Invariants, symmetries and divergences in maximally supersymmetric field theories

- N=8 Supergravity (MSG) 32 susies
- N=4 Super Yang-Mills (MSYM) 16 susies

in various spacetime dimensions

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  ectoplasm

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Introduction

- Supersymmetry is well-known to soften UV divergences in QFT. Indeed, D=4, N=4 SYM is UV finite. But what about the non-renormalisable case?
- This subject was once thought to have been closed (c. 1990). Such theories were expected to diverge at some loop order; the D=7 L=2 MSYM *Marcus,Sagnotti* divergence agreed with predictions; higher loop computations were deemed impossible, and in any case the focus had switched to string theory.
- More recently, new methods have allowed higher loop calculations to be made, and "unexpected" finiteness results have reopened the debate.
- In this talk it will be argued that there are no miracles – the finiteness results can be understood in terms of the symmetry properties of the invariants that can arise as possible counterterms.
- These can be divided into two classes: D-terms and F-terms. The former correspond to full superspace integrals and are numerous, while the latter correspond to subsuperspace integrals, and there are only a few of them. They are also known as short or BPS invariants.
- D-terms are not protected by any conventional field theory argument.

- The invariants we shall be interested in are expressions of the form F^4 or R^4, or higher powers, and possibly involving extra derivatives, together with their susy partners.

- There are further requirements on a putative counterterm than its just being supersymmetric. This can be easily understood using superspace non-renormalisation theorems (NRTs). Suppose a theory can be formulated in off-shell superfields, then any divergence will be given by a local integral over the full superspace. For gauge theories, the background field method also implies that the integrands should be gauge-invariant functions of the background fields (excluding prepotentials). *Grisaru, Rocek, Siegel; Grisaru, Siegel*

- For N=1 D=4 this rules out counterterms of the form $\int d^4x\, d^2\theta\, f(\varphi)$, where φ is a chiral field. This is the prototype F-term.

- For D=4 N=2 SYM a D-term counterterm would have to be of the form $\int d^4x\, d^8\theta\, L$, where L has dimension zero, and there are no such objects. So no divergences. (The action, $\int d^4x\, d^4\theta\, \mathrm{Tr}(W^2)$, is an F-term).

- For technical reasons there is a one-loop exception, so N=2 SYM is finite ***except*** at one loop. N=4 is finite because the one-loop divergences cancel.

PH, Stelle, Townsend

- In D=4 supergravity is finite at L=1,2, even for N=1. The first counterterm one can construct is R^4 which can occur at three loops. (The superficial degree of divergence of a diagram in gravity is (D-2)L +2). *Deser, Kay, Stelle*
- R^4 is a D-term for $N \leq 4$, but an F-term for N>4.
- So we might expect it to be protected in N>4, but there is a problem. The NRT applies for the number of susies, M say, that can be linearly realised off-shell (the algebra closes without the use of the equations of motion). In the early 1980s it was thought that the best one could do was to realise half of the susies linearly (so M=4 for N=8 MSG). *PH, Stelle, Townsend*
- This led to the prediction that MSYM would diverge first at L=1,2,3,4 loops in D=8,7,6,5 resp. The invariants are F^4,d^2F^4,d^2F^4 and F^4 resp. *PH, Stelle*
- In N=8 SG, the R^4 invariant can be written as a $d^{16}\theta$ integral, and so is apparently ***not*** protected if only M=4 can be linearly realised. *Kallosh; PH, Stelle, Townsend*
- If we had had an off-shell version of the theory with N=8 then there would have been no problem in declaring that the first divergence could occur at L=7. Indeed there is no problem in constructing such invariants even in the full theory and with E_7 symmetry. *PH, Lindstrom; Kallosh,*
- Note that the first divergence must also be invariant under the on-shell non-linear susies.

- Over the last decade or so new, unitarity based computational techniques have allowed explicit calculations to be pushed to a much higher loop order.

  Bern, Dixon, Kosower

- In MSYM it has been shown that ½ BPS invariants are finite, including F^4 at L=4 loops in D=5. More recently, a certain ¼ BPS counterterm, $Tr^2(d^2F^4)$, has been shown to have a zero coefficient in D=6 L=3.

  Bern,Dixon,Dunbar, Perelstein, Rozowsky; *Bern,Carrasco,Dixon,Johansson,Roiban*

- In supergravity, N=8 finite at L=3 in D=4 $(d^{16}\theta \rightarrow R^4)$ ½ BPS

  Bern,Carrasco,Dixon,Johansson,Kosower,Roiban

- MSG finite at L=4 in D=5 $(d^{28}\theta \rightarrow d^6R^4)$ 1/8 BPS

  Bern,Carrasco,Dixon,Johansson,Roiban

- But note that there ***are*** F-term divergences in D=6,7,8

- Recently, a manifestly supersymmetric approach based on pure spinor quantum mechanics has been proposed. Confirms and extends the MSG results of Bern et al.

  Bjornsson,Green

- In MSYM one can improve superspace NRT predictions by using the harmonic superspace formalism which allows ¾ susies (i.e. 12) to be linearly realised. This explains the D=5 L=4 finiteness result but not the L=3 D=6 one. *Galperin, Ivanov, Kalitsin, Ogievetsky, Sokatchev; PH, Stelle*

- In MSG the status of off-shell harmonic superspace formulations is unclear and difficult to analyse.

- Algebraic renormalisation theory (ART) offers another approach to restricting the allowed counterterms. The quantisation is in components. Each putative counterterm can be thought of as a D-form in D dimensions, L_D. Susy (BRST) variations then yield a series of forms $L_{D-1,1}$, $L_{D-2,2}$, etc, where the second index labels the susy ghost number. The whole series can be interpreted as a cocycle of the operator $d_0 + s$, the sum of the spacetime exterior derivative and the susy BRST operator. The algebraic NRT states that any allowed counterterm must have the same cocycle structure as that of the action.

- For MSYM this gives the same results as harmonic superspace, namely that ½ BPS operators are protected, but not ¼ BPS. ½ BPS operators are also protected by ART in MSG. *BHS*

- An explanation of the MSYM finiteness results using pure spinors has also been given. *Berkovits,Green,Russo,Vanhove; Bjornsson,Green*

- To summarise, the difficulty lies with the F-term counterterms in MSG. Which ones are protected and which are not? How do we construct them, and what are their cocycle structures?

- The discussion so far has made little mention of duality symmetries, but it turns out that these are the key to resolving these problems.

- The importance of dualities in this context has been emphasised for some time in the string theory approach.

  Green,Russo,Vanhove

- In the past year there have been a number of developments from various points of view

- The string theory discussion has been strengthened. *Green,Russo,Vanhove*

- It has been pointed out that E_7 can be preserved in perturbation theory, at the cost of manifest Lorentz invariance. Duality symmetries can therefore be expected to hold in all dimensions where there are no anomalies. Indeed, in odd dimensions, they are symmetries of the spacetime Lagrangian.

Bossard, Hillmann, Nicolai

- E_7 violation in N=8 D=4 R^4 . *Broedel, Dixon*

- The lack of E_7 invariance for R^4 has been confirmed by an argument using string theory and dimensional reduction. *Elvang, Kiermaier*

- In two recent papers the discussion has been extended to other short invariants.

BHS; *Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger*

- The upshot is that imposing duality symmetries is more restrictive than had been thought, and its implications for the onset of UV divergences are in line with explicit computations. It also makes predictions beyond current unitarity computations; for example, N=8 should be finite at six loops.

Linearised F-term invariants in MSG

Focus on D=4 N=8. In the linearised theory the field strength superfield is W_{ijkl} in the 70 of SU(8). It is an antisymmetric, self-dual fourth-rank tensor. It obeys the constraint $DW=\Lambda$, where Λ is the spinor in the 56 of SU(8).

Its components are $W \rightarrow \Lambda \rightarrow F \rightarrow \Psi \rightarrow R$ (Weyl tensor). All possible short invariants are given by integrating W^4 in various representations over subspaces with odd dimension 16 (R^4, ½ BPS), 24 (d^4R^4, ¼ BPS) and 28 (d^6R^4, 1/8 BPS).

For example, for the R^4 invariant, one takes W^4 in the 232,848 rep of SU(8), or [0004000] in Dynkin labels, or 4 x 4 square YT.

Kallosh; PH, Stelle, Townsend

The full classification was obtained using superconformal methods.

Drummond, Heslop, PH,Kerstan + DHH

It has also been obtained from scattering amplitudes.

Elvang, Freedman, Kiermaier

- These F-terms can also be written as harmonic superspace integrals.
- We have $D_i W_{jklm} = D_{[i} W_{jklm]}$, so that W_{1234} is annihilated by $D_1 \ldots D_4$. It is also killed by $\dot{D}^5, \ldots, \dot{D}^8$. So the integral $\int d^4x\, d^{16}\theta\, (W_{1234})^4$ is supersymmetric, where the integration is over the coords that W_{1234} depends on.
- This is not obviously SU(8) invariant but we can remedy this by making an SU(8) transformation of the integrand and then integrating over the group, or rather over the coset defined by the isotropy group S(U(4)xU(4)).
- The resulting expression is then an integral of a single-component object over a subsuperspace given by a subset of the odd coordinates, and is manifestly SU(8) invariant.
- Similar constructions can be made for the other two cases. This is the appropriate generalisation of chirality for this type of invariant.

Difficulties with non-linear F terms

Example: IIB SG in D=10. At the linearised level there is a chiral superfield φ (also satisfying a fourth-order reality constraint), and the R^4 invariant is the integral over chiral superspace of φ^4, $\int d^{10}x\, d^{16}\Theta\, \varphi^4$.

Although we know a good deal about the full invariant from component studies we do not know a complete formula for it.

Green, Sethi

The obstruction to constructing a straightforward superspace version is the absence of a chiral measure in the full theory. The tangent spaces split invariantly into three: the even part and two 16-dim odd parts, primed and unprimed, say, which are complex conjugates, the structure group being Spin(1,9) x U(1):

$$T=T_0 + T_1 + T'_1$$

In order for there to be a chiral structure we require that the graded commutator of two unprimed odd vectors should also lie in the unprimed space. But this is not the case due to the fact that there is a component of the dimension one-half torsion tensor, $T_{\alpha\beta}{}^{\gamma'}$, that is non-zero.

PH, West *de Haro, Sinkovics, Skenderis* *Berkovits, PH*

What happens in D=4? It turns out that, for similar reasons, there are no (harmonic superspace) measures for either ½ or ¼ BPS invariants in N=8, that is, R^4 (L=3) and $d^4 R^4$ (L=5). In N=6 there is also no L=3, R^4 measure.

BHS

Non-linear measures ***do*** exist for L=6 (d^6R^4) in N=8 and L=4 ($d^2 R^4$) in N=6, as well as for L=3 in N=5. These are integrations over 4(N-1) θs.

In these cases, however, there are no duality invariant integrands available. This is because the measures carry R-charges and the only dimension zero fields available to soak these up are the scalars that transform under the duality group.

Note that this discussion already implies that L=6 in N=8, L=4 in N=6 and L=3 in N=5 will be protected, provided that the lower-order counterterms have zero coefficients.

The fact that some measures do not exist, although suggestive, does not by itself rule out the corresponding invariants' being duality invariant nor does it necessarily imply that the corresponding divergences cannot occur.

ART & Ectoplasm

Basic idea: in D-dimensional spacetime a superinvariant can be obtained from a closed superspace D-form, $L_{D,}$ by

$$I=\int \varepsilon^{mn....} L_{mn.......}(x,0)$$

Voronov; Gates; Gates, Grisaru, Knutt-Wehlau, Siegel

Easy to see that this does define a susy invariant since, under a superspace diffeomorphism, a closed form changes by the d of something. Evaluated at $\theta=0$ this gives a total derivative. If L_D is exact I=0, so we are interested in cohomology (but purely algebraic).

In supergeometry we have an invariant splitting of the tangent bundle into even and odd, $T=T_O + T_{1,}$ with corresponding basis forms $E^A=(E^a,E^\alpha)$, where $E^A=dz^M E_M{}^A = dx^m E_m{}^A + d\theta^\mu E_\mu{}^A$. We can identify $E_m{}^a(x,o)=e_m{}^a$ and $E_m{}^\alpha(x,0)=\psi_m{}^\alpha$ (gravitino).

Then, for example for D=4, we have

$$I=\int \varepsilon^{mnpq}(e_q{}^d..e_m{}^a L_{abcd} + 4\, \psi_q{}^\delta..e_m{}^a L_{abc\delta} +........\psi_q{}^\delta..\Psi_m{}^\alpha L_{\alpha\beta\gamma\delta})$$

We can analyse closed forms with the aid of superspace cohomology:

space of n-forms, $\Omega^n = \Sigma_{p+q=n} \Omega^{p,q,}$, spaces of (p,q) forms

The exterior derivative splits into various cpts:

$$d=d_0 + d_1 + t_0 + t_1$$

with bi-degrees (1,0), (0,1),(-1,2), (2,-1) resp. The first two are even and odd differential operators, while the third is an algebraic operation involving the dimension-zero torsion (gamma matrix). The vector index on the torsion is contracted with one of the even indices on the form, while all (q+2) odd indices, on the torsion and the form, are symmetrised.

- t_1 is also algebraic and uses the dimension 3/2 torsion (gravitino field strength), but doesn't play a significant role.
- The equation $d^2=0$ splits into various cpts:

$$t_0^2=0$$

$$d_1 t_0 + t_0 d_1=0$$

$$d_1^2 + d_0 t_0 + t_0 d_0 = 0$$

The first of these means that we can define cohomology groups $H_t^{p,q}$.

We can also define a spinorial derivative, d_s, that acts on $H_t^{p,q}$, by $d_s[\omega_{p,q}]=[d_1\omega_{p,q}]$. It is easy to see, using the other two equations, that this is well-defined and squares to zero. We can therefore define the spinorial cohomology groups $H_s^{p,q}$.

These groups generalise spaces of multi pure spinors, and pure spinor cohomology respectively.

Bonora, Pasti, Tonin: PH; Berkovits; Cederwall, Nilsson, Tsimpis; PH, Tsimpis

The equation $dL_D=0$ can be analysed cpt by cpt, starting from the lowest dimension (smallest p) and solutions can be obtained in terms of spinorial cohomology groups. If the lowest non-zero cpt is $L_{p,q}$ say, then the full D-form will be given by $L_{p,q}$, $L_{p+1,q-1}$, ...$L_{D,0}$.

D-forms like this can be identified with the cocycles that are derived in ART.

We shall call a cocycle ***standard*** if all of its higher cpts are determined from the lowest non-zero one directly. Essentially this means that the higher cpts are given by acting on the lowest non-zero one by powers of D.

Otherwise, a cocycle is ***non-standard***.

- The algebraic NRT states that an invariant is protected if its cocycle has a ***different*** structure to that of the action.
- The action cocycle has the same structure as any cocycle for a full superspace integral → D terms ***not*** protected. These cocycles are always standard.
- Protected invariants have different cocycle structures to those for full superspace integrals.
- Invariants with non-standard cocycles are always protected.

To illustrate this let us consider N=1, D=10 superspace. Each standard cocycle has a lowest cpt of the form $L_{5,5}=\Gamma_{5,2}\, M_{0,3}$, with $d_s\,[M_{0,3}]=0$. In D<10 such a cocycle will reduce to one that has lowest cpt $L_{D-5,5}$. In MSYM, the action cocyle with have this structure for all D>4.

On the other hand, in e.g. D=6, the lowest cpt of the ½ BPS invariant F^4 turns out to be in $L_{0,6}$ rather than $L_{1,5}$. Both of these cocycles are standard, but the one associated with F^4 differs from that of the action, in fact it is longer, and we conclude that it is protected by ART.

In D=10, however, the F^4 invariants are no longer given by BPS superfields. Instead they come from Chern-Simons type forms. The eleven-form $W_{11}=H_3F^4$ can be written as dK_{10}, where K_{10} is a gauge-invariant 10-form, and also as dZ_{10}, where $Z_{10}=H_3\, Q_7$, and where Q_7 is a CS 7-form, $dQ_7=F^4$. In flat sspace $H_3 \sim \Gamma_{1,2}$. So the closed 10-form is $L_{10}=K_{10}-Z_{10}$. For the double-trace invariant we find that the lowest non-zero cpt of this is $L_{4,6}$ so that the cocycle has the following structure

$$\begin{array}{ccccccc} L_{4,6} \rightarrow & L_{5,5} & \rightarrow L_{6,4} \rightarrow & \ldots & \rightarrow L_{10,0} \\ & \uparrow & & & \\ & M_{0,3} & & & \end{array}$$

Bossard, Howe, Lindstrom, Stelle, Wulff

Duality symmetries in MSG

- Ectoplasm. We have seen that the ½ and ¼ BPS counterterms in D=4 MSG do not have corresponding non-linear measures. By itself, this does not necessarily mean that these invariants do not exist, nor that they necessarily violate E_7. But their cocycles must presumably be non-standard; if so, they can be ruled out as UV divergences by ART.

  We can also say something about their E_7 properties using our knowledge of the linearised invariants which do have standard cocycles. A full, non-linear cocycle for an E_7 invariant will have cpts $L_{0,4}$, $L_{1,3}$, ..., $L_{4,0}$ which are separately E_7 invariant. In other words, ***all*** cpts of L_4, wrt a preferred basis, will be constructed from superspace tensors that do not involve any factors of undifferentiated scalars. This is something we can test at the linearised level. The top component, $L_{4,0}$, whose integral is the linearised invariant, is shift invariant, but we also need to consider its "soul".

- Indeed, one can see, from the ectoplasm formula, that the linearised $L_{0,4}$ will make a contribution to the 8-pt function.

As an example consider R^4. In D=4 we can compute $L_{0,4}$ by evaluating $D^{12}(W^4)_{[000400]}$. One can explicitly check that this involves undifferentiated Ws, and so is not E_7 invariant even at the linearised level.

For the other two linearised BPS invariants, d^4R^4 and d^6R^4, this analysis does not give the same result. This is rather easy to see. For example, the d^4R^4 invariant can be written as the integral over 16 θs of $(\partial W)^4_{[0004000]}$, where the spacetime derivatives are contracted in pairs ("pseudo-1/2 BPS"). However, this certainly does ***not*** prove that the full invariants are E_7 symmetric.

We can also use this type of analysis for N<8. In fact, we find that the linearised N=6, L=3,4 and N=5, L=3 invariants are not invariant under linearised duality transformations. (The groups are $SO^*(12)$ and SU(5,1) resp.) This shows that these invariants cannot correspond to UV divergences.

- Dimensional reduction. Start with R^4 in D=11, no scalars, reduce to D=4, and then examine the scalar factor multiplying R^4 in D=4.

  The idea is to examine whether this factor is constant or not. If the latter, then the term cannot be E_7 invariant.

  We need to do two things:

  (1) dualise the antisymmetric tensors where necessary to obtain the complete set of scalars

  (2) impose SU(8) symmetry. This can be done by averaging over the group modulo the trivially preserved subgroup SO(7). *Elvang, Kirmaier*

  Step (2) is difficult to do in practise, but we can circumvent this problem by observing that such scalar pre-factors, f say, obey differential equations on the scalar coset manifolds, e.g. $SU(8)\backslash E_7$ for D=4. So we get equations of the form (cf. string theory *Green,Russo,Vanhove*)

$$(\Delta + k) f=0$$

  where k is a constant and Δ is the Laplacian on the coset. So for f to be constant we require that the constant k must vanish. If not, duality symmetry is violated.

This argument depends on the uniqueness of the R^4 invariant in D=4, because if there were a second one, it might be possible to find a duality-invariant combination. At the linearised level this would show up at more than four points, i.e. there would have to be an independent linearised invariant with the same dimension but with more fields.

So the claim that a given counterterm in D=4 is not E_7 invariant can be reduced to showing that the relevant constant term in the Laplace equation is non-zero.

The method can be adapted to discuss the duality properties of other invariants in D=4. In fact, we can unify the discussion by noting that R^4 in D=8(L=1), $d^4 R^4$ in D=7 (L=2) and d^6R^6 in D=6 (L=3) must have trivial scalar factors as they correspond to known divergences. We can then predict that the scalar factors for these invariants will be non-trivial in D=4 and hence that E_7 will be violated.

Explicitly, reducing from D to 4 for $d^{2n}R^4$, we find that the scalar factor $f_n(\varphi)$ satisfies the Laplace equation

$$(\Delta + (D-4)\,(D-2)^{-1} n(32-D-n))f_n(\varphi) = 0$$

The argument can be turned around to show that the only BPS divergences in higher dimensions are these three.

BPS invariants in MSG that could arise as counterterms for log divergences . Red: known divergences; yellow, known to be finite (unitarity); turquoise, predicted to be finite.

D / L	4	5	6	7	8	9	10	11
1					R^4			
2		R^4		d^4R^4	d^6R^4			
3	R^4		d^6R^4					
4		d^6R^4						
5	d^4R^4							
6	d^6R^4							

Conclusions

- It has taken some time, but we are now confident that the question of UV divergences for short invariants in MSG has been resolved. The key extra ingredient is the use of duality symmetries.
- The only such divergences are

  D=8, L=1: R^4 ; D=7, L=2: d^4R^4 and D=6,L=3: d^6R^4

- For D=4, N=8 we have been able to argue the absence of BPS divergences in four ways:

  - E_7 + dim red + uniqueness of linearised invariants for D=4
  - Ectoplasm and linearised E_7 for L=3
  - Ectoplasm + ART for L=3,5
  - Existence of 1/8 BPS measure + E_7 for L=6

- There are no BPS divergences in D=4 for N=5,6. This can also be seen from ectoplasm, ART, duality symmetries and measure properties.

- Remaining questions:
- Can there be truly unexpected, i.e. non-BPS, UV cancellations?
- In D=4 N=8 these invariants start at L=7. The unique E_7 invariant one is the volume of superspace. This is known to vanish for N=1,2,3 on-shell, but there is no reason why this should be so for N>3. For N=4 it should give R^4, and for higher N, R^4 with derivatives plus higher-point contributions.
- It has been suggested that N=4 SG might be UV finite at L=3, i.e. the R^4 counterterm should have a zero coefficient. This is a preliminary conjecture and has yet to be confirmed. *Dunbar, Hertle, Perkins*

  Moreover, pure spinor methods indicate that N=8 will diverge at L=7. *Bjornsson, Green*
- It has also been suggested that D=5 MSYM might be UV finite via a relationship to a superconformal theory in D=6. *Douglas; Lambert, Papageorgakis, Schmidt-Sommerfeld*
- We would also like to understand how to construct the complete BPS invariants, at least in principle. The absence of appropriate measures in most cases makes this a difficult problem to investigate, but it might be that superspace cohomology could help.