Pohlmeyer Reduction

and superstrings in $AdS_5 \times S^5$

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"Pohlmeyer reduction":

reformulation of gauge-fixed $AdS_5 \times S^5$ superstring

in terms of current-type variables

preserving 2d Lorentz invariance:

way towards exact solution of quantum $AdS_5 \times S^5$ superstring?

Some history

K. Pohlmeyer

Integrable Hamiltonian Systems

and Interactions through Quadratic Constraints.

Commun.Math.Phys. 46, 207 (1976)

[Cited 405 times in Spires]

Abstract: O(n)-invariant classical relativistic field theories in one time and one space dimension with interactions that are entirely due to quadratic constraints are shown to be closely related to integrable Hamiltonian systems.

Discovery of integrability (existence of infinite number of conservation laws) of *classical* O(3) sigma model via relation to sine-Gordon theory. Generalization to O(4) sigma model related to complex sine-Gordon theory. Integrability of O(n) model: Backlund transformations to generate solutions and higher conserved charges.

Extensions and generalizations:

• M. Luscher, K. Pohlmeyer, "Scattering of Massless Lumps and Non-local Charges in the Two-Dimensional Classical Nonlinear Sigma Model." Nucl.Phys. B137, 46 (1978) [Cited 246 times in Spires]

Finite-energy solutions of the field equations of the non-linear sigma-model are shown to decay asymptotically into massless lumps. By means of a linear eigenvalue problem connected with the field equations we then find an infinite set of dynamical conserved charges.

• K. Pohlmeyer and K. H. Rehren, "Reduction Of The Two-Dimensional O(N) Nonlinear Sigma Model," J. Math. Phys. 20, 2628 (1979).

We reduce the field equations of the two-dimensional O(n) nonlinear sigma-model to relativistic O(n) covariant differential equations involving n scalar fields.

• H. Eichenherr and K. Pohlmeyer, "Lax Pairs For Certain Generalizations Of The Sine-Gordon Equation," Phys. Lett. B 89, 76 (1979).

We derive the one-parameter family of isospectral linear eigenvalue problems which is the basic tool for treating certain generalized sine-Gordon equations by the inverse scattering method.

But why reduction relevant? Assumed classical 2d conformal invariance which is broken at quantum level.

• “The existence of an infinite number of conservation laws for classical O(N) model has been discovered by Pohlmeyer. However, since the quantum vacuum of the model appears to be crucially different from the classical one, the relation between the classical conservation laws and quantum ones cannot be straightforward. In particular, the conformal invariance of the classical theory which is of essential use in Pohlmeyer’s derivation is surely broken in a quantum case due to coupling constant renormalization. The presence of higher conservation laws in quantum O(N) model has been shown by Polyakov. Here we present briefly Polyakov’s derivation.”

[A. B. Zamolodchikov and A. B. Zamolodchikov, “Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models,” Annals Phys. 120, 253 (1979).]

• “It has been shown by Pohlmeyer that on the classical level the theory is completely integrable by the inverse scattering method. We shall show that this result has its non-trivial quantum counterpart.”

[A. M. Polyakov,“Hidden Symmetry Of The Two-Dimensional Chiral Fields,” Phys. Lett. B 72 (1977) 224.]

Pohlmeyer reduction (PR) was not used much in the next 20 years...

Technical issue:
equations of dim higher dim > 3 reduced models
(e.g. for $S^n = SO(n+1)/SO(n), \; n > 2$)
were apparently non-Lagrangian
Resolution suggested in:
[K. Bakas, Q. Park and I. Shin, “Lagrangian Formulation
of Symmetric Space sine-Gordon Models,” 1996]
$S^n = SO(n+1)/SO(n)$ sigma model is classically equivalent
to an integrable massive theory:
$G/H = SO(n)/SO(n-1)$ gauged WZW model + potential term

Fully justified/generalized recently:
[M. Grigoriev and A.T., “Pohlmeyer reduction of $AdS_5 \times S^5$
superstring sigma model.” (2008);
J. Miramontes, “Pohlmeyer reduction revisited,” 2008]

PR became important in the context of string theory:

Technical tool: classical string solutions

• construction of *classical* string solutions in constant-curvature spaces like de Sitter and anti de Sitter
[Barbashov, Nesterenko, 1981; de Vega, Sanchez, 1993]

• construction of *classical* string solutions in $AdS_5 \times S^5$ representing semiclassical closed string states in AdS/CFT context
[Hofman, Maldacena, 2006; Dorey et al, 2006; Jevicki, Spradlin, Volovich et al, 2007; ..., Hoare, Iwashita, AT, 2009; Hollowood, Miramontes, 2009; ...]

• construction of euclidean open-string world-surfaces related to $N = 4$ SYM scattering amplitudes at strong coupling
[Alday, Maldacena, 2009; Alday, Gaiotto, Maldacena, 2009; Dorn et al, 2009; Jevicki, Jin, 2009, ...]

Essential idea: reformulation/solution of quantum string theory

Quantum $AdS_5 \times S^5$ string theory is UV finite so PR
– reformulation in terms of integrable massive theory –
may lead to an equivalent theory also at quantum level
[Grigoriev and A.T, 2007; Mikhailov and Schafer-Nameki, 2007]

Advocated as a way to exact solution of $AdS_5 \times S^5$ superstring
• proof of UV finiteness of the reduced theory
[Roiban and A.T., 0902.2489]
• semiclassical expansion and relation between 1-loop
quantum partition functions of string theory and reduced theory
[Hoare, Iwashita and A.T., 0906.3800]
• derivation of tree-level S-matrix of reduced theory and its
similarity with $AdS_5 \times S^5$ magnon S-matrix
[Hoare and A.T., 0912.2958]

Pohlmeyer reduction: bosonic coset models

Original example: S^2-sigma model $\longrightarrow$ Sine-Gordon theory

$$L = \partial_+ X^m \partial_- X^m - \Lambda(X^m X^m - 1)\,, \qquad m = 1, 2, 3$$

Equations of motion:

$$\partial_+\partial_- X^m + \Lambda X^m = 0\,, \qquad \Lambda = \partial_+ X^m \partial_- X^m\,, \qquad X^m X^m = 1$$

Stress tensor: $\mathrm{T}_{\pm\pm} = \partial_\pm X^m \partial_\pm X^m$

$$\mathrm{T}_{+-} = 0\,, \qquad \partial_+ \mathrm{T}_{--} = 0\,, \qquad \partial_- \mathrm{T}_{++} = 0$$

implies $\mathrm{T}_{++} = f(\sigma_+)$, $\mathrm{T}_{--} = h(\sigma_-)$
using the conformal transformations $\sigma_\pm \to F_\pm(\sigma_\pm)$ can set

$$\partial_+ X^m \partial_+ X^m = \mu^2\,, \qquad \partial_- X^m \partial_- X^m = \mu^2\,, \qquad \mu = \text{const}\ .$$

3 unit vectors in 3-dimensional Euclidean space:

$$X^m\,, \qquad X^m_+ = \mu^{-1}\partial_+ X^m\,, \qquad X^m_- = \mu^{-1}\partial_- X^m\,,$$

X^m is orthogonal to X^m_+ and X^m_- ($X^m \partial_\pm X^m = 0$)
remaining $SO(3)$ invariant quantity is scalar product

$$\partial_+ X^m \partial_- X^m = \mu^2 \cos 2\varphi$$

then $\quad \partial_+ \partial_- \varphi + \frac{\mu^2}{2} \sin 2\varphi = 0$
following from sine-Gordon action (Pohlmeyer, 1976)

$$\widetilde{L} = \partial_+ \varphi \partial_- \varphi + \frac{\mu^2}{2} \cos 2\varphi$$

2d Lorentz invariant despite explicit constraints
Classical solutions and integrable structures
(Lax pair, Backlund transformations, etc) are directly related
e.g., SG soliton mapped into rotating string on S^2:
“giant magnon” in the $J = \infty$ limit (Hofman, Maldacena 06)

Analogous construction for S^3 model gives

Complex sine-Gordon model (Pohlmeyer; Lund, Regge 76)

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \cot^2\varphi\,\partial_+\theta\partial_-\theta + \frac{\mu^2}{2}\cos 2\varphi$$

φ, θ are $SO(4)$-invariants:

$\mu^2 \cos 2\varphi = \partial_+ X^m \partial_- X^m$

$\mu^3 \sin^2\varphi\,\partial_\pm\theta = \mp\frac{1}{2}\epsilon_{mnkl} X^m \partial_+ X^n \partial_- X^k \partial_\pm^2 X^l$

In the case of AdS_2 or AdS_3:

replace $\sin\varphi \to \sinh\phi$, etc.

Reduced eqs for $d > 3$ are non-Lagrangian (but see below)

String-theory interpretation: string on $R_t \times S^n$

conformal gauge plus $t = \mu\tau$ to fix conformal diffeomorphisms:

$\partial_\pm X^m \partial_\pm X^m = \mu^2$ are Virasoro constraints

e.g., reduced theory for string on $R_t \times S^3$

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \cot^2\varphi\,\partial_+\theta\partial_-\theta + \frac{\mu^2}{2}\cos 2\varphi$$

Similar construction for AdS_n case,

i.e. string on $AdS_n \times S^1_\psi$ with $\psi = \mu\tau$

e.g., reduced theory for string on $AdS_3 \times S^1$

$$\widetilde{L} = \partial_+\phi\partial_-\phi + \coth^2\varphi\,\partial_+\chi\partial_-\chi - \frac{\mu^2}{2}\cosh 2\phi$$

Comments:

- Virasoro constraints are solved by a special choice of variables related nonlocally to original coordinates
- Although the reduction is not explicitly Lorentz invariant the resulting Lagrangian turns out to be 2d Lorentz invariant
- The reduced theory is formulated in terms of manifestly $SO(n)$ invariant variables: “blind” to original global symmetry
- reduced theory is equivalent to the original theory as integrable system: the respective Lax pairs are gauge-equivalent
- PR may be thought of as a formulation in terms of physical d.o.f. – coset space analog of flat-space l.c. gauge (where 2d Lorentz symmetry is unbroken, but broken in curved space)

PR for string in AdS_d

solve Virasoro just for AdS_d stress tensor – no extra S^1
[de Vega, Sanchez 93; Jevicki et al 07]
string in AdS_d (in conformal gauge)
$Y \cdot Y = -Y_{-1}^2 - Y_0^2 + Y_1^2 + ... + Y_{d-1}^2 = -1$

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \Big[\partial Y \cdot \bar{\partial} Y + \Lambda(\sigma, \tau)(Y \cdot Y + 1)\Big]$$

$$\partial \bar{\partial} Y - (\partial Y \cdot \bar{\partial} Y) Y = 0$$

$$z = \tfrac{1}{2}(\sigma - \tau),\ \bar{z} = \tfrac{1}{2}(\sigma + \tau),\ \ \partial = \partial_\sigma - \partial_\tau,\ \bar{\partial} = \partial_\sigma + \partial_\tau$$

$$\partial Y \cdot \partial Y = \bar{\partial} Y \cdot \bar{\partial} Y = 0$$

New $SO(2, d-1)$ invariant variables to solve Virasoro algebraically:
introduce basis vectors

$$e_i = (Y, \partial Y, \bar{\partial} Y, B_4, \cdots, B_{d+1}), \qquad i = 1, 2, ..., d+1,$$

$$B_i \cdot B_j = \delta_{ij}, \quad B_i \cdot Y = B_i \cdot \partial Y = B_i \cdot \bar{\partial} Y = 0$$

Then define the scalar α and two sets of auxiliary fields

$$\alpha(z,\bar{z}) \equiv \ln(\partial Y \cdot \bar{\partial} Y), \quad u_i \equiv B_i \cdot \bar{\partial}^2 Y, \quad v_i \equiv B_i \cdot \partial^2 Y$$

get new form of equations of motion

$$\partial\bar{\partial}\alpha - e^{\alpha} - e^{-\alpha} \sum_{i=4}^{d+1} u_i v_i = 0,$$

$$\partial u_i = \sum_{j \neq i} (B_j \cdot \partial B_i) u_j, \qquad \bar{\partial} v_i = \sum_{j \neq i} (B_j \cdot \bar{\partial} B_i) v_j$$

case of AdS_3: one vector B_4, i.e. $\partial u = 0$, $\bar{\partial} v = 0$ and

$$\partial\bar{\partial}\alpha - e^{\alpha} - e^{-\alpha} u(\bar{z}) v(z) = 0$$

get standard sinh-Gordon eq. $\partial\bar{\partial}\widehat{\alpha} - \sinh\widehat{\alpha} = 0$ by

$$\alpha(z,\bar{z}) = \widehat{\alpha}(z,\bar{z}) + \ln\sqrt{-u(\bar{z})v(z)}$$

$$d\bar{z}' = \sqrt{2u(\bar{z})}\, d\bar{z}, \qquad dz' = \sqrt{-2v(z)}\, dz$$

higher-dim cases: related to Toda-type equations

useful for constructing various classical string solutions

employing inverse scattering method

(spiky strings; euclidean open-string surfaces, etc.)

AdS_4: eqs can be reduced to B_2 Toda system

$$\partial\bar{\partial}\widehat{\alpha} - e^{\widehat{\alpha}} + e^{-\widehat{\alpha}} \cos\beta = 0, \qquad \partial\bar{\partial}\beta - e^{-\widehat{\alpha}} \sin\beta = 0$$

PR for bosonic F/G-coset model

PR theory for string on $F/G \times R_t$:

G/H gauged WZW model + integrable potential

F/G-coset sigma model: symmetric space

$$\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{g} \,, \qquad [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g} \,, \qquad [\mathfrak{g}, \mathfrak{p}] \subset \mathfrak{p} \,, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{g}$$

$$J = f^{-1} df = \mathcal{A} + P \,, \qquad \mathcal{A} \in \mathfrak{g} \,, \qquad P \in \mathfrak{p} \,.$$

$$L = -\mathrm{Tr}(P_+ P_-) \,, \qquad f \in F$$

G gauge transformations $f \to f g$;

global F-symmetry: $f \to f_0 f$, $f_0 \in F$;

classical conformal invariance

$J = \mathcal{A} + P$ as fundamental variables

$D_+ P_- = 0 \,, \quad D_- P_+ = 0 \,, \qquad D = d + [\mathcal{A}, \ ]$ – EOM

$D_- P_+ - D_+ P_- + [P_+, P_-] + \mathcal{F}_{+-} = 0$ – Maurer-Cartan

$\mathrm{Tr}(P_+ P_+) = -\mu^2 \,, \quad \mathrm{Tr}(P_- P_-) = -\mu^2$ – Virasoro

Main idea: first solve EOM and Virasoro and then MC

special choice of G gauge condition and conformal diffs. $\to$

find reduced action giving eqs. resulting from MC

gauge fixing that solves the first Virasoro constraint

$$P_+ = \mu\, T = \text{const}\ , \qquad T \in \mathfrak{p} = \mathfrak{f} \ominus \mathfrak{g}, \qquad \text{Tr}(TT) = -1$$

choice of special element $T \to$ decomposition of algebra of F:

$$\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{g}\,, \qquad \mathfrak{p} = T \oplus \mathfrak{n}\,, \qquad \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}\,, \qquad [T, \mathfrak{h}] = 0\,,$$

$\mathfrak{h}$ is a centraliser of T in $\mathfrak{g}$

second Virasoro constraint is solved by

$$P_- = \mu\, g^{-1} T g\ , \qquad g \in G$$

EOM $D_- P_+ = 0$ is solved by $\mathcal{A}_- = (\mathcal{A}_-)_{\mathfrak{h}} \equiv A_-$

EOM $D_+ P_- = 0$ is solved by $\mathcal{A}_+ = g^{-1}\partial_+ g + g^{-1} A_+ g$

Thus new dynamical variables

$$G\text{-valued } g\ , \qquad \mathfrak{h}\text{-valued} \quad A_+,\ A_-, \qquad [T, A_\pm] = 0$$

remaining Maurer-Cartan eqs on $g, A_\pm$ follow from
G/H gWZW action with potential:

$$L = -\frac{1}{2}\mathrm{Tr}(g^{-1}\partial_+ g g^{-1}\partial_- g) + \text{WZ term}$$

$$-\mathrm{Tr}\Big(A_+\,\partial_- g g^{-1} - A_-\,g^{-1}\partial_+ g - g^{-1}A_+ g A_- + A_+ A_-\Big)$$

$$-\mu^2 \mathrm{Tr}(T g^{-1} T g)$$

Pohlmeyer-reduced theory for F/G coset sigma model
[Bakas, Park, Shin 95; Grigoriev, AT 07]
$\equiv$ PR theory for string on $R_t \times F/G$ or $F/G \times S^1_\psi$
equivalent eqs of motion; equivalent integrable structure (Lax pairs)
special case of non-abelian Toda theory:
"symmetric space Sine-Gordon model"
[Hollowood, Miramontes et al 96]
A_+, A_-: integrate out or gauge fix

Reduced equation of motion in the "on-shell" gauge $A_\pm = 0$:

Non-abelian Toda equations:

$$\partial_-(g^{-1}\partial_+ g) - \mu^2[T, g^{-1}Tg] = 0\ ,$$

$$(g^{-1}\partial_+ g)_{\mathfrak{h}} = 0\ , \qquad (\partial_- g g^{-1})_{\mathfrak{h}} = 0\ .$$

$F/G = SO(n+1)/SO(n) = S^n$: $\quad G/H = SO(n)/SO(n-1)$

parametrize g by k_m, $\quad \sum_{1=1}^n k_l k_l = 1$

get (in general non-Lagrangian) EOM for k_m

$$\partial_-\Big(\frac{\partial_+ k_\ell}{\sqrt{1-\sum_{m=2}^n k_m k_m}}\Big) = -\mu^2 k_\ell\ , \qquad \ell = 2, \ldots, n\ .$$

Linearising around the vacuum $g = 1 \quad (k_1 = 1,\ k_\ell = 0)$

$$\partial_+\partial_- k_\ell + \mu^2 k_\ell + O(k_\ell^2) = 0$$

massive spectrum: non-trivial S-matrix with H global symmetry?

$F/G = SO(n+1)/SO(n) = S^n$:

parametrization of g in Euler angles (gauge fixing)

$g = e^{T_{n-2}\theta_{n-2}} ... e^{T_1\theta_1} e^{2T\varphi} e^{T_1\theta_1} ... e^{T_{n-2}\theta_{n-2}}$

integrating out $H = SO(n-1)$ gauge field $A_\pm$

leads to reduced theory that generalizes SG and CSG

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + G_{pq}(\varphi,\theta)\partial_+\theta^p\partial_-\theta^q + \frac{\mu^2}{2}\cos 2\varphi$$

gWZW for $G/H = SO(n)/SO(n-1)$:

$$ds^2_{n=2} = d\varphi^2 \ , \qquad ds^2_{n=3} = d\varphi^2 + \cot^2\varphi \, d\theta^2$$

$$ds^2_{n=4} = d\varphi^2 + \cot^2\varphi \, (d\theta_1 + \cot\theta_1 \tan\theta_2 \, d\theta_2)^2 + \tan^2\varphi \, \frac{d\theta_2^2}{\sin^2\theta_1}$$

and similar for $n = 5$

Bosonic strings on $AdS_n \times S^n$

straightforward generalization:

$L = \mathrm{Tr}(P_+^A P_-^A) - \mathrm{Tr}(P_+^S P_-^S)\,,$

$\mathrm{Tr}(P_\pm^S P_\pm^S) - \mathrm{Tr}(P_\pm^A P_\pm^A) = 0$

fix conformal symmetry by

$\mathrm{Tr}(P_\pm^S P_\pm^S) = \mathrm{Tr}(P_\pm^A P_\pm^A) = -\mu^2$

direct sum PR systems for S^n and AdS_n

linked by Virasoro – common μ

e.g. for string in $F/G = AdS_2 \times S^2$:

$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \partial_+\phi\partial_-\phi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi)$

for string in $F/G = AdS_3 \times S^3$:

$$\widetilde{L} = (\partial\varphi)^2 + \cot^2\varphi\,(\partial\theta)^2 + (\partial\phi)^2 + \coth^2\phi\,(\partial\chi)^2 \\ + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi)$$

String Theory in $AdS_5 \times S^5$

bosonic coset $\frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$

generalized to GS string: supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$

$$S = T \int d^2\sigma \big[G_{mn}(x)\partial x^m \partial x^n + \bar{\theta}(D + F_5)\theta\partial x + \bar{\theta}\theta\bar{\theta}\theta\partial x\partial x + ... \big],$$

tension $T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$

Conformal invariance: $\beta_{mn} = R_{mn} - (F_5)^2_{mn} = 0$

Classical integrability of coset model (Pohlmeyer et al)

applies also to classical $AdS_5 \times S^5$ superstring

Quantum integrability:

explicit 1- and 2-loop computations

and comparison to Bethe ansatz

[work of last 8 years]

Aims:

solve string theory in $AdS_5 \times S^5$

use conformal invariance,

global (super)symmetry and integrability

find S-matrix and justify Bethe Ansatz for the spectrum

from first principles

then understand the theory in finite volume:

spectrum of closed string theory from TBA

would constitute proof of AdS/CFT

Green-Schwarz superstring:

Superstring in curved type II supergravity background

$\int d^2\sigma\, G_{MN}(Z)\partial Z^M \partial Z^N + \ldots\,, \qquad Z^M = (x^m, \theta^I_\alpha)$

$m = 0, 1, \ldots 9, \quad \alpha = 1, 2 \ldots, 16, \quad I = 1, 2$

Explicit form of action is generally hard to find

$AdS_5 \times S^5$: coset space symmetry facilitates explicit construction

Algebraic construction of unique κ-invariant action in flat space

$R^{1,9} = \frac{F}{G} = \frac{\text{Poincare}}{\text{Lorentz}}$

Flat superspace = $\frac{\widehat{F}}{G} = \frac{\text{SuperPoincare}}{\text{Lorentz}}$

structure of action is fixed by superPoincare algebra $(\mathcal{P}, \mathcal{M}, \mathcal{Q})$

$[\mathcal{P}, \mathcal{M}] \sim \mathcal{P}, \;\; [\mathcal{M}, \mathcal{M}] \sim \mathcal{M}, \;\; [\mathcal{M}, \mathcal{Q}] \sim \mathcal{Q}, \;\; \{\mathcal{Q}, \mathcal{Q}\} \sim \mathcal{P}$

$f^{-1}df = J^m \mathcal{P}_m + J^I_\alpha \mathcal{Q}^\alpha_I + J^{mn}\mathcal{M}_{mn}$

Supercoset action= $\int \mathrm{Tr}(f^{-1}df)^2_{F/G}$ + fermionic WZ-term

$I = \int d^2\sigma (J^m J^m + a \bar{J}^I J^I) + b \int \; J^m \wedge \bar{J}^I \Gamma_m J^J s_{IJ}$

$J^m = dx^m - i\bar{\theta}^I \Gamma^m \theta^I, \qquad J^I_\alpha = d\theta^I_\alpha$

unitarity and right fermionic spectrum iff $a = 0, \; b = \pm 1$

$AdS_5 \times S^5 = \frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$

Killing vectors and Killing spinors of $AdS_5 \times S^5$:

$PSU(2,2|4)$ symmetry

replace $\frac{\widehat{F}}{G} = \frac{\text{SuperPoincare}}{\text{Lorentz}}$ in flat GS case by

$$\frac{\widehat{F}}{G} = \frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$$

generators: $(\mathcal{P}_q, \mathcal{M}_{pq}); (\mathcal{P}'_r, \mathcal{M}'_{rs}); \mathcal{Q}^I_\alpha, \quad m = (q, r)$

$$[\mathcal{P}, \mathcal{P}] \sim \mathcal{M}, \ \ [\mathcal{P}, \mathcal{M}] \sim \mathcal{P}, \ \ [\mathcal{M}, \mathcal{M}] \sim \mathcal{M},$$

$$[\mathcal{Q}, \mathcal{P}_q] \sim \gamma_q \mathcal{Q}, \ \ [\mathcal{Q}, \mathcal{M}_{pq}] \sim \gamma_{pq} \mathcal{Q}$$

$$\{\mathcal{Q}^I, \mathcal{Q}^J\} \sim \delta^{IJ}(\gamma \cdot \mathcal{P} + \gamma' \cdot \mathcal{P}') + \epsilon^{IJ}(\gamma \cdot \mathcal{M} + \gamma' \cdot \mathcal{M}')$$

$PSU(2,2|4)$ invariant action:

$\int \mathrm{Tr}(f^{-1}df)^2_{F/G}$ + WZ-term

$J = f^{-1}df = J^m \mathcal{P}_m + J^I_\alpha \mathcal{Q}^\alpha_I + J^{mn}\mathcal{M}_{mn}$

$$I = \frac{\sqrt{\lambda}}{2\pi}\Big[\int d^2\sigma (J^m J^m + a\bar{J}^I J^I) + b\int J^m \wedge \bar{J}^I \Gamma_m J^J s_{IJ}\Big]$$

as in flat space $a = 0,\ \ b = \pm 1$ required by κ-symmetry

unique action with right symmetry and right flat-space limit

Formal argument for UV finiteness (2d conformal invariance):

1. global symmetry –

   only overall coefficient of J^2 (radius) can run

2. non-renormalization of WZ term (homogeneous 3-form)

3. preservation of κ-symmetry at the quantum level

   – relates coefficients of J^2 and WZ terms

Equivalent form of the GS action:

$\frac{F}{G} = AdS_5 \times S^5 = \frac{SU(2,2)}{Sp(2,2)} \times \frac{SU(4)}{Sp(4)}$

generalized to

$\frac{\widehat{F}}{G} = \frac{PSU(2,2|4)}{Sp(2,2)\times Sp(4)}$

basic superalgebra $\widehat{\mathfrak{f}} = psu(2,2|4)$

bosonic part $\mathfrak{f} = su(2,2) \oplus su(4) \cong so(2,4) \oplus so(6)$

admits Z_4-grading:

$$\widehat{\mathfrak{f}} = \mathfrak{f}_0 \oplus \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \mathfrak{f}_3 \,, \qquad\qquad [\mathfrak{f}_i, \mathfrak{f}_j] \subset \mathfrak{f}_{i+j \bmod 4}$$

$\mathfrak{f}_0 = \mathfrak{g} = sp(2,2) \oplus sp(4)$

$\mathfrak{f}_2 = AdS_5 \times S^5$

current $J = f^{-1}\partial_a f, \ f \in \widehat{F}$ (notation change: $J_0 \to \mathcal{A}$, etc)

$$J_a = f^{-1}\partial_a f = \mathcal{A}_a + Q_{1a} + P_a + Q_{2a}$$

$$\mathcal{A} \in \mathfrak{f}_0, \quad Q_1 \in \mathfrak{f}_1, \quad P \in \mathfrak{f}_2, \quad Q_2 \in \mathfrak{f}_3 \,.$$

GS Lagrangian:

$$L_{\rm GS} = \frac{1}{2}{\rm STr}(\sqrt{-g}g^{ab}P_a P_b + \varepsilon^{ab}Q_{1a}Q_{2b})\,,$$

fermionic currents in WZ term only

conformal gauge: $\sqrt{-g}g^{ab} = \eta^{ab}$

$$L_{\rm GS} = {\rm STr}[P_+P_- + \frac{1}{2}(Q_{1+}Q_{2-} - Q_{1-}Q_{2+})]$$

$${\rm STr}(P_+P_+) = 0\,, \qquad {\rm STr}(P_-P_-) = 0$$

Equations of motion in terms of currents: 1-st order form

$$\begin{aligned} \text{EOM}: \quad & \partial_+P_- + [\mathcal{A}_+, P_-] + [Q_{2+}, Q_{2-}] = 0\,, \\ & \partial_-P_+ + [\mathcal{A}_-, P_+] + [Q_{1-}, Q_{1+}] = 0\,, \\ & [P_+, Q_{1-}] = 0\,, \qquad [P_-, Q_{2+}] = 0\,. \end{aligned}$$

$$\text{MC}: \quad \partial_-J_+ - \partial_+J_- + [J_-, J_+] = 0\,.$$

How to solve quantum string theory in $AdS_5 \times S^5$?

GS string on supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$

not of known solvable type (cf. free oscillators; WZW)

analogy with exact solution of $O(n)$ model (Zamolodchikovs) or

principal chiral model (Polyakov-Wiegmann, ...) ?

2d CFT – no quantum mass generation

one problem of direct approaches:

lack of manifest 2d Lorentz symmetry

S-matrix depends on two rapidities, not on their difference,

symmetry constraints on it are not obvious...

An alternative approach?

Classically equivalent 2d Lorentz invariant action

describing same physical degrees of freedom?

formulation in terms of currents rather than coordinate fields!

“Pohlmeyer reduction”

Integrable + 2d conformally invariant (UV finite) model –
fermionic generalization of non-abelian Toda theory

- intimately related (at least classically) to $AdS_5 \times S^5$ GS model
- contains fermions with standard kinetic terms
- has 2d Lorentz invariant S-matrix

for an equivalent set of 8+8 physical massive excitations

- remarkable UV finite massive integrable model:

exact solution?

- deserves study regardless possible relation

to $AdS_5 \times S^5$ superstring at the quantum level

PR theory for $AdS_5 \times S^5$ superstring

[Grigoriev, AT 07; Mikhailov, Schafer-Nameki 07]

• start with GS equations in terms of currents

• solve conformal gauge constraints algebraically introducing new set of field variables directly related to the currents

• fix κ-symmetry gauge

• reconstruct the action for resulting field equations in terms of new current variables

• this implies classical equivalence of original and “reduced” sets of equations so the reduced theory is also integrable

GS action: start with $f \in \widehat{F} = PSU(2,2|4)$

current J $\equiv f^{-1}df$

split according to Z_4 decomposition of $\widehat{\mathfrak{f}}$ =alg $\widehat{F}$

$\mathrm{J}_a = f^{-1}\partial_a f = \mathcal{A}_a + Q_{1a} + P_a + Q_{2a}$,

$\mathcal{A} \in \widehat{\mathfrak{f}}_0, \quad Q_{1,2} \in \widehat{\mathfrak{f}}_{1,3}, \quad P \in \widehat{\mathfrak{f}}_2$

$\mathcal{A} \in \mathfrak{g}$= alg $G = Sp(2,2) \times Sp(4)$

conformal gauge

$$L = \mathrm{Str}\Big[P_+P_- + \tfrac{1}{2}(Q_{1+}Q_{2-} - Q_{1-}Q_{2+})\Big]$$

$$\mathrm{Str}(P_+P_+) = 0\ , \qquad \mathrm{Str}(P_-P_-) = 0$$

solve Virasoro algebraically; fix κ-symmetry gauge;

find action for new current variables

Virasoro can be solved by fixing a special

G-gauge and residual conformal diffeomorpism gauge

$$P_+ = \ \mu\, T\ , \qquad\quad P_- = \ \mu\, g^{-1}Tg\,, \qquad \mu = \mathrm{const}$$

$$g \in G = Sp(2,2) \times Sp(4)$$

μ= arbitary scale parameter – remnant of fixing
residual conformal diffeomorphisms, cf. p^+ in l.c. gauge

T is a fixed constant matrix, e.g., diag$(I, -I, I, -I)$, $\mathrm{Str}\, T^2 = 0$

$H \in G$ that commutes with T, $[T, h] = 0$, $h \in H$:

$H = SU(2) \times SU(2) \times SU(2) \times SU(2)$

P_- is invariant under $g \to hg$ if $h \in H$

implies extra H gauge invariance of e.o.m. for g

$g \in G = Sp(2,2) \times Sp(4)$

A_+, A_- in $\mathfrak{h} = su(2) \oplus su(2) \oplus su(2) \oplus su(2)$

as new independent bosonic variables

impose partial κ-symmetry gauge

$$Q_{1-} = 0 \,, \qquad Q_{2+} = 0 \,,$$

define independent fermionic variables

$$\Psi_1 = Q_{1+} \in \widehat{\mathfrak{f}}_1 \,, \qquad \Psi_2 = g Q_{2-} g^{-1} \in \widehat{\mathfrak{f}}_3$$

residual κ-symmetry fixed by $\Psi_{1,2} T = -T \Psi_{1,2}$
define new fermionic variables

$$\Psi_{_R} = \frac{1}{\sqrt{\mu}} \Psi_1^{||} \,, \qquad \Psi_{_L} = \frac{1}{\sqrt{\mu}} \Psi_2^{||}$$

expressed in terms of real Grassmann
2×2 matrices $\xi_{R,L}$ and $\eta_{R,L}$: 8+8=16 components

Remarkably, exists local Lagrangian reproducing
resulting classical superstring equations:

gauged WZW model for

$$\frac{G}{H} = \frac{Sp(2,2)}{SU(2)\times SU(2)} \times \frac{Sp(4)}{SU(2)\times SU(2)}$$

with integrable potential and coupled to fermions:

$$L_{tot} = L_B + L_F = L_{\rm gWZW}(g,A) + \mu^2\,{\rm Str}(g^{-1}TgT)$$

$$+\,{\rm Str}\left(\Psi_{_L} TD_+\Psi_{_L} + \Psi_{_R} TD_-\Psi_{_R} + \ \mu\, g^{-1}\Psi_{_L} g\Psi_{_R}\right)$$

fields are represented by 8×8 supermatrices, e.g.,

$g = {\rm diag}(a,b)$, $a\in Sp(2,2)$, $b\in Sp(4)$

$D_\pm\Psi = \partial_\pm\Psi + [A_\pm,\Psi]$, $A_\pm \in \mathfrak{h} = su(2)\oplus ... \oplus su(2)$

$T = \frac{i}{2}{\rm diag}(1,1,-1,-1,1,1,-1,-1)$;

$[T,h]=0$, $h\in H = [SU(2)]^4$,

invariant under H gauge transformations

$$g' = h^{-1}gh, \quad A'_\pm = h^{-1}A_\pm h + h^{-1}\partial_\pm h, \quad \Psi'_{_{L,R}} = h^{-1}\Psi_{_{L,R}} h$$

$$[T,h]=0, \quad h\in H = [SU(2)]^4$$

integrable model

classically equivalent to GS theory:

Lax pair encoding equations of motion

$$\mathcal{L}_- = \partial_- + A_- + z^{-1}\sqrt{\mu}g^{-1}\Psi_{_L}g + z^{-2}\mu g^{-1}Tg\,,$$
$$\mathcal{L}_+ = \partial_+ + g^{-1}\partial_+ g + g^{-1}A_+ g + z\sqrt{\mu}\Psi_{_R} + z^2\mu T$$

Comments:

- gWZW model coupled to the fermions interacting minimally and through the "Yukawa" term
- 2d Lorentz invariant action with $\Psi_{_R}, \Psi_{_L}$ as 2d Majorana spinors with standard kinetic terms
- 8 real bosonic and 16 real fermionic independent variables; fermions link bosons from $Sp(2,2) \times Sp(4)$: transform under both groups
- 2d supersymmetry? yes, at least at quadratic level and in $AdS_2 \times S^2$ truncation limit: $n = 2$ super sine-Gordon model
- μ-dependent interactions are equal to GS Lagrangian; gWZW produces MC eqs.: path integral derivation via change from fields to currents?
- linearisation of e.o.m. in the gauge $A_{\pm} = 0$ around $g = \mathbf{1}$: gives 8+8 bosonic and fermionic d.o.f. with mass μ – same as in BMN limit

H gauge field $A_\pm$ can be gauged away on e.o.m. –
fermionic generalization of non-abelian Toda equations:

$$\partial_-(g^{-1}\partial_+ g) + \mu^2[g^{-1}Tg, T] + \mu[g^{-1}\Psi_{_L} g, \Psi_{_R}] = 0,$$

$$T\partial_-\Psi_{_R} + \tfrac{1}{2}\mu(g^{-1}\Psi_{_L} g)^{\parallel} = 0 \ ,$$

$$T\partial_+\Psi_{_L} + \tfrac{1}{2}\mu(g\Psi_{_R} g^{-1})^{\parallel} = 0 \ ,$$

$$(g^{-1}\partial_+ g - \tfrac{1}{2}[[T, \Psi_{_R}], \Psi_{_R}])_{\mathfrak{h}} = 0 \ ,$$

$$(g\partial_- g^{-1} - \tfrac{1}{2}[[T, \Psi_{_L}], \Psi_{_L}])_{\mathfrak{h}} = 0$$

fermions carry representations of both $Sp(2,2)$ and $Sp(4)$:
“intertwine” the two bosonic reduced sub-theories
Model resembles WZW models based on supergroups
rather than 2d supersymmetric WZW model
but fermions here have 1-st order kinetic term – a “hybrid”

Example: superstring on $AdS_2 \times S^2$

PR Lagrangian: same as $n = 2$ supersymmetric sine-Gordon!

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \partial_+\phi\partial_-\phi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi)$$
$$+ \beta\partial_-\beta + \gamma\partial_-\gamma + \nu\partial_+\nu + \rho\partial_+\rho$$
$$- 2\mu\left[\cosh\phi\ \cos\varphi\ (\beta\nu + \gamma\rho) + \sinh\phi\ \sin\varphi\ (\beta\rho - \gamma\nu)\right]\ .$$

equivalent to

$$\widetilde{L} = \partial_+\Phi\partial_-\Phi^* - |W'(\Phi)|^2 + \psi^*_{\scriptscriptstyle L}\partial_+\psi_{\scriptscriptstyle L} + \psi^*_{\scriptscriptstyle R}\partial_-\psi_{\scriptscriptstyle R}$$
$$+\left[W''(\Phi)\psi_{\scriptscriptstyle L}\psi_{\scriptscriptstyle R} + W^{*\prime\prime}(\Phi^*)\psi^*_{\scriptscriptstyle L}\psi^*_{\scriptscriptstyle R}\right]\ .$$

bosonic part is of $AdS_2 \times S^2$ bosonic reduced model if

$$W(\Phi) = \mu\cos\Phi\ , \qquad |W'(\Phi)|^2 = \frac{\mu^2}{2}(\cosh 2\phi - \cos 2\varphi)\ .$$

$$\psi_{\scriptscriptstyle L} = \nu + i\rho\,, \qquad \psi_{\scriptscriptstyle R} = -\beta + i\gamma\ ,$$

UV finiteness of reduced theory

[R. Roiban, A.T., 2009]

Reduction procedure may work at quantum level

only in conformally invariant case (like $AdS_5 \times S^5$ case)

Consistency requires that reduced theory is also UV finite

gWZW+ free fermions is finite; μ-terms may renormalize;

fermions should cancel bosonic renormalization

indeed true in $AdS_2 \times S^2$ case ($n = 2$ sine-Gordon)

true also in general:

$\mathrm{STr}(g^{-1}TgT) = \mathrm{Tr}(a^{-1}TaT) - \mathrm{Tr}(b^{-1}TbT)$

$\to \cos 2\varphi - \cosh 2\phi$

$\cos 2\varphi$ is “relevant”, $\cosh 2\phi$ - “irrelevant”

bosonic 1-loop correction $\sim (\cos 2\varphi + \cosh 2\phi)$

but fermions cancel this divergence

directly verified at 1-loop and 2-loop order

Thus μ is not renormalized, remains arbitrary

conformal symmetry gauge fixing parameter at quantum level

Some of open questions

- Quantum equivalence of reduced theory and GS theory?
  Path integral argument for equivalence?
  Transformation may work only in quantum-conformal
  case like $AdS_5 \times S^5$
- Indication of equivalence: semiclassical expansion
  near counterparts of rigid strings in $AdS_5 \times S^5$
  leads to same characteristic frequencies
  – same 1-loop partition function
- Tree-level S-matrix for elementary excitations?
  [Hoare, AT, 09]
  Relation to magnon S-matrix in BA?
- Quantum integrability? Exact solution?
- Solve reduced theory $\to$ solve $AdS_5 \times S^5$ superstring

Step towards exact solution: S-matrix

Integrable theory – determined by 2-particle S-matrix

superstring:

$$\frac{\widehat{F}}{G} = \frac{PSU(2,2|4)}{Sp(2,2) \times Sp(4)}$$

reduced theory

$$\frac{G}{H} = \frac{Sp(2,2)}{SU(2) \times SU(2)} \times \frac{Sp(4)}{SU(2) \times SU(2)}$$

fields may be represented by 8×8 supermatrices

in fundamental representation of $PSU(2,2|4)$

diagonal 4×4 blocks bosonic

and off-diagonal 4×4 blocks being fermionic

g in $G = Sp(2,2) \times Sp(4)$ and $A_\pm$ in algebra of $H = [SU(2)]^4$

fermionic fields $\psi_{_L}, \psi_{_R}$ from particular components

of the fermionic $\mathfrak{psu}(2,2|4)$ superstring currents expand the action around the trivial vacuum

$g = \mathbf{1},\ A_\pm = 0,\ \psi_R = \psi_L = 0$

find the tree-level two-particle scattering amplitude

for the 8+8 massive elementary excitations

Resulting 2-particle S-matrix ($\mathbb{S} = 1 + \frac{i}{k}\mathbb{T}$)

generic integrable theory with non-simple $G_1 \times G_2$ symmetry

and with fields in bi-fundamental representation:

S-matrix should exhibit group factorization property

happens in the light-cone gauge $AdS_5 \times S^5$ superstring S-matrix

is invariant under the product supergroup $PSU(2|2) \times PSU(2|2)$

[Kloze,MacLoughlin,Roiban,Zarembo; Arutyunov,Frolov,Zamaklar06]

field contents of the light-cone superstring and reduced theory

are identical in how they transform under

the bosonic symmetry group $[SU(2)]^4$

Remarkably, here get exactly the same factorisation structure

as in the superstring case

S-matrix group factorisation property

$$\mathbb{S} = \widetilde{\mathbb{S}} \otimes \widetilde{\mathbb{S}}\,, \qquad \mathbb{T} = \mathbb{I} \otimes \widetilde{\mathbb{T}} + \widetilde{\mathbb{T}} \otimes \mathbb{I}\,,$$

fields are $\Phi_{A\dot{A}}$, with $A=(a|\alpha),\ \ \dot{A}=(\dot{a}|\dot{\alpha})$, i.e. factorization

$$\mathrm{S}^{C\dot{C},D\dot{D}}_{A\dot{A},B\dot{B}}=(-1)^{[\dot{A}][B]+[\dot{C}][D]}\mathrm{S}^{CD}_{AB}\mathrm{S}^{\dot{C}\dot{D}}_{\dot{A}\dot{B}}\ ,$$

$$\mathbb{T}|\Phi_{A\dot{A}}(p_1)\Phi_{B\dot{B}}(p_2)\rangle=\frac{1}{4\sinh\vartheta}\Big[(-1)^{[\dot{A}]([B]+[D])}T^{CD}_{AB}\delta^{\dot{C}}_{\dot{A}}\delta^{\dot{D}}_{\dot{B}}$$
$$+(-1)^{\left([\dot{A}]+[\dot{C}]\right)[D]}\delta^{C}_{A}\delta^{D}_{B}T^{\dot{C}\dot{D}}_{\dot{A}\dot{B}}\Big]|\Phi_{C\dot{C}}(p_1)\Phi_{D\dot{D}}(p_2)\rangle$$

$[a]=[\dot{a}]=0$ and $[\alpha]=[\dot{\alpha}]=1$
explicit form of T^{CD}_{AB} can be written in terms of 10 functions K_i

$$T^{cd}_{ab}=K_1\ \delta^c_a\delta^d_b+K_2\ \delta^d_a\delta^c_b\,,$$

$$T^{\gamma\delta}_{\alpha\beta}=K_3\ \delta^\gamma_\alpha\delta^\delta_\beta+K_4\ \delta^\delta_\alpha\delta^\gamma_\beta\,,$$

$$T^{\gamma\delta}_{ab}=K_5\ \epsilon_{ab}\epsilon^{\gamma\delta}\ ,\quad T^{cd}_{\alpha\beta}=K_6\ \epsilon_{\alpha\beta}\epsilon^{cd}\ ,$$

$$T^{\gamma d}_{a\beta}=K_7\ \delta^d_a\delta^\gamma_\beta\,,\quad T^{c\delta}_{\alpha b}=K_8\ \delta^\delta_\alpha\delta^c_b\,,$$

$$T^{c\delta}_{a\beta}=K_9\ \delta^c_a\delta^\delta_\beta\,,\ \ T^{\gamma d}_{\alpha b}=K_{10}\ \delta^\gamma_\alpha\delta^d_b\,.$$

Pohlmeyer reduced theory is 2-d Lorentz-invariant:

K_i depend only on difference of two rapidities $\vartheta = \theta_1 - \theta_2$

$$K_1 = -K_3 = \sinh^2 \frac{\vartheta}{2},$$

$$K_2 = -K_4 = -\cosh \vartheta,$$

$$K_5 = K_6 = -\sinh \frac{\vartheta}{2},$$

$$K_7 = K_8 = -\cosh \frac{\vartheta}{2},$$

$$K_9 = -K_{10} = 0\,.$$

in the light-cone superstring $\mathbb{T}$-matrix

K_i depend separately on the two rapidities

$$K_1 = -K_3 = (\sinh\theta_1 - \sinh\theta_2)^2$$

$$K_2 = -K_4 = 4\sinh\theta_1 \sinh\theta_2$$

$$K_5 = K_6 = 4\sinh\theta_1 \sinh\theta_2 \sinh\frac{\theta_1 - \theta_2}{2}$$

$$K_7 = K_8 = 4\sinh\theta_1 \sinh\theta_2 \cosh\frac{\theta_1 - \theta_2}{2}$$

$$K_9 = -K_{10} = -\sinh^2\theta_1 + \sinh^2\theta_2 \,.$$

vanishing of K_9 and K_{10} reflects the fact that the bosonic part of the reduced theory is the direct sum of the "AdS" and "sphere" parts (which separate as in the conformal gauge) while in the light-cone gauge superstring action the corresponding sets of the bosonic fields were coupled

Comments:

•Main conclusion: exists special 2d Lorentz covariant S-matrix
corresponding to the reduced theory –
local UV finite massive integrable theory
whose algebraic structure is very similar to that of the S-matrix
of the $AdS_5 \times S^5$ superstring theory in the S^5 light-cone gauge

•reduced theory S-matrix has same type of group factorisation
as the superstring theory S-matrix –
suggests extra hidden fermionic symmetry
(like extension to $PSU(2|2) \times PSU(2|2)$)

•relation between the two S-matrices?

•hidden 2d supersymmetry? (present in $AdS_2 \times S^2$ case)

•Yang-Baxter equation: satisfied modulo similarity transf.
or twisting (due to gauge symmetry factorization)

•towards exact quantum solution of the reduced theory:
draw lessons from examples of massive integrable

deformations of coset CFT's studied in literature

●need to generalize to the case of non-abelian H

issues of vacua and topological (?) solitons....

many open questions...

Conclusion

Pohlmeyer reduction is a very fruitful idea