Viscosity and conductivity in general theories of gravity.

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Motivation

- Recently, great interest in applying AdS/CFT methods to "real" systems.
- Macroscopic, thermal averaged evolution: transport coefficients.
- Insights into quasiparticle excitation structure.
- Examples: condensed matter systems close to quantum critical points, ultra-cold atom gases, graphene...

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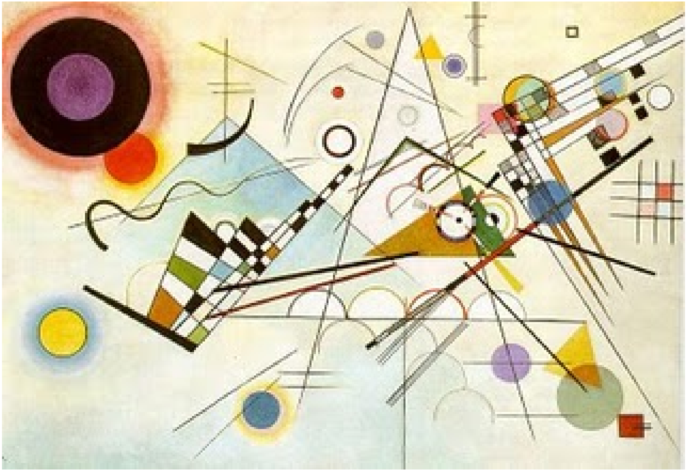
- Important application: the strongly coupled quark-gluon plasma.

Quark-gluon plasma (artist's depiction)



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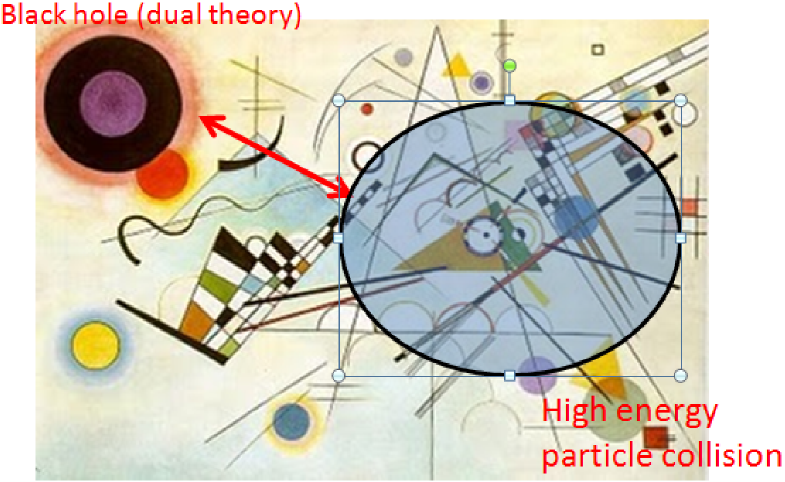
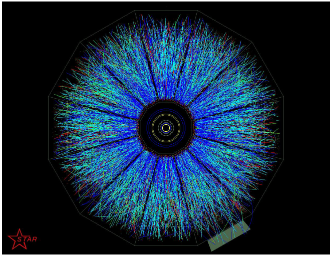
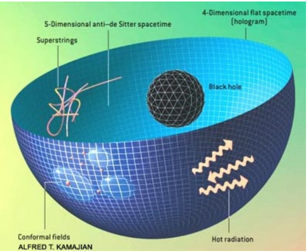


Quark-gluon plasma as seen by strings (?)

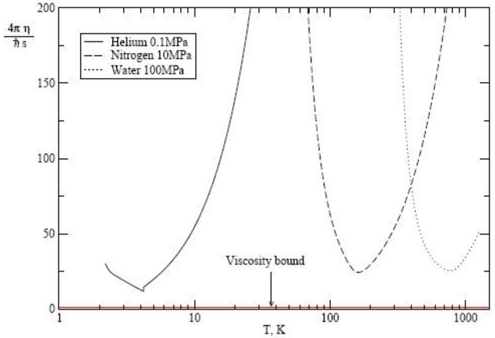


$\stackrel{?}{\simeq}$



- QGP close to deconfinement: strongly coupled, near-perfect, near-conformal plasma.
- Hydrodynamic simulations indicate $\eta/s \simeq 0.04 - 0.16$: the most ideal fluid in nature.
- Gauge-gravity duality predicts for a large class of theories $\eta/s = 1/4\pi \simeq 0.08$!

The viscosity bound



- Kovtun, Son, Starinets: a new universal lower bound in nature?

$$\frac{\eta}{s} \geq \frac{1}{4\pi}$$

The viscosity bound

- Theories with Einstein gravity dual $\Rightarrow \eta/s = 1/4\pi$.
- $\mathcal{N} = 4$ SYM (Type IIB SUGRA + R^4 correction):

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 + 15\frac{\zeta(3)}{\lambda^{3/2}}\right) > \frac{1}{4\pi}$$

- Including D7's leads to quadratic curvature terms, which can contribute negative corrections.
- Violation of bound for superconformal gauge theories with $c > a$ [Buchel,Myers,Sinha '08].

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- Is there a bound of the form $\eta/s \geq \mathcal{O}(1)/4\pi$?
  Study shear viscosity in general higher derivative theories!

Main points of this talk:

- It is possible to find universal properties of transport coefficients in general higher derivative theories. These effectively descend from the universality of black hole horizons.

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Main points of this talk:

- It is possible to find universal properties of transport coefficients in general higher derivative theories. These effectively descend from the universality of black hole horizons.
- The pole method: a class of transport coefficients can be obtained straightforwardly by computing the residue of a simple pole of an off-shell lagrangian. This requires only information about the horizon.
- There are simple, generic, Wald-like formulae for the shear viscosity and conductivity. This opens up new perspectives in the holographic study of possible new fundamental bounds.

Outline

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Hydrodynamics as Effective Field Theory

- Hydrodynamics describes slow spatial and temporal variations of fields with respect to some microscopic scale ℓ.
- Hydrodynamic fields are (approximately) conserved currents, as these necessarily have low frequency and momentum modes. Ex.:

$T^{\mu\nu}$	Stress energy tensor
J^{μ}	Abelian charge current

- Fast, non-hydrodynamic modes are integrated out. Effective description in terms of a set of transport coefficients.
- Ex.:

η	Shear viscosity
σ	Conductivity
τ	Relaxation time

Derivative expansion of transverse modes.

- Exchange T^{00} and momentum density T^{0i} for 4-velocity and energy density.

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + T_\perp^{\mu\nu}$$
$$u_\mu T_\perp^{\mu\nu} = 0$$

- $T_\perp^{\mu\nu}$ are fixed and can be written in a derivative expansion. To first order we have:

$$T_\perp^{\mu\nu} = P(\varepsilon)\Delta^{\mu\nu} - \eta(\varepsilon)\sigma^{\mu\nu} - \zeta(\varepsilon)\Delta^{\mu\nu}(\nabla\cdot u),$$

- η is the shear viscosity, and ζ the bulk viscosity which vanishes in conformal theories.

Linearized theory

- Perturbing the metric by $h_{xy}(t,z)$ about equilibrium state:

$$T_{xy} = -P\, h_{xy} - \eta \dot{h}_{xy}$$

- Linearized response theory implies:

$$G_R^{xy,xy}(\omega) = P - i\eta\omega + \mathcal{O}(\omega^2)$$

with $G_R^{xy,xy}(\omega) \simeq \langle T^{xy}(\omega) T^{xy}(-\omega)\rangle$.

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with $G_R^{xy,xy}(\omega) \simeq \langle T^{xy}(\omega) T^{xy}(-\omega) \rangle$.

- **Kubo formula**:

$$\boxed{\eta = -\lim_{\omega \to 0} \frac{1}{\omega} \mathrm{Im}\, {G_R}^{xy}_{\ \ xy}(\omega)}$$

Gauge/gravity dictionnary

- Strongly coupled large N gauge theories $\Leftrightarrow$ gravity/string theories.
- Best known example:

$$\begin{array}{ccc} \mathcal{N}=4\ SU(N)\ \text{SYM} & \leftrightarrow & \text{Type IIB superstring on } AdS_5\times S^5 \\ \lambda = g_{YM}^2 N & \leftrightarrow & R/l_s \\ \lambda/N & \leftrightarrow & g_s \\ \mathcal{O}(x^\mu) & \leftrightarrow & \phi_{\mathcal{O}}(r,x^\mu) \end{array}$$

- Partition function maps onto on-shell gravitational action in the bulk.

$$\mathcal{Z} = \exp\left(-S_E\right)$$

- Operators are sourced via $\int d^4x \mathcal{O}(x^\mu)\phi_{\mathcal{O}}(r=+\infty, x^\mu)$.

Finite temperature

- In the gravity sector we obtain the effective action:

$$S_{5D} = \frac{1}{16\pi G_N}\int d^5x\sqrt{-g}\left(R+\frac{12}{L^2}\right)$$

- Vacuum described by AdS_5. Finite temperature $\longrightarrow$ Black hole!

$$\begin{aligned} ds^2 &= \frac{r_0^2}{L^2}(-f(u)dt^2+dx^2)+\frac{L^2\,du^2}{4\,u^2} \\ f(u) &= 1-u^2, \qquad T=\frac{r_0}{L^2\pi}. \end{aligned}$$

- Field theory thermodynamics $\leftrightarrow$ Black hole thermodynamics.
- Hydrodynamics $\leftrightarrow$ Small frequency and momentum modes.

General setup

- Focus on d dimensional field theories which have effective $d+1$ gravitational description at strong coupling.
- Gravitational sector:

$$S = -\frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{-g}\left(R + \frac{12}{L^2} + \gamma L^2 (R_{abcd})^2 + \dots\right)$$

- Treat higher derivative corrections perturbatively $\gamma \ll 1$, except in special cases.
- Assume that at finite temperature effective gravitational description is

$$ds^2 \quad = \quad = \frac{L^2}{z} e^{2g(z)} dz^2 - z\, e^{2f(z)} dt^2 + e^{2\rho(z)} dx^i dx_i$$

- Functions f, g, ρ regular $\Rightarrow$ Horizon at $z = 0$.

Real-time correspondence

- Studying hydrodynamics requires real-time correlation functions - Policastro,Son,Starinets '02.
- Simple case: massless scalar field.

$$S_\phi^{(2)} = -\frac{1}{2}\int d^d x\, dz \frac{\sqrt{-g}}{\kappa}\,(\nabla\phi)^2$$

- Near horizon equation for $\phi(t,z) = \phi(z)e^{-i\omega t}$:

$$\phi''_\omega(z) + \frac{\phi'(z)}{z} + \left(\frac{\omega}{4\pi T}\right)^2 \frac{\phi(z)}{z^2} = 0$$

- **Prescription:** Infalling boundary condition (retarded propagator!)

$$\Rightarrow \quad \phi(z) \simeq \phi_0 \exp\left(-i\frac{\omega}{4\pi T}\log z\right).$$

Real-time correlation functions

- Evaluating the on-shell action we obtain a boundary term

$$S_\phi^{(2)} = \frac{1}{2}\int d^d x \left[-\frac{\sqrt{-g}}{\kappa} g^{zz}\phi'(z)\phi(z) \right]_{z=0}^{z=1}$$

- **Prescription**: take $z=1$ piece. Rewriting in terms of canonical momentum get

$$G_R(\omega) = \lim_{\omega\to 0, z\to 1} \frac{\Pi(z)}{\phi(z)}$$

- Transport coefficient via Kubo formula:

$$\xi = \lim_{\omega\to 0} \mathrm{Im}\, \frac{\Pi(z=1)}{\omega\, \phi(z=1)}$$

Bulk flow is trivial.

- At $q = 0$, the equation of motion is:

$$\partial_z \Pi(z) = \mathcal{O}(\omega^2)$$

- In low frequency limit $\Pi(z)$ is constant!
- Also, at $\omega = 0$, solution is $\phi(z) = \text{Constant}$.

$$\Rightarrow \omega \partial_z \phi(z) = \mathcal{O}(\omega^2)$$

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Conclusion: Massless scalars have trivial bulk flow at small frequencies.

- In particular we may compute transport coefficient at the *horizon*:

$$\xi = \lim_{\omega \to 0} \frac{\Pi(z=0)}{i\omega\phi(z=0)}$$

Transport coefficient of a massless scalar.

- Recall near horizon behaviour,

$$\phi_\omega(z) \simeq \phi_0 \exp\left(-i\frac{\omega}{4\pi T}\log z\right)$$

- Using definition of canonical momentum,

$$\Pi(z) = -\frac{\sqrt{-g}}{\kappa} g^{zz}\phi'(z)$$

- We conclude

$$\xi = \lim_{\omega\to 0}\frac{\Pi(z=0)}{i\omega\phi_\omega(z=0)} = \frac{A_h}{\kappa}.$$

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- Dividing by entropy density:

$$\frac{\xi}{s} = \frac{4G_N}{\kappa}$$

- Shear viscosity corresponds to

$$\kappa = 16\pi G_N \quad \Rightarrow \quad \frac{\eta}{s} = \frac{1}{4\pi}$$

Lessons learned

This simple two derivative case has taught us valuable lessons

- **Lesson 1**
  $\text{Im} G_R(\omega) \quad \Leftrightarrow \quad$ Canonical momentum $\quad \Pi(z)$
- **Lesson 2**
  Low frequency limit + Zero “mass” $\longrightarrow$ Trivial bulk flow
- **Lesson 3**
  Universal horizon behaviour: Scaling solution

$$\phi(z) \simeq z^{-i\omega/(4\pi T)}$$

Outline

Setup

- General action for $\phi(t,z) = \phi_\omega(z) e^{i\omega t}$:

$$S^{(2)}_\phi = \int \prod_{i=1}^{d-1} dx^i \int \frac{d\omega}{2\pi} \left(S_{(z)} + S_{(t)} + S_B\right) .$$

$$S_{(z)} = \int_0^1 dz \left(\sum_{n,m\geq 0} A_{n,m}(z) \phi_\omega^{(n+1)}(z) \phi_{-\omega}^{(m+1)}(z) \right) ,$$

- Always possible to write in this form for massless perturbations.
- $S_{(t)} \propto \omega^2$ and S_B contains boundary terms. Three types:
  - $S_B^1 \simeq (B_0(z)\phi^2) \quad \rightarrow \quad$ no contribution to Im G_R.
  - $S_B^2 \simeq \mathcal{O}(\omega^2) \quad \rightarrow \quad$ don't contribute in low frequency limit.
  - $S_B^3 \simeq (B_{n,m}(z)\phi_\omega^{(n+1)}\phi_{-\omega}^{(m+1)}) \quad \rightarrow \quad \phi(z)$ always appears differentiated!

The generalized canonical momentum

- Generalize definition of canonical momentum:

$$\Pi_\omega(z) \equiv \frac{\delta S_z}{\delta(\partial_z \phi_{-\omega})}$$

- Radial action becomes after integration by parts:

$$S_{(z)} = \int_0^1 dz \left(\frac{1}{2} \Pi_\omega(z) \phi'_{-\omega}(z) \right)$$

- Equation of motion

$$\partial_z \Pi_\omega(z) = \omega^2 F(z, \phi, \phi', \ldots).$$

- The Green's function is given by the value of the on-shell action:

$$G_R(\omega) = \lim_{z \to 1} \frac{\Pi_\omega(z)}{\phi_\omega(z)} + \text{Boundary terms.}$$

Bulk flow is still trivial

Transport coefficient is determined by horizon quantities.

- Just as in the two derivative case we have
  - $\partial_z \Pi_\omega(z) = \mathcal{O}(\omega^2)$.
  - $\omega \partial_z \phi(z) = \mathcal{O}(\omega^2)$
- Bulk flow is trivial!
- Relevant boundary terms are of form $B_{n,m}(z)\phi_\omega^{(n+1)}\phi_{-\omega}^{(m+1)} = \mathcal{O}(\omega^2)$
- **Conclusion:**

$$\boxed{\xi = \lim_{\omega \to 0} \frac{\Pi(z=0)}{i\omega\phi_\omega(z=0)}}.$$

- In higher derivative theories, the important quantity is the generalized canonical momentum.

The most important slide of this talk.

- Infalling observer must see regular ϕ at the horizon.
- At horizon, perturbation must be function of Eddington-Finkelstein coordinates:

$$\partial_z\phi = \pm\sqrt{-\frac{g_{zz}}{g_{tt}}}\partial_t\phi = \mp\frac{i\omega}{4\pi T}\frac{\phi_0}{z}.$$

- Composing the two possible behaviours we get

$$\boxed{\phi_k''(z) + \frac{\phi_k'(z)}{z} + \frac{\omega^2}{(4\pi T)^2}\frac{\phi_k(z)}{z^2} = 0.}$$

- This is exactly the near-horizon equation of motion of the two derivative case.
- Argument is completely **general**: near horizon behaviour is **universal** and fixed by **regularity.**

Near horizon behaviour

- Form of the equation of motion implies near horizon action:

$$S^{(2)}_{\phi} = \int \prod_{i=1}^{d-1} dx^i \int \frac{d\omega}{2\pi} \int_0^1 dz \, \frac{-\sqrt{-g}}{2\tilde{\kappa}} \left(g^{zz} \phi'_{\omega}(z) \phi'_{-\omega}(z) + g^{tt} \omega^2 \phi_{\omega}(z) \phi_{-\omega}(z) \right),$$

- This implies the canonical momentum at the horizon

$$\Pi_{\omega}(z) = i\omega \frac{A_h}{\tilde{\kappa}} \frac{\phi_0}{z}.$$

- And therefore the transport coefficient:

$$\xi = \frac{A_h}{\tilde{\kappa}}.$$

- Higher derivative structure is packaged into the single coefficient $\tilde{\kappa}$.

Computing the canonical momentum

How to obtain the value of the canonical momentum?

1. Find the effective action and compute $\delta S/\delta\phi'(z)$. Evaluate near the horizon.
2. Use equations of motion on action to reduce it to two derivative form and read off $\tilde{\kappa}$.

Problems:

1. Must know explicit form of effective action for perturbation.
2. Must manipulate this action to find out $\Pi(z)$
3. In general higher derivative theories this quickly gets messy!

Putting the lagrangian off-shell

- Consider plugging into the action an off-shell perturbation:

$$\phi_\omega(z) = \phi_0 \exp\left(-i\alpha \log z\right).$$

- In the near horizon limit we get

$$S_\phi^{(2)} = \int \prod_{i=1}^{d-1} dx^i \int \frac{d\omega}{2\pi} \int dz\, \frac{A_h}{2\tilde{\kappa}} \left(\frac{\omega^2}{(4\pi T)^2} - \alpha^2 \right) \frac{4\pi T}{z} \phi_0^2.$$

- On-shell lagrangian is zero as it reduces to boundary term:

$$\partial_z(\Pi_\omega(z)\phi_{-\omega}(z)) = 0 \qquad \text{(Can. mom. is constant!)}$$

- **Residue of simple pole** $\leftrightarrow \tilde{\kappa}$.

The pole method.

- We can exploit the pole to obtain $\tilde{\kappa}$!
- Pole method formulae:

$$\xi = 8\pi T \lim_{\omega \to 0} \frac{\mathrm{Res}_{z=0}\, \mathcal{L}^{(2)}_{\phi = z^{i\omega/(4\pi T)}}}{\omega^2} \qquad \textbf{Radial formula}$$

$$\xi = -8\pi T \lim_{\omega \to 0} \frac{\mathrm{Res}_{z=0}\, \mathcal{L}^{(2)}_{\phi = e^{-i\omega t}}}{\omega^2} \qquad \textbf{Time formula}.$$

- Works for any lagrangian, on any non-extremal black hole background.
- No detailed knowledge of effective action necessary: simply evaluate a covariant lagrangian on perturbed background, and extract residue.

Shear viscosity and conductivity

Important transport coefficients are η, σ.

- Shear viscosity read off from correlator of $T^{\mu\nu}$

$$G_R^{xy,xy}(\omega) = -i\int dt\,\theta(t)\langle T^{xy}(t)T^{xy}(0)\rangle e^{-i\omega t},$$
$$\eta = \lim_{\omega\to 0}\frac{\mathrm{Im}\,G_R^{xy,xy}(\omega)}{i\omega}.$$

- Shear viscosity read off from correlator of J^{μ}

$$G_R(\omega)^{x,x} = -i\int dt\,\theta(t)\langle J^{x}(t)J^{x}(0)\rangle e^{-i\omega t},$$
$$\sigma = \lim_{\omega\to 0}\frac{\mathrm{Im}\,G_R^{x,x}(\omega)}{i\omega}.$$

Shear viscosity and conductivity

- In gauge-gravity duality, T^{xy}, J^x couple to perturbations

$$\begin{aligned} dx_2 &\rightarrow dx_2 + \phi(t,z)dx_1\,, \\ A_x(t,z) &= \psi(t,z) \end{aligned}$$

- ϕ is always massless by $SO(2)$ symmetry.
- ψ is effectively massless if background is uncharged.

- Gauge-invariance guarantees action automatically depends only on differentiated perturbations.
- No need to integrate by parts to put it into required form!

Example: $(\nabla R_{abcd})^2$

- Let us compute the shear viscosity in a simple case. Take $\gamma \ll 1$:

$$\mathcal{S} = -\frac{1}{16\pi G_N} \int d^5x \sqrt{-g} \left(R + \frac{12}{L^2} + \gamma L^4 \nabla_a R_{bcde} \nabla^a R^{bcde} \right) ,$$

- Solution at $\gamma = 0$: AdS-Schwarzschild

$$ds^2 = \frac{L^2 dz^2}{4z(1-z)^2(2-z)} + \frac{r_0^2}{L^2(1-z)} \left(-z(2-z)dt^2 + \sum_i (dx_i)^2 \right)$$

- Higher derivative term implies $O(\gamma)$ correction to g_{tt}, g_{zz}
- Lowest order results only depend on g_{xx}, $\xi = g_{xx}^{(d-1)}/\kappa$.

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- Higher derivative term implies $O(\gamma)$ correction to g_{tt}, g_{zz}
- Lowest order results only depend on g_{xx}, $\xi = g_{xx}^{(d-1)}/\kappa$.
- To $\mathcal{O}(\gamma^2)$ correction is irrelevant!

Example: $(\nabla R_{abcd})^2$

- Effective action for ϕ:

$$\mathcal{L}^{(2)}_{\phi} = -\frac{1}{32\pi G_N}\left(A\phi'_{\omega}\phi'_{-\omega} + B\phi'_{\omega}\phi''_{-\omega} + C\phi''_{\omega}\phi''_{-\omega}\right.$$
$$+ \left. D\phi^{(3)}_{\omega}\phi'_{-\omega} + E\phi^{(3)}_{\omega}\phi''_{-\omega} + F\phi^{(3)}_{\omega}\phi^{(3)}_{-\omega}\right)$$

- Canonical momentum is then:

$$\Pi_{\omega}(z) = \tilde{A}\phi'_{\omega}(z) - (\tilde{B}\phi'_{\omega}(z))' + (E\phi''_{\omega}(z))''$$
$$\tilde{A} = A - \frac{1}{2}B' + \frac{1}{2}D'', \qquad \tilde{B} = C - \frac{1}{2}E' - D$$

- Plugging in the near horizon $\phi_{\omega} = z^{-i\omega/4r_0}$ gives

$$\Pi_{\omega}(0) = i\omega\frac{r_0^3}{16\pi L^3 G_N}(1-1024\gamma)\phi_0 + \mathcal{O}(\omega^2),$$
$$\Rightarrow \eta = \frac{1}{16\pi G_N}\left(\frac{r_0^3}{L^3}\right)(1-1024\gamma).$$

Example: $(\nabla R_{abcd})^2$

- Alternatively, use the pole method.
- Evaluate gravitational lagrangian on perturbed background,

$$dx_2 \rightarrow dx_2 + z^{-i\omega/4\pi T} dx_1 .$$

- Expand near horizon:

$$\mathcal{L} = \frac{1}{16\pi G_N}\left(... + \frac{(\omega r_0)^2}{8L}\frac{1-1024\gamma}{z} + \text{Regular}\right)$$

and we read off η from the residue of the simple pole.

Outline

The Wald formula for the entropy

- Recall we're considering backgrounds of type

$$\begin{aligned} ds^2 &= g_{ab}dx^a dx^b = \frac{L^2}{z} e^{2g(z)} dz^2 + g_{\mu\nu} dx^\mu dx^\nu \\ g_{\mu\nu} &= -z\, e^{2f(z)} dt^2 + e^{2\rho(z)} dx^i dx_i . \end{aligned}$$

- Wald's formula for the entropy:

$$S = \int_H d\Sigma \left(\frac{\delta \mathcal{L}}{\delta R_{abcd}} \right) \epsilon_{ab} \epsilon_{cd}$$

- For the above backgrounds we obtain the entropy density

$$s = \frac{A_h}{4\, G_N} X^{zt}{}_{zt} \Big|_{z=0}, \qquad X^{abcd} = -\frac{32\pi G_N}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta R_{abcd}}$$

- Can a similar formula be found for transport coefficients?

Two derivative case: conductivity

- Strategy: compute the simple pole in a general uncharged background.

$$S = -\frac{1}{2g_{d+1}^2}\int d^d x\, dz \sqrt{-g}\left(\frac{1}{4}M^{abcd}F_{ab}F_{cd}\right).$$

  with M^{abcd} some arbitrary tensor.
- Turn on a small perturbation $A_{x_1}(t,z) \equiv \psi(t,z)$, obtain:

$$S_\psi^{(2)} = -\frac{1}{2g_{d+1}^2}\int d^d x\, dz\, \sqrt{-g}\, g^{xx}\left(M^{zx_1}{}_{zx_1}\, g^{zz}\,\partial_z\psi\,\partial_z\psi + M^{tx_1}{}_{tx_1}\, g^{tt}\,\partial_t\psi\,\partial_t\psi\right)$$

- Plugging in near horizon solution, read off the pole. Radial and time formula give:

$$\boxed{\sigma = \frac{e^2}{g_{d+1}^2}(g_{xx})^{d-3}\, M^{zx_1}{}_{zx_1}\Big|_{z=0}} \quad \text{or} \quad \boxed{\sigma = \frac{e^2}{g_{d+1}^2}(g_{xx})^{d-3}\, M^{tx_1}{}_{tx_1}\Big|_{z=0}}$$

- Equality is insured by horizon regularity.

A formula for the shear viscosity

- We consider a general lagrangian:

$$S = -\frac{1}{16\pi G_N}\int d^d x\sqrt{-g}\,\mathcal{L}(R_{abcd}, \tilde{F}^{(q)}_{ab}, \Phi^{(r)}, \ldots)$$

- Add a shear mode perturbation to the background:

$$dx_2 \to dx_2 + A_m(x^n)dx^m,$$

- The curvatures transform as

$$\begin{aligned} R_{mnpq} &= \hat{R}_{mnpq} - \frac{3}{4}e^{2\rho}P[F_{mn}F_{pq}] \\ R_{mymy} &= \hat{R}_{mymy} + \frac{1}{4}e^{4\rho}F_{mp}F_n^{\ p} \\ R_{mnpy} &= -\frac{1}{6}e^{-\rho}\left[2\nabla_p(e^{3\rho}F_{mn}) + \nabla_n(e^{3\rho}F_{mp}) - \nabla_m(e^{3\rho}F_{np})\right] \end{aligned}$$

A formula for the shear viscosity

- We look for the simple pole in the off-shell lagrangian. Defining:

$$X^{abcd} = -\frac{32\pi G_N}{\sqrt{-g}}\frac{\delta \mathcal{L}}{\delta R_{abcd}}, \qquad Y^{abcd,efgh} = \frac{\delta X^{abcd}}{\delta R_{efgh}}$$

- The final result is

$$\boxed{\eta = \frac{A_h}{16\pi G_N}\left(2X^{zy}{}_{zy} - X^{xy}{}_{xy} - \frac{4\,e^{-2g_0}}{L^2}\left[\partial_z \alpha^{zt} - \alpha^{zt}\tilde{R}\right]\right)\Bigg|_{z=0}}$$

where

$$\begin{aligned} \tilde{R} &= \left(e^{2g_0}L^2 R + (d+3)\partial_z \rho\right) \\ \alpha^{zt} &= Y^{xz}{}_{zy}{}^{,yz}{}_{zx} - Y^{xz}{}_{zy}{}^{,yt}{}_{tx} \end{aligned}$$

Application: Gauss-Bonnet theory

- As an example, consider Gauss-Bonnet gravity in 5 dimensions

$$S = -\frac{1}{16\pi G_N}\int d^4xdz\sqrt{-g}\left(R+\frac{12}{L^2}+W\right)$$
$$W = \frac{\lambda}{2}L^2\left(R^2_{abcd}-4R^2_{ab}+R^2\right)$$

- Planar black hole solution

$$ds^2 = \frac{r_0^2}{L^2}\left(-N^2 f(u)dt^2+dx^i dx_i\right)+\frac{du^2}{4\,u^2 f(u)}$$
$$f(u) = \frac{1}{2\lambda}\left(1-\sqrt{1-4\lambda\left(1-u^2\right)}\right), \qquad N^2=\frac{1}{2}\left(1+\sqrt{1-4\lambda}\right).$$

- The Ricci scalar is given by

$$R\Big|_{z=0} = -\frac{20}{L^2}\left(1+\frac{8}{5}\lambda\right)$$

- Next step is to compute the various coefficients, X, Y.

Application: Gauss-Bonnet theory

- Computing quantities in formula yields

$$X^{xy}{}_{xy}\Big|_{z=0} = 1-4\lambda-32\lambda^2, \qquad X^{zy}{}_{zy}\Big|_{z=0} = 1-8\lambda$$

$$\partial_z \alpha^{zt}\Big|_{z=0} = 0, \qquad \alpha^{zt}\Big|_{z=0} = \frac{\lambda L^2}{4},$$

- Final result

$$\eta = \frac{A_h}{16\pi G_N}(1-4\lambda),$$

- This agrees with previous calculations in the literature.

Outline

1. Transport and gauge/gravity duality
   - Equilibrium Hydrodynamics
   - Gauge/gravity basics
   - Two-derivative case
2. Higher derivative case
   - Canonical momentum method.
   - The pole method
   - Applications
3. Wald like formulae for transport coefficients.
   - Two-derivative case
4. Universality at extremality
   - Shear viscosity
   - Conductivity

Extremal black holes

- Theories at zero temperature but finite chemical potential.
- Holographic description: extremal charged black hole (e.g. extremal AdS-RN).

$$ds^2 = \frac{L^2}{z^2}\, e^{2g(z)} dz^2 + \left(-z^2\, e^{2f(z)} dt^2 + e^{2\rho(z)} dx^i dx_i\right) ,$$

- Near horizon $AdS_2 \times R^{d-1}$ factor supported by flux:

$$\begin{aligned} ds^2 &= -v_1 \left(-z^2 d\tau^2 + \frac{dz^2}{z^2}\right) + v_2(dx^2). \\ F_{z\tau} &= Q. \end{aligned}$$

- Double pole at horizon! Problems?

Horizon regularity

- Same argument as before. Infalling observer must see regular fields

$$\partial_z \phi_0 = \pm \sqrt{-\frac{g_{zz}}{g_{tt}}} \partial_t \phi_0 = \mp \frac{i\omega}{\mu} \frac{\phi_0}{z^2},$$

- Equation of motion is completely determined and coincides with that of scalar in AdS_2:

$$\phi''(z) + \frac{2}{z^2} \phi'(z) + \frac{\omega^2}{z^4} \phi(z) = 0.$$

- Solution is

$$\phi(z) = \phi_0 \exp\left(\pm \frac{i\omega}{z} \right).$$

Zero temperature limit is continuous

- Generalized canonical momentum at the horizon:

$$\Pi(z) = -\frac{\sqrt{-g}}{\tilde{\kappa}} g^{zz} \partial_z \phi(z) = \frac{i\omega}{\tilde{\kappa}} \sqrt{-g} g^{zz} \sqrt{-\frac{g_{zz}}{g_{tt}}} \phi_0 .$$

- The nature of the pole is irrelevant.
- Zero temperature limit is **continuous**.
- Automatically implies universality of $\eta/s = 1/4\pi$ for all extremal backgrounds.

Pole method and formulae

- Since transition is continous analytic formula for η is still valid.
- Simplification because of AdS_2 near horizon:

$$\eta = \frac{v_2^3}{16\pi G_N}\left(2X^{zy}{}_{zy} - X^{xy}{}_{xy} - 8\frac{\alpha^{zt}}{v_1}\right).$$

- To use pole method, simply consider any finite temperature AdS_2! Now we have simple pole, but result is independent of this temperature.

Perturbation equations

- Background is charged: non-trivial flow for gauge perturbations.

$$\begin{aligned} a_x(t,z) &= A(z)e^{-i\omega\tau} \\ h_t^x(t,z) &= H(z)e^{-i\omega\tau} \end{aligned}$$

- Gauge constraint plus equation of motion:

$$\begin{aligned} A_x''(z) + \frac{2}{z}A_x'(z) + \frac{\omega^2 + z^2 v_2 H'(z)/v_1}{z^4}A_x(z) &= 0, \\ QA(z) + v_2 H'(z) &= 0. \end{aligned}$$

- Equation of motion for a massive scalar field with $m^2 = Q^2/(v_1 L^2)$.

$$A''(z) + \frac{2}{z}A'(z) + \frac{\omega^2 - 2z^2}{z^4}A(z).$$

Universal scaling of conductivity

- Solution to the equation with infalling boundary conditions.

$$A(z) = \exp(i\omega/z)(z - i\omega) \simeq z + \frac{\omega^2}{2z} + \frac{1}{3}\frac{i\omega^3}{z^2} + \ldots$$

- IR CFT Green's function is determined by

$$\mathcal{G}_\omega = i\omega^3.$$

- Non-analytic behaviour of full Green's function is completely fixed by this:

$$\text{Im} G_R(\omega) \propto i\omega^3 \Rightarrow \text{Re}(\sigma) \propto \omega^2.$$

- Universal scaling!
- Higher derivative corrections do not modify this result! (Proof?).
- Scaling dimension of IR CFT operator is protected.

Summary

- Transport in strongly coupled field theories using gauge-gravity duality: QGP, AdS/CMT.
- Finite coupling, N: higher derivative corrections.
- Universal near-horizon behaviour.
- Shear viscosity and DC conductivity easily computed with pole method.
- Analytic, Wald-like formulae.

Prospects and comments

- Our results are straightforwardly extended to extremal black holes (ask me in question time!).
- We can presumably rewrite our results in the language of **absorption cross-sections**. This would give

$$\sigma_{\text{abs}} = \kappa\xi = \frac{\kappa}{\tilde{\kappa}}\frac{A_h}{.}$$

- Shear viscosity $\rightarrow \sigma_{\text{abs}}$ of s-wave gravitons.
- Can our analytic formulae be covariantized? What lessons do they teach us about the relation between *n*-point functions and transport coefficients?

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Thank you!