CD}). \]
Application to Yang-Mills

\[ P^C_A \bar{P}^D_B \mathcal{F}_{CD} \] is DFT field strength which is fully covariant with respect to

- \( \mathbb{O}(D, D) \) T-duality
- Gauge symmetry
  - Double gauge = Diffeomorphism + one form gauge symmetry
  - Yang-Mills gauge
Yang-Mills action

- Our double field formulation of Yang-Mills action is
  \[ S_{YM} = g_{YM}^{-2} \int d^{2D} e^{-2d} \text{Tr} \left( P^{AB} \bar{P}^{CD} F_{AC} F_{BD} \right), \]

- Manifestly this action is invariant under $O(D, D)$ T-duality, double-gauge and Yang-Mills gauge transformation.

- Corresponding $D$-dimensional action of the Double field YM action?
Yang-Mills in components

- Decompose the vector potential into chiral and anti-chiral ones,

\[
\mathcal{V}_A = V_A^+ + V_A^-,
\]

\[
V_A^+ = P_{A}^{B} \mathcal{V}_B,
\]

\[
V_A^- = \bar{P}^{A}_{B} \mathcal{V}_B.
\]

- Their general forms are

\[
V_A^+ = \frac{1}{2} \begin{pmatrix} A^{+\lambda} \\ (g+B)_{\mu\nu}A^{+\nu} \end{pmatrix},
\]

\[
V_A^- = \frac{1}{2} \begin{pmatrix} -A^{-\lambda} \\ (g-B)_{\mu\nu}A^{-\nu} \end{pmatrix}.
\]
Yang-Mills in components

- With the field redefinition,

\[ A_\mu := \frac{1}{2}(A^\mu_\uparrow + A^\mu_\downarrow), \quad \phi_\mu := \frac{1}{2}(A^\mu_\uparrow - A^\mu_\downarrow), \]

we get a general form of double gauge field

\[ \mathcal{V}_A = \begin{pmatrix} \phi^\lambda \\ A_\mu + B_{\mu\nu}\phi^\nu \end{pmatrix}. \]

- \( A_\mu \) and \( \phi_\mu \) will be YM gauge connection and YM gauge covariant one-form respectively.
Yang-Mills in components

- Turning off the $\tilde{x}$-dependence reduces the action to

$$S_{YM} \equiv g_{YM}^{-2} \int dx^D \sqrt{-g} e^{-2\phi} \text{Tr} \left( -\frac{1}{4} \hat{f}_{\mu\nu} \hat{f}^{\mu\nu} \right),$$

where

$$\hat{f}_{\mu\nu} := f_{\mu\nu} - D_\mu \phi_\nu - D_\nu \phi_\mu + i [\phi_\mu, \phi_\nu] + H_{\mu\nu\lambda} \phi^\lambda,$$

and

$$\text{Tr} \left( \hat{f}_{\mu\nu} \hat{f}^{\mu\nu} \right) = \text{Tr} \left( f_{\mu\nu} f^{\mu\nu} + 2D_\mu \phi_\nu D^\mu \phi^\nu + 2D_\mu \phi_\nu D^\nu \phi^\mu - [\phi_\mu, \phi_\nu][\phi^\mu, \phi^\nu] + 2if_{\mu\nu} [\phi^\mu, \phi^\nu] + 2 (f^{\mu\nu} + i[\phi^{\mu}, \phi^{\nu}]) H_{\mu\nu\sigma} \phi^\sigma + H_{\mu\nu\sigma} H^{\mu\nu\tau} \phi^\sigma \phi^\tau \right).$$

- For T-duality, we need YM gauge covariant 1-form field.
- Similar to topologically twisted Yang-Mills, but differs in detail.
- Curved $D$-branes are known to convert adjoint scalars into one-form,

$$\phi^a \rightarrow \phi_\mu, \quad \text{Bershadsky}$$
More on fully covariant quantities

- Even power of the field strength, \( \hat{F}^{AB} := P_A^C P_B^D F_{CD} \),
  
  \[
  \text{Tr} \left( \hat{F}^{A_1 B_1} \hat{F}^{A_2 B_2} \hat{F}^{A_3 B_2} \cdots \hat{F}^{A_n B_n} \hat{F}^{A_1 B_n} \right).
  \]

- For the Abelian group, DBI type action
  
  \[
  \det \left( \eta_{AB} + \kappa \hat{F}_{AC} \hat{F}^C_B \right) = \det \left( \eta_{AB} + \kappa \hat{F}_{CA} \hat{F}^C_B \right),
  \]
  
  \( \kappa \) is a constant.
  
  No square root is necessary since this is a scalar, not a density.
Concluding remarks

- \textbf{O}(D, D) T-duality, diffeomorphism, one-form gauge symmetry fixes the low energy effective action,

\[
S_{\text{eff.}} = \int dx^D e^{-2d} \mathcal{H}^{AB} S_{AB}.
\]

- Non-Abelian Yang-Mills field is incorporated in DFT formulation.
Concluding remarks

- Yet, string theory interpretation of the YM theory is not clear.
- D-brane in DFT Albertsson, Dai, Kao, Lin
- Fermion in DFT Coimbra, Strickland-Constable, Waldram
- Extension to RR fields Hull, Kwak, Zwiebach
- Application to ‘doubled sigma model’ and generalization to $\mathcal{M}$-theory are of interest Hull, Berman, Perry, ... ... etc...

- Concluding:
  Perhaps, our formalism may provide some clue to a new framework for string theory, beyond Riemann.
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  Perhaps, our formalism may provide some clue to a new framework for string theory, beyond Riemann.

  Thank you.