T-duality Invariant Formalisms at the Quantum Level

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T-duality I - Overview

- T-duality is one of the most remarkable features of string theory
- Two string theories defined in different backgrounds may be physically identical
- Simplest example is the bosonic string on S^1 of radius R dual to the string on S^1 radius α'/R
- Extends to toroidal T^d compactifications with $O(d, d, \mathbb{Z})$ duality group
- T-duality is not an obvious symmetry

T-duality II - Buscher Procedure

Bosonic sigma-model in background fields

$$S = \frac{1}{2\pi\alpha'} \int d^2\sigma \, G_{ij}(X)\partial_\alpha X^i \partial^\alpha X^j + \epsilon^{\alpha\beta} B_{ij}(X)\partial_\alpha X^i \partial_\beta X^j$$

with an invariance/isometry generated by a vector k

$$\mathcal{L}_k G_{ij} = \mathcal{L}_k H = 0$$

Gauge the isometry with Lagrange multiplier for flat connection

- Recover ungauged sigma model after integrating out the Lagrange multiplier
- Integrating out the gauge field gives T-dual sigma-model
- Dilaton transformation due to path integral measure

Motivation

1. Can we better understand T-duality?
2. Can we make the T-duality symmetry manifest?
3. Possible applications of T-duality
    - String compactifications (T-folds, non-geometric backgrounds, mirror symmetry)
    - Scattering amplitudes (fermionic T-duality and AdS-CFT)
    - Supergravity (solution generation, generalised geometry)

Today we will look at Duality Invariant String Theory

Doubled Formalism I

For toroidal T^d fibrations we have the Doubled Formalism [Hull]

- Extend the fibration to a T^{2d} by doubling the coordinates

$$\mathbb{X}^I = (x^i, \tilde{x}_i)$$

- $O(d,d)$ then has a natural action

$$\mathbb{X}'^I = (\mathcal{O}^{-1})^I{}_J \mathbb{X}^J$$

where $\mathcal{O}$ preserves the $O(d,d)$ metric

$$\eta_{IJ} = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$$

- Further restrict $\mathcal{O} \in O(d,d,\mathbb{Z})$ to preserve periodicities of $\mathbb{X}^I$

Doubled Formalism II

Geometric data packaged into $O(d,d)/O(d)\times O(d)$ coset form

$$\mathcal{H}_{IJ}(y)=\begin{pmatrix} g-bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1}\end{pmatrix}$$

The $O(d,d,\mathbb{Z})$ duality transformations are now transparent

$$\mathcal{H}'=\mathcal{O}^T\mathcal{H}\mathcal{O}$$

Compare with the fractional linear transformation

$$E_{ij}=g_{ij}+b_{ij}\to(a.E+b)(cE+d)^{-1}\,,\quad \mathcal{O}=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$$

Duality transformations on the same footing as geometrical transition functions so can describe T-folds

Doubled Formalism III

Lagrangian for Doubled Formalism

$$\mathcal{L} = \frac{1}{4}\mathcal{H}_{IJ}(y) d\mathbb{X}^I \wedge * d\mathbb{X}^J + \frac{1}{2}\Omega_{IJ} d\mathbb{X}^I \wedge d\mathbb{X}^J + \mathcal{L}(y)$$

- Unconventional normalisation of kinetic term
- Topological term - not needed for today
- Standard action for base coordinates y
- Constraint for correct number of degrees of freedom

$$d\mathbb{X}^I = \eta^{IJ}\mathcal{H}_{JK} * d\mathbb{X}^K$$

- Classically equivalent to standard string

Chirality Constraints

Consider simplest case $d = 1$ i.e. a circle of radius R then

$$\mathcal{L} = \frac{1}{4}R^2 dX \wedge *dX + \frac{1}{4}R^{-2} d\tilde{X} \wedge *d\tilde{X}$$

Change basis

$$P = RX + R^{-1}\tilde{X} \ , \quad Q = RX - R^{-1}\tilde{X} \ ,$$

Then

$$\mathcal{L} = \frac{1}{8}dP \wedge *dP + \frac{1}{8}dQ \wedge *dQ$$

Constraint becomes a chirality constraint

$$\partial_- P = 0 \ , \quad \partial_+ Q = 0$$

Implementing The Constraints

- Pasti-Sorokin-Tonin procedure allows a Lorentz covariant way to implement chirality constraints at the expense of introducing some auxiliary fields (closed 1-forms)
- PST symmetry allows gauge fixing of auxiliary fields u, v to give Floreanini-Jackiw action

$$S = \frac{1}{4}\int d^2\sigma \left[\partial_1 P \partial_- P - \partial_1 Q \partial_+ Q\right]$$

- Equivalent to Tseytlin's duality invariant string

$$S = \frac{1}{2}\int d^2\sigma \left[-(R\partial_1 X)^2 - (R^{-1}\partial_1 \tilde{X})^2 + 2\partial_0 X \partial_1 \tilde{X}\right]$$

Quantum Aspects of Duality Invariant String

- What is the quantum behaviour of the duality invariant string?
  - Partition function (Berman, Copland; Chowdhury)
  - Canonical Quantisation (Hackett-Jones, Moutsopoulos)
  - Doubled string field theory (Hull, Zwiebach)
- What are the beta-functions and how do they constrain the geometry?
  - Weyl anomaly of string theory gives equations of motion of Supergravity

Background Field Expansion I

We work with the Doubled action in Tseytlin form:

$$\mathcal{L} = -\frac{1}{2}\mathcal{H}_{IJ}(y)\partial_1\mathbb{X}^I\partial_1\mathbb{X}^J + \frac{1}{2}\eta_{IJ}\partial_0\mathbb{X}^I\partial_1\mathbb{X}^J + \mathcal{L}(y)$$

Background field expansion

- Covariant expansion in the tangent ξ to the geodesic between classical and quantum values
- Expand to quadratic order in ξ
- Calculate effective action by exponentiation and Wick contraction
- Regulate UV divergences produce $1/\epsilon$ poles for 1-loop beta-function

Background Field Expansion II

Non-Lorentz invariant structure complicates matters upon Wick contraction since

$$\lim_{z\to 0}\langle \xi^I(z)\xi^J(0)\rangle \sim \frac{1}{\epsilon}\mathcal{H}^{IJ} + \theta\eta^{IJ}$$

Two sources of anomalies

1. Weyl anomaly parametrised by the UV divergent quantity $1/\epsilon$ related to scale of z
2. Lorentz anomaly parametrised by finite quantity θ related to the argument of z

Background Field Expansion III

Then to find the effective action

$$S_{eff} =< S_{int} > + < (S_{int})^2 > + \dots$$

One encounters strange contractions like

$$\begin{aligned} \langle \xi^A \partial_0 \xi^B \xi^C \partial_0 \xi^D \rangle \qquad & \sim -\frac{1}{2}(\mathcal{H}^{A[C}\mathcal{H}^{D]B} + 3\eta^{A[C}\eta^{D]B})\frac{1}{\epsilon} \\ & -(\mathcal{H}^{A[C}\eta^{D]B} + \eta^{A[C}\mathcal{H}^{D]B})\Theta \ , \end{aligned}$$

And again must keep track of both Lorentz and Weyl anomaly contributions

Beta functions of the Duality Invariant String

1. There is no Lorentz anomaly at one-loop (non-trivial cancellations)
2. The Weyl anomaly vanishes providing the background fields obey a consistent set of equations:

$$\begin{aligned}
\beta_{IJ} &= -\frac{1}{2}\hat{\nabla}\mathcal{H}_{IJ} + \frac{1}{2}\left(\hat{\nabla}_a\mathcal{H}\mathcal{H}^{-1}\hat{\nabla}^a\mathcal{H}\right)_{IJ} - \frac{1}{2}\hat{\nabla}_a\mathcal{H}_{IJ}\hat{\nabla}^a\Phi \\
\beta_{ab} &= \hat{R}_{ab} + \frac{1}{8}\hat{\nabla}_a\mathcal{H}_{IJ}\hat{\nabla}_b\mathcal{H}^{IJ} - \hat{\nabla}_a\hat{\nabla}_b\Phi \\
\beta_{\Phi} &= \frac{\alpha'}{2}\left(-2\hat{\nabla}^2\Phi - (\hat{\nabla}\Phi)^2 + \frac{1}{8}\hat{\nabla}_a\mathcal{H}_{IJ}\hat{\nabla}^a\mathcal{H}^{IJ}\right)
\end{aligned}$$

- These equations have a space-time interpretation as the equations of motion of a toroidally reduced gravity theory!

Generalised T-duality

- Generalisation of T-duality to non-abelian isometries [de la Ossa, Quevedo]
    - Loss of isometry after Buscher dualisation
    - Dualisation procedure invalid on higher genus world sheets [Giveon, Rocek]
- Nonetheless expect cases for which T-duality can be generalised and these backgrounds of are particular interest for compactification
- Poisson–Lie T-duality is a key generalisation of T-duality
    - Beautiful mathematical structure: Drinfeld Double
    - Manifestly duality invariant formalism [Klimcik, Severa]

Poisson Lie T-duality Invariant Theory I

Key mathematical structure is the Drinfeld Double

- Lie-algebra $d = g \oplus \tilde{g}$
- Sub algebras g and $\tilde{g}$ are maximally isotropic with respect to inner product $\eta_{AB} = \langle T_A | T_B \rangle$
- Write generators as $T_A = (T_a, \tilde{T}^a)$ and commutators:

$$
\begin{aligned}
[T_a, T_b] &= i f_{ab}{}^c T_c \ , \\
[\tilde{T}^a, \tilde{T}^b] &= i \tilde{f}^{ab}{}_c \tilde{T}^c \ , \\
[T_a, \tilde{T}^b] &= i \tilde{f}^{bc}{}_a T_c - i f_{ac}{}^b \tilde{T}^c \ .
\end{aligned}
$$

- Doubled torus $T^{2d} = T^d \oplus T^d$ is an example

Poisson Lie T-duality Invariant Theory II

Klimcik and Severa proposed a duality invariant theory whose action can be written as a chiral WZW model together with an extra term:

$$S = \frac{1}{2}\int_\Sigma d^2\sigma \langle h^{-1}\partial_1 h | h^{-1}\partial_0 h\rangle - \frac{1}{2}\int_\Sigma d^2\sigma \langle h^{-1}\partial_1 h | \mathcal{H} | h^{-1}\partial_1 h\rangle + \frac{1}{12}\int_B d^3\sigma \epsilon^{\alpha\beta\gamma}\langle h^{-1}\partial_\alpha h | [h^{-1}\partial_\beta h, h^{-1}\partial_\gamma h]\rangle$$

- h maps the worldsheet into the group of Drinfeld Double
- $\mathcal{H}$ is a constant matrix and contains d^2 parameters specifying the theory and $\mathcal{H}_{AB} = \langle T_A | \mathcal{H} | T_B\rangle$ is the $O(d,d)$ coset representative we had before
- Non-manifestly Lorentz covariant structure as before

Poisson Lie T-duality Invariant Theory III

- By parametrising the group element of the double in two inequivalent ways as $h = g\tilde{g}$ and $h = \tilde{g}g$ one can solve some constraint type equations for g or $\tilde{g}$ leaving a Lorentz invariant action for the remaining fields
- The resultant geometries are in general extremely complicated
- Vector fields of the target space obey a group structure $[K_a, K_b] = f_{ab}{}^c K_c$ and do not generate a strict isometry but instead result in $\mathcal{L}_{K_a} E_{ij} = \mathcal{L}_{K_a}(g_{ij} + b_{ij}) = \tilde{f}^{bc}{}_a K^k_b K^l_c E_{ki} E_{lj}$
- Evidence for Poisson Lie T-duality
  - There exists a canonical equivalence between the dual sigma models in phase space [Sfetsos]
  - Pairs of dual models have equivalent systems of RG equations for the moduli contained in $\mathcal{H}$ [Sfetsos, Siampos]

Background Field Expansion I

The Poisson–Lie duality invariant action can be written as

$$\mathcal{L} = -\frac{1}{2}\mathcal{H}_{AB} L^A_I L^B_J \partial_1 \mathbb{X}^I \partial_1 \mathbb{X}^J + \frac{1}{2}\left(\eta_{AB} + B_{AB}\right) L^A_I L^B_J \partial_0 \mathbb{X}^I \partial_1 \mathbb{X}^J$$

- Dressed by the left-invariant forms $L^A(\mathbb{X}) = L^A_I(\mathbb{X}) d\mathbb{X}^I$
- Maurer-Cartan equations $dL^A = -\frac{1}{2} f_{BC}{}^A L^B \wedge L^C$
- Field strength $H_{IJK} = (dB)_{IJK} = f_{ABC} L^A_I L^B_J L^C_K$
- Spin-connection and field strength (Torsion) are proportional

Background Field Expansion II

As before expand to second order quantum fluctuations ξ^A and find

$$S^{(2)} = S_{kin} + S_{int}$$

Kinetic term for fluctuations same as abelian case described before hence $\langle \xi^A \xi^B \rangle \sim \frac{1}{\epsilon}\mathcal{H}^{AB} + \theta\eta^{AB}$ and interaction terms

$$S_{int} = \frac{1}{2}\int d\sigma d\tau \left(I_{AB}\xi^A\xi^B + J_{AB}\xi^A\partial_1\xi^B + K_{AB}\xi^A\partial_0\xi^B \right) \ ,$$

with

$$I_{AB} = -L_1^C L_1^D \left[f_{AC}{}^E f_{BD}{}^F \mathcal{H}_{EF} + (2 f_{AF}{}^E \mathcal{H}_{EC} + f_{AC}{}^E \mathcal{H}_{EF}) f_{BD}{}^F \right] \ ,$$
$$J_{AB} = (f_{BA}{}^C \mathcal{H}_{CE} + 2 f_{EA}{}^C \mathcal{H}_{CB}) L_1^E \ ,$$
$$K_{AB} = -f_{ABC} L_1^C \ .$$

Beta-function for Poisson-Lie Duality Invariance

1. Lorentz anomaly cancels
2. The theory is renormalisable (i.e. absorb counter terms into redefinition of $\mathcal{H}$)
3. Concise expression for RG equation [also Avramis et al.]

$$\frac{d\mathcal{H}_{AB}}{dt} = \frac{1}{4}(\mathcal{H}_{AC}\mathcal{H}_{BF} - \eta_{AC}\eta_{BF})(\mathcal{H}^{KD}\mathcal{H}^{HE} - \eta^{KD}\eta^{HE})f_{KH}{}^{C} f_{DE}{}^{F} ,$$

   with $t = \ln m$ where m is the energy scale.
4. Agrees with the RG found in the T-dual pairs for specific examples
5. This has been extended to show agreement in general (laborious but easy)
6. Process of constraining and quantising commute

Hidden Lorentz Invariance

A key feature of these duality invariant theories was a lack of manifest Lorentz invariance

- However this is illusory - there is classical Lorentz invariance
- And no Lorentz anomaly at the quantum level
- In the abelian case this was an artifact of gauge fixing choice

Can we understand the origin of the Lorentz invariance better?

Hidden Lorentz Invariance

We considered an arbitrary general sigma model of the form

$$S = \frac{1}{2}\int d\sigma d\tau \left(C_{MN}(X)\partial_0 X^M \partial_1 X^N + M_{MN}(X)\partial_1 X^M \partial_1 X^N \right) ,$$

Not invariant Lorentz transformations

$$\delta X^M = -\sigma \partial_\tau X^M - \tau \partial_\sigma X^M ,$$

However if the generalised (torsionful) spin-connection defined by C_{MN} is zero then

1. Equation of motion becomes first order: $0 = S_{MN}\partial_0 X^N + M_{MN}\partial_1 X^N$ where $S_{MN} = \frac{1}{2}(C_{MN} + C_{NM})$
2. Action is on-shell invariant provided that $M_{MP}S^{PQ}M_{QN} = S_{MN}$

These conditions are exactly solved by the group geometry of the duality invariant theory (but allows more general group structure).

Coset Constructions I

- Interesting classes of sigma-models have been obtained from WZW through the coset construction
- In this one considers a subgroup $H \subset G$ and gauges its action in the WZW model
- By solving for the non-propagating gauge fields one finds resultant exact CFT's defined on interesting spaces
- Classic example: Witten's cigar 2d black hole defined as $SL(2, \mathbb{R})/U(1)$

Do the theories we have been considering admit new coset constructions?

Coset Constructions II

The first requirement is that we can gauge the theory. For the WZW pieces that is unchanged however we need that

$$S_{NL}[h] = -\frac{1}{2}\int_{\Sigma} d^2\sigma \langle h^{-1}\partial_1 h|\mathcal{H}|h^{-1}\partial_1 h\rangle$$

can be gauged. Obvious approach is to try

$$S_{GNL}[h, A] = -\frac{1}{2}\int_{\Sigma} d^2\sigma \langle h^{-1}D_1 h|\mathcal{H}|h^{-1}D_1 h\rangle$$

Gauge invariance is not automatic! Constrains the choice of $\mathcal{H}$:

$$0 = f_{Ai}{}^{E}\mathcal{H}_{EB} + f_{Bi}{}^{E}\mathcal{H}_{EA}$$

in which $i = 1 \ldots \dim H$ and $A = 1 \ldots \dim G$

Coset Constructions III

- Viewed as a truncation of parameter space the gauge invariance conditions

$$0 = f_{A_i}{}^{E}\mathcal{H}_{EB} + f_{B_i}{}^{E}\mathcal{H}_{EA}$$

  is preserved by the RG equations
- The projection of $\mathcal{H}$ into the subgroup completely decouples
- The effective geometries that arise depend on $\dim G - \dim H$ coordinates and seem to be consistent, if complicated, sigma models

Conclusions

- Duality invariant frameworks are an interesting approach to string theory
- They shed light on the nature of duality and have applications to e.g. non-geometric backgrounds
- These frameworks seem to be consistent at a quantum level
- Some promising progress in constructing new theories through coset constructions

Many interesting directions for more research

- Aspects of compactification
- Extension to U-duality and perhaps, M-theory
- Application to AdS-CFT and fermionic T-duality