MONOTONE INTEGRATED LARGE EDDY SIMULATION OF
BuoYant TurBuLeNT JeTS wITh OFF-SOURCE HeATING

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Declaration

This dissertation is the result of my own work and includes nothing that is the outcome of work done in collaboration except where specifically indicated in the text.

No parts of this dissertation have been submitted for any other qualification.

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Abstract

Condensation of water vapour carried by buoyant atmospheric plumes is responsible for cloud formation and leads to a release of latent heat. This release provides a secondary source of buoyancy away from the origin of the plume, and field observations suggest that the laterally entraining plume model does not capture the resulting behaviour correctly. To investigate this problem in the laboratory Bhat and Narasimha (1996) reproduced analogous off-source heating by using electrodes to heat an acidic jet in a deionised ambient. It was observed that the heating significantly disrupted the eddy structure of the jet and the mass flux was drastically reduced. J. C. R. Hunt (1994) proposed that a tendency for the jet to break up into distinct eddies, which would increase the relative turbulent intensity, would lead to a decrease in entrainment.

This dissertation describes a numerical investigation of volumetrically heated jets. The numerical approach taken employs a method known as Monotone Integrated Large Eddy Simulation (MILES). It is a natural form of large eddy simulation that can capture inherently the correct flow of energy through the inertial range, and provides physical decay at the grid-scale cut-off. The method does not use an explicit turbulence model and hence makes no assumptions about the structure of the flow. The extent to which turbulent statistics can be recovered with such an approach is first explored, in the context of homogeneous turbulence. Open shear flows are then considered, first in a temporally evolving circular shear layer. Comparisons are made with an existing study in the literature (Basu and Narasimha 1999), and the validity and benefits of adaptive mesh refinement (AMR) are demonstrated. Finally, spatially evolving jets with and without off-source heating are investigated. The results confirm observations made in experimental and numerical studies, but also provide further insight. Measurements, particularly of second-order correlations, help to clarify the understanding of this flow, and identify further requirements necessary for an analytic model to capture fully the phenomena observed.
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Chapter 1

Introduction
Turbulent jets and plumes occur in many geophysical and engineering situations. In the simplest forms, these flows provide archetypal examples of turbulent mixing by free shear. However, even minor modifications to the source and/or ambient fluid can bring about drastic changes in behaviour that are difficult to model.

In 1956, Morton, Taylor and Turner published a landmark paper [60] that has formed the basis of many works. The underlying assumption, due to Taylor, is that the rate of entrainment of ambient fluid is proportional to a characteristic velocity within the jet or plume, an assumption that has proved successful in a range of applications, for example see Turner’s review paper [86].

One example where changes in conditions lead to anomalous results is cumulus clouds, where observations have raised questions about the effects of heat release on entrainment behaviour. Clouds form due to the condensation of water vapour carried by atmospheric plumes. There is an associated latent heat release that enhances the buoyancy of the plume. Based on Taylor’s entrainment hypothesis, the acceleration of fluid to the buoyancy should lead to an increase in entrainment. However, field observations suggest a decrease in entrainment, and Paluch [63] reports that it is reduced almost to zero.

By considering the effects of turbulent momentum flux, J. C. R. Hunt [39] proposed that an increase in relative turbulent intensity could lead to a decrease in entrainment.

In order to study this problem, Bhat, Narasimha and Arakeri [15] pioneered an analogous laboratory experiment. An acidic jet was injected into a deionised ambient, and electrodes were used to selectively heat the conducting jet fluid. A dramatic change in structure of the jet was observed, along with a drastic reduction in the spread rate of the jet. It was concluded that that the assumption of a constant entrainment coefficient was not valid for this flow, and consideration of the relative widths of the velocity and scalar profiles was also required. Furthermore, a decrease in the relative turbulent intensity was observed, and so it was concluded that Hunt’s proposal was not verified by these results. More recent experiments and numerical simulations by Agrawal and Prasad [2] have provided a more detailed picture of the velocity and temperature fields.

The primary aim of this dissertation is to investigate the laboratory experiment using numerical simulations. The broad range of time and length scales in turbulent flows places huge demands on computational resources, and so the numerical approach is a crucial factor in the study.

The most accurate method, known as Direct Numerical Simulation (DNS), is to simulate the Navier-Stokes equations in their entirety, resolving all scales of motion. This is often prohibitively expensive, and so different approaches have been formulated to circumvent this issue.

One such method is Large Eddy Simulation (LES). The idea is to use a spatial filter to separate the large
and small scales. Then, only the large scales are resolved, and the effects of the small scales are captured using turbulence models. These models have to be constructed carefully, and are often computationally expensive and flow dependent.

An approach that is receiving increasingly more attention is to use an implicit turbulence model. The general idea is to use a carefully constructed scheme so that the numerical error acts as a turbulence model in such a way as to emulate the physical effects of small scale viscous dissipation. This approach was introduced by Boris et. al. [18], and referred to as Monotone Integrated Large Eddy Simulation (MILES), but has more recently come to encompass a broader range of scales under the umbrella of Implicit Large Eddy Simulation (ILES).

The numerical code that is used throughout this study is IAMR, which was written at the Center for Computational Sciences and Engineering at the Lawrence Berkeley National Laboratory. It is an incompressible, variable density Navier-Stokes solves that is capable of MILES calculations. A finite-volume approach is taken with a two-step predictor corrector method. The flow variables are advected by a specialised upwind method before an approximate projection is used to enforced the divergence free constraint. The method is second-order accurate in space and time. Adaptive mesh refinement (AMR) is achieved using a hierarchy of grids, and the code is capable of parallel execution.

1.2 Thesis Organisation

A theoretical background is given in chapter 2 that introduces the concepts required for a numerical study of spatially evolving turbulent jets with off-source heating. Chapter 3 explores the extent to which turbulent structure and statistics can be recovered using an implicit numerical approach, specifically IAMR. Chapter 4 uses an analogous temporally evolving free shear flow, which has been used to investigate the effect of heat release, as further code validation without the computational expense of spatially evolving flows. Chapter 5 presents results from spatially evolving turbulent jets and plumes without off-source heating as a preliminary study of these benchmark cases before more complicated flows are investigated. Chapter 6 presents the crux investigation of the effects of off-source heating on spatially evolving turbulent jets. Chapter 7 summarises the conclusions of the investigation, and outlines relevant further work.
Chapter 2

Technical Background
2.1 Equations of Motion

This section presents the equations of motion that will be used throughout this dissertation. A comprehensive account of the derivation of these equations can be found in Batchelor [11]. These equations describe the motion of a considerable proportion of fluids, certainly of those encountered on a daily basis.

The physical properties described by these equations are incompressibility (often referred to as continuity), conservation of mass and momentum, and a constitutive law that describes the effect of molecular viscosity. The effect of heat transfer will be described by a simple transport equation, which will be coupled to the momentum equation using the Boussinesq approximation.

By considering the rate of change of mass in a volume of fluid, an equation for conservation of mass can be written as

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,
\]

where \(\rho\) and \(\mathbf{u}\) are the density and velocity of the fluid.

The discussion will be restricted to incompressible fluids. Batchelor [11] defines a fluid to be incompressible if the density of a fluid element is not affected by changes in pressure, and is usually taken to imply that the rate of change of density, following the fluid is zero,

\[
\frac{D\rho}{Dt} = \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \rho = 0,
\]

which coupled with (2.1) implies

\[
\nabla \cdot \mathbf{u} = 0.
\]

This divergence-free constraint will be referred to as incompressibility, and is appropriate for both liquid-phase laboratory experiments and for atmospheric flows if the fluid speed is much less than the speed of sound.

The key equation for describing the motion of fluids is the Navier-Stokes equation. This is essentially Newton’s second law applied to fluid elements: conservation of momentum. For the \(i\)th component of momentum, the transport equation, including the body force due to gravity, can be written as

\[
\frac{\partial}{\partial t} (\rho u_i) + \nabla \cdot (\rho u_i \mathbf{u}) = \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) (\rho u_i) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} - \rho g \delta_{iz},
\]

where \(p\) is the pressure, \(\tau_{ij}\) are the viscous stresses, \(g\) is acceleration due to gravity, \(\delta_{ij}\) is the Kronecker delta, and the Einstein summation convention is assumed. Note that incompressibility has been used to change between the conservative and non-conservative forms.

The constitutive law that describes the effects of viscosity is Newton’s law of viscosity

\[
\tau_{ij} = 2\mu S_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),
\]
where $S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$ is the rate of strain tensor and $\mu$ is the dynamic viscosity coefficient (assumed to be constant). Fluids that obey (2.5) are known as Newtonian fluids. Substituting (2.5) into (2.4) gives

$$
\left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) (\rho \mathbf{u}) = -\nabla p + \mu \nabla^2 \mathbf{u} - \rho g e_z,
$$

(2.6)

If the viscous effects are neglected, taking $\mu = 0$, then the resulting equations are the incompressible Euler equations.

**The Boussinesq Approximation**

Let $\rho_r$ be a constant reference density, and $\rho' = \rho - \rho_r$ be the density difference from the reference. The Boussinesq approximation, introduced in 1903 [19], assumes that density variations, $\rho'$, are small compared with the reference density, $\rho_r$, so that the density variation can be neglected in all terms except the buoyancy term (i.e. when multiplied by $g$)

$$
\left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} - g' e_z,
$$

(2.7)

where $\nu = \frac{\mu}{\rho_r}$ is the kinematic viscosity coefficient $P = \frac{\rho}{\rho_r} - g z$ is the modified pressure, and $g' = g \frac{\rho'}{\rho_r}$ is known as reduced gravity.

For later application to the heated jets, the reduced gravity can be written in terms of a temperature. Introducing the coefficient of volumetric thermal expansion

$$
\beta_T = -\frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_P,
$$

(2.8)

where $T$ denotes temperature. Then for small temperature variations, to leading order

$$
\frac{\rho'}{\rho_r} = -\beta_T (T - T_0),
$$

(2.9)

where $T_0$ is a reference temperature corresponding to $\rho_0$; without loss of generality, $T_0$ is henceforth taken to be zero. Substituting into (2.7) yields

$$
\left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + \alpha_T T e_z
$$

(2.10)

where $\alpha_T = g \beta_T$, so $\alpha_T T$ is equivalent to the the reduced gravity $g'$. A transport equation for temperature can be written as

$$
\left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) T = \kappa_T \nabla^2 T,
$$

(2.11)

where $\kappa_T$ is the coefficient of thermal diffusivity.

**Axisymmetric Equations**

For the discussion of axisymmetric jets and plumes, the equations of motion (2.3), (2.10) and (2.11), along with a transport equation for a scalar can be written in cylindrical polar coordinates $(r, \theta, z)$, assuming
axisymmetry, no swirl, and the Boussinesq approximation, as
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_r}{\partial r} \right) + \frac{\partial u_z}{\partial z} = 0,
\]
\[
\left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) u_r - \frac{u_r^2}{r} = -\frac{\partial P}{\partial r} + \nu \left( \nabla^2 u_r - \frac{u_r}{r^2} \right),
\]
\[
\left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) u_z = -\frac{\partial P}{\partial z} + \nu \nabla^2 u_z + \alpha_T T,
\]
\[
\left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) T = \kappa_T \nabla^2 T + \alpha_c c,
\]
\[
\left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) c = \kappa_c \nabla^2 c,
\]
where \( \mathbf{u} \cdot \nabla = u_r \frac{\partial}{\partial r} + u_z \frac{\partial}{\partial z}, \nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2}, \) is the tracer concentration and \( \alpha_c \) is the heating rate coefficient, that will be used for the heated jets.

**Non-dimensionalisation**

Let \( L, U_0 \) and \( T_0 \) be characteristic length, velocity and temperature scales, respectively, and write \( x_i = L x_i^*, \ u = U_0 u^* \) and \( T = T_0 T^* \). A characteristic time scale can be derived as the ratio \( \frac{T_0}{c}, \) and so time can be written as \( t = \frac{T_0}{c} t^* \). Substituting these expressions into (2.3) and (2.10) gives
\[
\left[ \frac{U_0}{L} \right] \nabla^* \cdot \mathbf{u}^* = 0
\]
\[
\left[ \frac{U_0}{L} \right] \left( \frac{\partial}{\partial t^*} + \mathbf{u}^* \cdot \nabla^* \right) \left( \frac{[U_0]}{[T_0] \ [T^*]} \mathbf{u}^* \right) = -\nabla P + \nu \left[ \frac{[U_0]}{[T_0]} \right] \nabla^* \mathbf{u}^* + \alpha_T [T_0] T^* \mathbf{e}_z
\]
[where \( \nabla^* = \mathbf{e}_i \frac{\partial}{\partial x_i} \). Now, writing \( P = \frac{[U_0]}{[T_0]} \ P^* \), the non-dimensionalised incompressible Navier-Stokes equations can be written as
\[
\nabla^* \cdot \mathbf{u}^* = 0
\]
\[
\left( \frac{\partial}{\partial t^*} + \mathbf{u}^* \cdot \nabla^* \right) \left( \frac{\mathbf{u}^*}{T^*} \right) = -\nabla P^* + \frac{1}{\Re} \nabla^* \mathbf{u}^* + \text{Ri}_0 T^* \mathbf{e}_z
\]
\[
\frac{1}{\Re} \left( \frac{\partial}{\partial t^*} + \mathbf{u}^* \cdot \nabla^* \right) \left( \frac{\mathbf{u}^*}{T^*} \right) = -\nabla P^* + \frac{1}{\Re} \nabla^* \mathbf{u}^* + \text{Ri}_0 T^* \mathbf{e}_z
\]
where \( \text{Re} = \frac{U_0 L}{\nu} \) is the Reynolds number, which is the ratio of inertial to viscous forces, \( \text{Pr} = \frac{c}{\kappa} \) is the Prandtl number, which is the ratio of molecular to thermal diffusion, and \( \text{Ri}_0 = \frac{\alpha_T L T_0}{\kappa c} \) is a Richardson number, which is a ratio of buoyancy to thermal diffusion (the suffix 0 and the indefinite article, are used here as this is one of many dimensionless groups that are referred to as a Richardson number, see the discussions in Turner’s book [85] or Batchelor [10]).
The basic aim of numerical methods is to be able to solve a given set of equations and initial and/or boundary conditions, particularly those that are difficult or impossible analytically. Digital computers are inherently designed to handle discrete sets of integers, and although technology has developed to work with floating point numbers, the discrete nature of data storage remains. The consequence for solving the equations of motion for fluid flow, as well as other forms of differential equations over continuous fields, is that each flow variable has to be represented in a discrete way. It seems somewhat paradoxical that the discrete molecules that constitute the fluids in question are first modelled mathematically as a continuum before being broken down once more into a finite number of individual items. The discretisation process applies not only to the representation of the fluid variables in space and time, but also to the representation of the equations of motion, and forms the basis of numerical methodology. The aim of this section is to introduce the relevant concepts necessary for the approach taken in later chapters.

Discretising Advection

For the purpose of introducing the relevant concepts, the equations of motion will be restricted to the one-dimensional linear advection and Burgers’ equations, which are

\[
\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0, \tag{2.21}
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0, \tag{2.22}
\]

respectively, where \(a\) is a constant assumed for now to be positive, \(u\) is the flow variable, and \(x\) and \(t\) are the space and time coordinates, respectively. These can be written in conservation form

\[
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0, \tag{2.23}
\]

where \(f(u) = au\) for linear advection, and \(f(u) = \frac{1}{2}u^2\) for Burgers’ equation.

Consider the linear advection equation over the domain \([0, 1]\). Boundary conditions will be discussed later in the section. The simplest way to discretise the domain is to split it into \(N\) evenly spaced intervals, or cells, of length \(\Delta x = \frac{1}{N}\). The data can then be represented either on cell nodes or cell centres. Firstly, for cell nodes, let \(i\) be an integer in the range \([0, N]\), and let the position of the cell nodes be defined as \(x_i = i\Delta x\). The data can be represented as \(u_i^n = u(x_i, t_n)\). Alternatively, for cell centres, let \(i\) be an integer in the range \([0, N - 1]\), and define \(x_i = (i + \frac{1}{2})\Delta x\). The data can now be represented in the same way as before, or by a cell average

\[
u_i^n = \frac{1}{\Delta x} \int_{x_i - \frac{\Delta x}{2}}^{x_i + \frac{\Delta x}{2}} u(x, t_n) \, dx. \tag{2.24}\]

Discretisation is also necessary in time, so firstly consider a general time point \(t = t_n\), the suffix \(n\) denoting the \(n\)th time step.
Figure 2.1: Schematics of the numerical stencils for three different numerical schemes, formed using the backward, centred, and forward spatial derivatives, respectively.

The task is to take the data at time $t = t_n$, and advance forward in time to get the data at $t = t_{n+1} = t_n + \Delta t$, using the chosen representation of the data and the equation of motion. To do this, the partial derivatives must be derived in terms of the available data. This can be achieved using Taylor expansions. For example, suppose the function $u(x, t)$ is sufficiently smooth to expand in space about $x = x_i$. Then

$$u(x_i + \Delta x, t_n) = u(x_i, t_n) + (\Delta x) \frac{\partial}{\partial x} u(x_i, t_n) + \frac{(\Delta x)^2}{2} \frac{\partial^2}{\partial x^2} u(x_i, t_n) + O((\Delta x)^3),$$

(2.25)

which can be rearranged to form an approximation of the $x$ derivative of $u$

$$\frac{\partial}{\partial x} u(x_i, t_n) = \frac{u(x_i + \Delta x, t_n) - u(x_i, t_n)}{\Delta x} + O(\Delta x).$$

(2.26)

This is known as a first-order forward finite-difference approximation, and can be written more concisely as

$$\frac{\partial u^n_i}{\partial x} = \frac{u^{n+1}_i - u^n_i}{\Delta x} + O(\Delta x).$$

(2.27)

By reversing the sign of $\Delta x$ in the Taylor expansion, the backward derivative can be evaluated

$$\frac{\partial u^n_i}{\partial x} = \frac{u^n_i - u^{n-1}_i}{\Delta x} + O(\Delta x).$$

(2.28)

Note that by taking the average of the above two approximations, the second-order term in the Taylor expansion cancels, and the resulting approximation is

$$\frac{\partial u^n_i}{\partial x} = \frac{u^{n+1}_i - u^{n-1}_i}{2\Delta x} + O((\Delta x)^2).$$

(2.29)

Temporal derivatives can be formed in the same way, so by taking a first-order forward difference in time and a first-order backward difference in space, the linear advection equation can be approximated by

$$\frac{u^{n+1}_i - u^n_i}{\Delta t} + a \frac{u^n_i - u^{n-1}_i}{\Delta x} = 0,$$

(2.30)

which can be solved for $u^{n+1}_i$

$$u^{n+1}_i = u^n_i - c \left( u^n_i - u^{n-1}_i \right),$$

(2.31)

where $c = a \frac{\Delta t}{\Delta x}$. This is known as an explicit update expression for $u^n_i$, as it allows the variable $u$ to be evaluated at the next point in time in terms of known quantities. Similarly, the centred and backward differences can be used to approximate the spatial derivative, and each equation can be solved for $u^{n+1}_i$ to give a different numerical scheme.

Figure 2.1 depicts the stencils for the different schemes. This outlines the data points that are used for the update, i.e. the numerical domain of dependence.
Stability analysis of the backward scheme gives rise to the restriction \( c \leq 1 \), where \( c \) is known as the CFL number; CFL is an acronym for Courant, Friedrichs and Lewy, and sometimes known more simply as a Courant number. This restriction is a natural consequence of an explicit scheme. Similar analysis of the forward scheme implies that it is unconditionally unstable. These results are related to the propagation of information. For \( a > 0 \), the flow variable, \( u \), is propagating in the positive \( x \) direction with speed \( a \). If the stencil of the scheme does not involve points in space where the information is coming from, then that information cannot be transported correctly, hence the forward difference leads to an unconditionally unstable scheme. The stability condition, \( c < 1 \), enforces that information cannot propagate over a distance greater than a single cell width in a single time step. Essentially, the numerical domain of dependence must contain the true domain of dependence.

Further consideration of the truncation error can give insight into the behaviour of the scheme. Consider the scheme constructed using the backward finite-difference approximation to the spatial derivative in the linear advection equation. Replacing the finite-differences with the Taylor series from which they are derived gives

\[
0 = \left( \frac{u_i^{n+1} - u_i^n}{\Delta t} \right) + a \left( \frac{u_{i+1}^n - u_{i-1}^n}{2 \Delta x} \right) = \left( \frac{\partial u}{\partial t} + \frac{\Delta t}{2} \frac{\partial^2 u}{\partial x^2} \right) + a \left( \frac{\partial u}{\partial x} - \frac{\Delta x}{2} \frac{\partial^2 u}{\partial x^2} \right) + O((\Delta x)^3). \tag{2.32}
\]

Repeated use of the linear advection equation can be used to replace temporal derivatives with spatial derivatives, and the equation can be rearranged to give

\[
\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} \approx a \frac{\partial^2 u}{\partial x^2}, \tag{2.33}
\]

where \( \alpha = \frac{a(\Delta x)}{2}(1 - c) \). This advection-diffusion equation is known as the modified equation. The extra term means that first-order accurate schemes like this one are diffusive, i.e. they tend to reduce extrema in the data and, over time, smooth out the solution. Analysis of second-order schemes can lead to modified equations involving dispersive terms, i.e. \( \beta \partial_{xxx}u \). Such schemes can suffer from phase errors, or time lagging.

For non-linear equations, such as Burgers’ equation, the formation of shocks causes further problems for numerical schemes. It can be shown that shock speeds can vary depending on the formulation of the equations, see Toro [84] section 3.3. The result is that it is necessary to solve the conservative form of the equations, otherwise the shock speeds and strengths, and therefore positions, can be miscalculated.

A natural way to do this is to use a finite-volume approach. For the general one-dimensional equation (2.23), consider the control volume \( [x_i - \frac{\Delta x}{2}, x_i + \frac{\Delta x}{2}] \times [t_n, t_n + \Delta t] \). To admit weak solutions, the integral form of the equations can be used

\[
\int (u \, dx - f \, dt) = 0, \tag{2.35}
\]
where integration is over the boundary of the control volume. The result is

\[ \int_{x_i - \frac{\Delta x}{2}}^{x_i + \frac{\Delta x}{2}} u(x, t_{n+1}) \, dx - \int_{x_i - \frac{\Delta x}{2}}^{x_i + \frac{\Delta x}{2}} u(x, t_n) \, dx = \int_{t_n}^{t_{n+1}} f \left( u \left( x_i - \frac{\Delta x}{2}, t \right) \right) \, dt - \int_{t_n}^{t_{n+1}} f \left( u \left( x_i + \frac{\Delta x}{2}, t \right) \right) \, dt. \]

(2.36)

The two terms on the left-hand side define the cell averages as in (2.24), so defining intercell fluxes

\[ F^n_{i-\frac{1}{2}} = \int_{t_n}^{t_{n+1}} f \left( u \left( x_i - \frac{\Delta x}{2}, t \right) \right) \, dt, \]

(2.37)

equation (2.36) can be written

\[ u^{n+1}_i = u^n_i + \Delta t \frac{\Delta x}{2} \left( F^n_{i-\frac{1}{2}} - F^n_{i+\frac{1}{2}} \right). \]

(2.38)

To construct a finite-volume scheme, the intercell fluxes, or more often approximations to the intercell fluxes, need to be evaluated. This can be done in many different ways, depending on the complexity of the equation. One such method is to treat the data as piecewise constant and treat each cell interface as a Riemann problem. If the Riemann problem is soluble then construction of the intercell fluxes follows. An excellent introduction to Riemann solvers is Toro [84].

An important property of numerical schemes, particularly for systems that have discontinuities or steep gradients, is monotonicity. An explicit scheme written in the form

\[ u^{n+1}_i = \sum_k \beta_k u^n_k \]

(2.39)
is said to be monotone if \( \beta_k \geq 0 \) for all \( k \). It turns out that a scheme that is not monotone can lead to spurious oscillations in the regions of steep gradients or discontinuities, whereas monotone schemes do not.

A subset of monotone schemes are total variation diminishing (TVD) schemes. The total variation of a set of data, \( u^n = u^n_i \), is defined as

\[ TV (u^n) = \sum_i \left| u^n_{i+1} - u^n_i \right|. \]

(2.40)

A scheme is said to be TVD if

\[ TV (u^{n+1}) \leq TV (u^n) \quad \forall n. \]

(2.41)

All of the schemes outlined above are known as linear schemes, because the coefficients \( \beta_k \) are all constants, and do not depend on the data. A theorem due to Godunov states that linear schemes that are monotone are at most first-order accurate. So first-order schemes are highly diffusive, but non-monotone second-order schemes can lead to spurious oscillations in the regions of steep gradients and discontinuities. The solution is to construct high-order non-linear schemes. By configuring the scheme so that the coefficients, \( \beta_k \), react to the flow variables, high-order monotone schemes can be constructed.

A method of doing this is to use flux limiters. The idea is to evaluate both a low-order monotone flux and a high-order flux, and interpolate between the two depending on the local data. Let \( F^n_{i+\frac{1}{2}}^{LO} \) and \( F^n_{i+\frac{1}{2}}^{HI} \)
be the low and high-order fluxes, respectively, and define a new intercell flux

\[ F_{i+\frac{1}{2}} = \phi_{i+\frac{1}{2}} \left( F_{i+\frac{1}{2}}^{HI} - F_{i+\frac{1}{2}}^{LO} \right) + F_{i+\frac{1}{2}}^{LO}, \]

where \( \phi_{i+\frac{1}{2}} \) if the flux limiter. Then it is clear that \( F_{i+\frac{1}{2}} = F_{i+\frac{1}{2}}^{LO} \) for \( \phi_{i+\frac{1}{2}} = 0 \), and \( F_{i+\frac{1}{2}} = F_{i+\frac{1}{2}}^{HI} \) for \( \phi_{i+\frac{1}{2}} = 1 \). By defining \( \phi_{i+\frac{1}{2}} \) appropriately the intercell flux can be varied between the high-order flux in smooth regions of the data and the low-order flux in the regions of steep gradients or discontinuities.

Other methods can be used, but the general idea is essentially the same, and the result is a method of constructing high-order non-oscillatory schemes that are well behaved for all kinds of data and sustain a high level of accuracy.

**Boundary Conditions**

With the spatial discretisation outlined above, boundary conditions are often implemented using what are known as ghost cells. These are cells that lie outside the physical domain of interest, but lie on the stencil of the cells within the domain. Consider a one-dimensional domain of length \( L \), discretised by \( N \) cells, indexed 1 to \( N \). Different kinds of boundary conditions demand different treatments, and in complicated situations there is no straightforward solution that will give the desired effect - boundary conditions are a research area in their own right. Basic types of conditions are explained here. The simplest case is Dirichlet boundary conditions. A constant temperature boundary, for example, can be achieved by setting the ghost cells to be that constant temperature. An abstract type of boundary condition is periodic. This means that each flow variable in a periodic domain obeys the restriction \( u(x + L) = u(x) \), and is implemented by setting \( u_{N+1} = u_1 \) and \( u_0 = u_N \). More generally \( u_{N+i} = u_i \). Transmissive boundaries (outflow) can be achieved naively by matching the ghost cells to the last cell within the domain, i.e. \( u_i = u_N \) for \( i > N \). More complicated boundary conditions can be specified to impose, for example, that the derivative on the boundary is zero. This, of course, demands knowledge of the approximation to the derivative used for the scheme. This can be achieved by writing the discretisation of the derivative at the boundary, setting it equal to zero (or the desired value), and solving for the value in the ghost cell.

**Spectral Methods**

Rather than discretising the flow variables onto individual points in space, spectral methods represent flow variables using Fourier series, specifically Fourier coefficients. The invention of the fast Fourier transform means that transforming between real space and spectral space can be achieved quickly and accurately.

Consider the linear advection equation (2.21) over the periodic domain \([0, 2\pi]\). The discrete Fourier
The transform of the velocity can be written as
\[ u(x, t) = \sum_{\kappa=0}^{\infty} \hat{u}_\kappa(t) e^{i\kappa x}. \] (2.43)

The equation of motion can be written as
\[ \frac{\partial}{\partial t} \sum_{\kappa=0}^{\infty} \hat{u}_\kappa e^{i\kappa x} + a \frac{\partial}{\partial x} \sum_{\kappa=0}^{\infty} \hat{u}_\kappa e^{i\kappa x} = 0. \] (2.44)

Assuming that differentiation and summation commute, and evaluating the spatial derivative, the equation becomes
\[ \sum_{\kappa=0}^{\infty} \left( \frac{\partial \hat{u}_\kappa}{\partial t} + a \kappa \hat{u}_\kappa \right) e^{i\kappa x} = 0, \] (2.45)
which reduces the PDE (2.21) to an ODE
\[ \frac{\partial \hat{u}_\kappa}{\partial t} = -a \kappa \hat{u}_\kappa, \] (2.46)
for each Fourier coefficient \( \hat{u}_\kappa \). In this simple case, this equation can be solved exactly to give
\[ u(x, t) = \sum_{\kappa=0}^{\infty} \hat{u}_{\kappa0} e^{i\kappa(x-\kappa t)}, \] (2.47)
where \( \hat{u}_{\kappa0} = \hat{u}_\kappa(0) \), and implies that \( u(x, t) = u(x - \kappa t, 0) \). In more complicated situations, the ODE (2.46) can be solved using any standard ODE solver, such as a Runge-Kutta method. It is not possible to solve for infinitely many Fourier coefficients, and so only a finite number of modes are solved, which introduces spatial errors.

Non-linearity further complicates matters as a convolution has to be evaluated, which is computationally expensive in spectral space. For this reason, pseudo-spectral methods can be used. For example, consider the term \( u \frac{\partial u}{\partial x} \). The spectral representation of each term (i.e. with Fourier coefficients \( \hat{u}_\kappa \) and \( i\kappa \hat{u}_\kappa \)) is transformed to real space, multiplied together, and the result transformed back into spectral space. With the fast Fourier transform, this is computationally much less expensive than evaluating the convolution.
In this section the phenomenology, nomenclature, definitions and tools used to describe and analyse turbulent flows are introduced. A comprehensive account of all manifestations of turbulence is well beyond the scope of this work, so only a relevant subset of topics is considered. Indeed, many authors have written accounts ranging from Batchelor’s concise and almost impenetrable classic on homogeneous isotropic turbulence [9], through frequently quoted texts such as Tennekes and Lumley [83] and Frisch [31], to more recent, reader-friendly textbooks like Pope [65] and Davidson [26]. Each book approaches the subject from a different perspective and contains something seldom found in other mathematical texts: opinion. What is presented here is intended to be unbiased and, where there are differing schools of thought, arguments for both sides are given.

The first difference of opinion arises when the issue of a definition of turbulence is raised, and most authors tend to refrain from attempting to do so. However, an unequivocal prerequisite for a region of fluid to be classed as turbulent is that it has a distribution of intense, fluctuating vorticity across a broad range of space and time scales. It is this separation of scales that makes the study of turbulence inherently difficult. More often than not, the only tangible way to characterise a turbulent flow is to describe it statistically.

In order to help the understanding of the concepts introduced in this section, it is useful to give two contextual examples of archetypal turbulent flows. The first is basic decaying turbulence. Consider a volume of fluid, neglecting any boundaries, that is given some kind of motion or stirred randomly, and then allowed to evolve freely. If the resulting flow does not have any position or direction that is distinguishable from any other, then it is known as homogeneous isotropic decaying turbulence; terms that will be defined below. The second flow is grid turbulence. This is generated by uniform flow passing over a mesh or grid, the exact details of which are not important for this general discussion. The essential point is that the no slip condition on the grid leads to separation, which introduces instabilities into the flow. These instabilities eventually lead to fluctuating velocities distributed across the entire fluid: turbulence. Both of these examples rely on the Reynolds number being sufficiently large that the inertia dominates the stabilising effect of viscosity. Exactly what constitutes sufficient depends on the exact details of the flow. With care, laminar flows can persist at high Reynolds numbers, but in contrast, turbulent flows cannot be realised at low Reynolds numbers.

The remainder of this section is organised as follows. First, the basic tools of probability are defined, and different kinds of averaging are discussed. Second, the tools used to describe the structure of turbulence are introduced. The basic phenomenology of turbulent flows is outlined, followed by a description of some experimental results. Finally, the averaged equations of motion are used to highlight what is known as the closure problem of turbulence, and a selection of closure strategies is given.
2.3.1 Statistical Description

Let $u$ be a random variable over $\mathbb{R}$. The cumulative distribution function (CDF) is defined as

$$F_u(v) = P\{u < v\}, \quad (2.48)$$

that is the probability that $u$ is less than $v$. It is a non-decreasing function, $F_u(v) \to 0$ as $v \to -\infty$, and $F_u(v) \to 1$ as $v \to \infty$.

The probability density function (PDF) can then be defined as

$$f_u(v) = \frac{d}{dv}F_u(v), \quad (2.49)$$

which is non-negative, $f_u(v) \to 0$ as $v \to \pm \infty$, and $\int_{\mathbb{R}} f_u(v) \, dy = 1$.

The mean of $u$ is defined to be

$$\langle u \rangle = \int_{-\infty}^{\infty} vf_u(v) \, dv. \quad (2.50)$$

Depending on the context, the mean may also be referred to as the expectation or simply the average. Different kinds of averaging will be discussed below.

The fluctuation from the mean is then defined as $u' = u - \langle u \rangle$, and clearly $\langle u' \rangle = \langle u - \langle u \rangle \rangle = \langle u \rangle - \langle u \rangle = 0$.

More generally, the $n$th-order central moment of $u$ is defined as

$$M_n = \int_{-\infty}^{\infty} (v - \langle u \rangle)^n f_u(v) \, dv. \quad (2.51)$$

Note that by specifying central moment the integrand involves the fluctuation $v - \langle u \rangle$ as opposed to $v$, which would simply determine the $n$th-order moment. Of course, the two coincide for a zero-mean random variable.

The second-order moment is known as the variance, the square root of which is the standard deviation. The $n$th-order central moment can be standardised by dividing by the standard deviation to the power $n$. The third-order standardised moment is known as the skewness, and indicates a lack of symmetry, and the fourth-order standardised moment is the flatness or kurtosis.

Now suppose that the random variable $u$ is a function of time. This is what is known as a random process, and is characterised at each point in time by a one-time CDF and related one-time PDF

$$F_u(v,t) = P\{u(t) < v\}, \quad (2.52)$$

$$f_u(v,t) = \frac{d}{dv}F_u(v,t). \quad (2.53)$$

More generally, the $n$-point joint CDF can be defined as

$$F^n_u(v_1,t_1;v_2,t_2;\ldots;v_n,t_n) = P\{u(t_1) < v_1; u(t_2) < v_2; \ldots; u(t_n) < v_n\}, \quad (2.54)$$
To characterise a random process fully the CDF must be known for all time instants. A random process is said to be statistically steady if all n-point statistics are invariant under temporal translation.

These ideas generalise naturally to multicomponent random variables over multidimensional space, $u(x, t)$. The one-point, one-time joint CDF and PDF are defined as follows

$$F_u(v, x, t) = P\{u_i(x, t) < v_i, i = 1, 2, 3\},$$

$$f_u(v, x, t) = \frac{\partial^3 F_u(v, x, t)}{\partial v_1 \partial v_2 \partial v_3},$$

and now moments are evaluated by triple integrals. If the statistics of $u(x, t)$ are independent under translation (in $x$) then the field is said to be homogeneous, and if they are also invariant under rotation then it is said to be isotropic.

When describing fluid flow using these tools, different kinds of averaging are appropriate. A general flow where the statistics vary in both time and space has to be averaged over many different realisations of the same flow, to give statistics that are functions of both space and time. This is known as ensemble averaging. Grid turbulence, in the frame of reference of the grid, is statistically steady and so temporal averaging is appropriate at each point in space, and so statistics are functions of spatial position. Homogeneous isotropic decaying turbulence can be averaged over the entire volume to give statistics that depend on time only. This is spatial averaging. Of course, mixtures of the above can be used. Grid turbulence, for example, has a streamwise direction, and at each streamwise position, the statistics will be homogeneous over the perpendicular plane. This can be exploited so that averaging can be performed spatially, as well as temporally. The resulting statistics are then a one-dimensional function of streamwise position. There is no standard notation that is used for each kind of averaging. Here, ensemble averaging will be denoted using $\langle \cdot \rangle$. Other kinds of averaging are used at different points throughout this dissertation. In each case, an overbar will be used, and defined appropriately.

### 2.3.2 Structural Description

#### Correlation functions

Consider a general flow that is evolving in both time and space, and so ensemble averaging is appropriate. Generalisation to spatial or temporal averaging is often trivial. Let the fluctuation from the mean be defined as $u'(x, t) = u(x, t) - \langle u(x, t) \rangle$. This is known as Reynolds decomposition. Using this fluctuation, correlation functions can be defined, and are a useful tool in analysing the structure of a turbulent flow.

The one-time, two-point, second-order velocity correlation tensor is defined as

$$R_{ij}(r, x, t) = \langle u'_i(x, t) u'_j(x + r, t) \rangle.$$  

This is a measure of how the velocity components at different points in space, at a particular time, are correlated. Similarly, the correlation in time can be measured, the two-time, one-point, second-order
velocity correlation is

\[ R_{ij}^t(s, x, t) = \langle u'_i(x, t)u'_j(x, t + s) \rangle. \] (2.58)

In the special case of homogeneous isotropic turbulence, the statistics are independent of \( x \), and (2.57) can be used to define two more functions, the longitudinal and lateral velocity correlation functions

\[ f(r) = u^2 R_{xx}(re_x), \] (2.59)
\[ g(r) = u^2 R_{yy}(re_x), \] (2.60)

where \( u \), without an suffix, denotes the RMS velocity fluctuation \( u^2 = \langle u_i'^2 \rangle \) for any \( i = 1, 2, 3 \) (under the assumption of isotropy). The functions \( R_{xx} \) and \( R_{yy} \) are really time-dependent functions, but in the ideal case of homogeneous isotropic turbulence, the normalisation by \( u^2 \) makes \( f(r) \) and \( g(r) \) self-similar. This then allows a length scale to be defined

\[ l = \int_0^\infty f(r) \, dr, \] (2.61)

which is known as the integral length scale.

**Energy spectra**

In homogeneous turbulence, the second-order velocity correlation tensor forms a Fourier transform pair with what is called the energy wavenumber spectrum tensor

\[ \Phi_{ij}(\kappa, t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} R_{ij}(r, t) e^{-i\kappa \cdot r} \, dr, \] (2.62)
\[ R_{ij}(r, t) = \int_{\mathbb{R}^3} \Phi_{ij}(\kappa, t) e^{i\kappa \cdot r} \, d\kappa, \] (2.63)

where \( \kappa \) is a wavenumber vector, and \( i^2 = -1 \). The energy wavenumber spectrum function can then be defined as

\[ E(\kappa, t) = \frac{1}{2} \int_{S(\kappa)} \Phi_{ii}(\kappa, t) \, dS(\kappa), \] (2.64)

integrating over the spherical shell, \( S(\kappa) \), of radius \( \kappa = |\kappa| \). Integrating this over all wavenumbers gives

\[ \int_0^\infty E(\kappa, t) \, d\kappa = \frac{1}{2} \langle u'_i u'_i \rangle, \] (2.65)

which is the total kinetic energy. Therefore, \( E(\kappa, t) \) can be thought of as the contribution to the kinetic energy from wavenumbers \( \kappa \).

In numerical studies, the velocity field is known at every point in space, which makes the calculation of the energy wavenumber spectra straightforward. However, in laboratory experiments, knowledge of the entire velocity field is extremely difficult to measure, but it is relatively simple to measure the evolution of a signal at a single point, or pair of points. Therefore, for homogeneous turbulence the one-dimensional energy wavenumber spectra is the relevant diagnostic diagnostics, and for statistically steady turbulence, the energy frequency spectrum is used.
The one-dimensional energy wavenumber spectrum is defined as twice the Fourier transform of $R_{ij}(r_1 e_1, t)$

$$E_{ij}(\kappa_1, t) = \frac{1}{\pi} \int_{\mathbb{R}} R_{ij}(r_1 e_1, t)e^{-i\kappa_1 r_1} dr_1,$$  \hspace{1cm} (2.66)

and similarly for $E_{ij}(\kappa_2, t)$ and $E_{ij}(\kappa_3, t)$.

The energy frequency spectrum function can be defined as

$$E_{ij}^t(x, \kappa) = \frac{1}{\pi} \int_{\mathbb{R}} R_{ij}^t(x, s)e^{-i\kappa s} ds.$$ \hspace{1cm} (2.67)

**Structure functions**

Further information can be derived from what are known as structure functions. The second-order structure function is defined as

$$D_{ij}(r, x, t) = \langle [u'_i(x + r, t) - u'_i(x, t)][u'_j(x + r, t) - u'_j(x, t)] \rangle,$$ \hspace{1cm} (2.68)

and the third-order as

$$D_{ijk}(r, x, t) = \langle [u'_i(x + r, t) - u'_i(x, t)][u'_j(x + r, t) - u'_j(x, t)][u'_k(x + r, t) - u'_k(x, t)] \rangle,$$ \hspace{1cm} (2.69)

generalising naturally to higher orders.

**2.3.3 Phenomenology**

The fundamental underlying idea behind the study of turbulence is the energy cascade, which is accredited to Richardson. His view of a turbulent flow field is a distribution of eddies of different sizes, and that these unstable eddies undergo a series of break ups, transferring their energy to smaller and smaller scales. This process continues until an eddy is sufficiently small that viscous effects become important and the eddies are diffused.

In a physical context, such as grid turbulence, the stages of decay are threefold. Large scale structures are generated by the grid. These structures are dominated by inertial effects and are transported and deformed by the non-linear advection term, transferring energy to smaller and smaller scales. This is known as the energy cascade. Finally, the energy cascade is terminated by the viscous dissipation. An essential point of this decay is that viscosity can play a role in the generation of the large scales, is responsible for destruction of energy at the small scales, but the energy cascade process itself is inviscid.

A significant consequence of this is that the rate at which energy is dissipated can only be determined by the large scales, moreover the flow geometry, and not by the viscosity. It is this principle that underpins what is probably the most significant result in the study of turbulence.
Kolmogorov’s 1941 theory

In 1941, Kolmogorov published two papers, in which he proposed three hypotheses. The papers were originally published in Russian, [48] and [47], but have been translated into English [51]. The ideas presented below will be referred to as \( k_{41} \), for convenience. Note that Kolmogorov does not assume that the large scales are either homogeneous or isotropic. The hypotheses assume that the Reynolds number is sufficient for the large scales to be dominated by inertial and not viscous effects. More specifically, there exists a separation of scales such that the size of large scale motions, \( l \), and the Kolmogorov length scale at which diffusive processes occur, \( \eta \), are far apart. Under this assumption the hypotheses can be stated as:

- **Local Isotropy Hypothesis**
  - Small scale turbulent motions, \( r \ll l \), are statistically isotropic.

- **First Similarity Hypothesis**
  - The statistics of the small scales, \( r \ll l \), have a universal form that is determined uniquely by the rate of energy dissipation and the viscosity.

- **Second Similarity Hypothesis**
  - Small scale motions larger than the Kolmogorov length scale, \( \eta \ll r \ll l \), are independent of viscosity and therefore depend solely on the rate of energy dissipation.

The scales described by the first similarity hypothesis, \( r \ll l \), are known as the universal equilibrium range. The second similarity hypothesis separates this range into the inertial range, \( \eta \ll r \ll l \), and the dissipation range \( r \sim \eta \). In isothermal flows the scales smaller than \( \eta \) are dynamically unimportant and are not considered here.

Kolmogorov used these hypotheses to define what are now known as the Kolmogorov scales. Length, velocity, and time scales can be formed in terms of the rate of energy dissipation, \( \varepsilon \), and the viscosity, \( \nu \). They are

\[
\eta \sim \left( \frac{\nu^3}{\varepsilon} \right)^{\frac{1}{4}}, \quad u_\eta \sim (\varepsilon \nu)^{\frac{1}{4}}, \quad \tau_\eta \sim \left( \frac{\nu}{\varepsilon} \right)^{\frac{1}{4}}.
\]  

(2.70)

Using these scales to normalise the second-order velocity correlation function, as defined in (2.68), Kolmogorov predicted that in the universal equilibrium range

\[
D_{11}(r) = \left\langle \left[ u'_1(x + re_1, t) - u'_1(x, t) \right]^2 \right\rangle = (\varepsilon r)^{\frac{3}{2}} F \left( \frac{r}{\eta} \right)
\]

(2.71)

for some (universal) dimensionless function \( F \). Moreover, in the inertial range, where viscous effects are neglected, there can be no dependence on \( \eta \), and so

\[
D_{11}(r) = \left\langle \left[ u'_1(x + re_1, t) - u'_1(x, t) \right]^2 \right\rangle = C_2 (\varepsilon r)^{\frac{3}{2}},
\]

(2.72)
for some (universal) dimensionless constant $C_2$. Similar treatment of the energy spectrum function yields

$$E(\kappa) = \varepsilon^\frac{2}{3} \kappa^{-\frac{4}{3}} \varphi(\kappa \eta),$$

(2.73)
in the universal equilibrium range, for some (universal) function $\varphi(\kappa \eta)$. In the inertial range, independence from $\eta$ gives rise to Kolmogorov’s famous minus five-thirds law

$$E(\kappa) = C \varepsilon^\frac{2}{3} \kappa^{-\frac{5}{3}},$$

(2.74)

for some (universal) dimensionless constant $C$.

More generally, Kolmogorov’s second similarity hypothesis implies that in the inertial range the $n$th-order structure function takes the form

$$D_n(r) = \langle [u_1'(x + re_1, t) - u_1'(x, t)]^n \rangle = C_n (\varepsilon r)^\frac{n}{3}.$$  

(2.75)

It is also possible to show that, if the turbulence is globally isotropic, then the third-order structure function takes the exact result

$$D_3(r) = -\frac{4}{5} (\varepsilon r),$$

(2.76)

which is known as Kolmogorov’s four-fifths law. This result has its roots in the Navier-Stokes equations, from which an evolution equation for $R_{ij}$ can be derived. Under the assumption of isotropy, this can be written as

$$\frac{\partial}{\partial t} (u^2 r^4 f(r, t)) = u^3 \frac{\partial}{\partial r} (r^4 K(r, t)) + 2 \nu u^2 \frac{\partial}{\partial r} \left( r^4 \frac{\partial}{\partial r} f(r, t) \right)$$

(2.77)

where $f(r, t)$ is the second-order longitudinal velocity correlation function defined by (2.59), and

$$K(r, t) = u^3 R_{xx}(r, t) = u^3 \langle u_x'(x, t)^2 u_x'(x + re_x, t) \rangle,$$

(2.78)

is the third-order longitudinal velocity correlation function. This is known as the Karman-Howarth equation. For the derivation of (2.76) from (2.77), see [26], for example.

**Taylor microscale**

A second length scale can be defined using $g(r)$, namely the Taylor microscale

$$\lambda_g = \sqrt{-\frac{1}{2} g''(0)},$$

(2.79)

which in isotropic turbulence is related to $\lambda_f$, analogously defined, by $\lambda_f = \lambda_g \sqrt{2}$. With a little manipulation, see [65], this can be written as

$$\lambda_g^2 = u^2 \left\langle \left( \frac{\partial u_1}{\partial x_1} \right)^2 \right\rangle^{-1}.$$ 

(2.80)

It will be shown in section 2.3.5 that the rate of energy dissipation can be written as $\varepsilon = 2\nu \langle S_{ij} S_{ij} \rangle$, where $S_{ij}$ is the rate of strain tensor $S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$. This in turn can be written as

$$\varepsilon = 15\nu \left\langle \left( \frac{\partial u_1}{\partial x_1} \right)^2 \right\rangle,$$

(2.81)
and so, the Taylor microscale can be written (and is often defined as)

$$\lambda_g^2 = \frac{15 \nu u^2}{\varepsilon}.$$  \hspace{1cm} (2.82)

The Taylor microscale can be used to define a Reynolds number

$$\text{Re}_\lambda = \frac{u \lambda_g}{\nu},$$ \hspace{1cm} (2.83)

which is often the preferred measure in studies of turbulence. The Taylor microscale can be related to the Kolmogorov length scale and the integral length scale (2.61) by

$$\lambda_g = \sqrt{10 \eta^{\frac{4}{3}} l^\frac{1}{3}},$$ \hspace{1cm} (2.84)

and the Taylor Reynolds number can be related to the integral scale Reynolds number by

$$\text{Re}_\lambda = \left(\frac{20}{3} \text{Re}\right)^\frac{1}{2}.$$ \hspace{1cm} (2.85)

**Intermittency**

A problem with the original K41 theory is what is known as intermittency, and is due to the intermittent nature of viscous dissipation. In essence, in a turbulent flow there are intense regions of vorticity interspersed with regions of almost irrotational fluid. This means that the local energy cascade will vary depending on the local large scale conditions. The end result is that (2.75) becomes less and less accurate with increasing $n$. This led Kolmogorov to revise his theory in 1962 [50], specifically the second similarity hypothesis, so that, rather than using the global energy dissipation rate, a local average was used. Specifically, for a sphere of radius $r$, with volume $V_r$, define

$$\varepsilon_r = \frac{1}{V_r} \int_{V_r} 2 \nu S_{ij} S_{ij} \, dV_r,$$ \hspace{1cm} (2.86)

where $S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$ is the rate of strain tensor. Using this local average dissipation rate, (2.75) can be written as

$$D_n(r) \sim \left( \frac{\varepsilon_r^\frac{4}{3}}{r} \right)^{\frac{n}{3}} r^{\zeta_n}.$$ \hspace{1cm} (2.87)

The next step is to model the statistical behaviour of $\left( \frac{\varepsilon_r^\frac{4}{3}}{r} \right)$. Two prominent models are the log-normal model of Kolmogorov [50] and $\beta$-model of Frisch et al. [32]. Both models lead to the expression

$$D_n(r) = C_n \left( \frac{\varepsilon_r}{r} \right)^{\frac{4}{3}} \left( \frac{1}{7} \right)^{\frac{n}{3}} \sim r^{\zeta_n},$$ \hspace{1cm} (2.88)

where $\zeta_n = \frac{n}{3} - \mu_\frac{4}{3}$.

Based on the work of Obukhov, Kolmogorov then assumed that the PDF for $\varepsilon_r$ has a log-normal distribution, which, combined with (2.88), gives

$$\mu_\frac{4}{3} = \frac{\mu}{18} \left( 3n - n^2 \right),$$ \hspace{1cm} (2.89)
where $\mu$ is known as the intermittency exponent.

The $\beta$-model uses space filling arguments and a fractal approach to define

$$\beta = \frac{2D}{3}, \quad D \leq 3, \quad (2.90)$$

to describe the fractional reduction in volume from one generation of eddies to the next, where $D$ is the so-called fractal dimension. This leads to a scaling exponent for (2.88) that can be written as

$$\zeta_n = (3 - D) + \frac{n}{3}(D - 2), \quad (2.91)$$

but is more commonly written as

$$\mu_n = \frac{\mu}{3}(n - 3). \quad (2.92)$$

The accuracy of these models will be considered in section 2.3.4.

**Evolution of the large scales**

Consider homogeneous isotropic decaying turbulence and the Karman-Howarth equation (2.77). Insight into the evolution of the large scales can be gained by integrating this equation with respect to $r$ to give

$$\frac{d}{dt} \left( u^2 \int_0^\infty r^4 f(r, t) \, dr \right) = \lim_{r \to \infty} \left( u^3 r^4 K(r) + 2\nu u^2 r^4 \frac{\partial}{\partial r} f(r, t) \right). \quad (2.93)$$

Now, if it is assumed that $f(r)$ and $K(r)$ are exponentially small for large $r$, then it follows immediately that

$$I = 8\pi u^2 \int_0^\infty r^4 f(r, t) \, dr = \text{const}, \quad (2.94)$$

where $I$ is known as Loitsyansky’s integral. Moreover, if it is assumed that $f(r)$ is self similar, then

$$u^2 l^5 = \text{const}. \quad (2.95)$$

Writing the energy equation as $\frac{d}{dt} \left( \frac{1}{2} u^2 \right) = -\varepsilon$, and using the estimate $\varepsilon = \frac{u^3}{2\tau}$, then power-laws can be derived for the evolution of the kinetic energy and the integral length scale

$$u^2 \propto t^{\frac{15}{14}}, \quad l \propto t^{\frac{2}{7}}. \quad (2.96)$$

Now, observing that the second-order longitudinal velocity correlation function and energy wavenumber spectrum function can be written as a Fourier transform pair

$$R(r) = \frac{1}{2} R_{ii}(r) = \int_0^\infty E(\kappa) \frac{\sin(\kappa r)}{\kappa r} \, d\kappa, \quad (2.97)$$

$$E(\kappa) = \frac{2}{\pi} \int_0^\infty R(r)(kr) \sin(\kappa r) \, dr, \quad (2.98)$$

and Taylor expanding $\sin(\kappa r)$ about $\kappa = 0$, then

$$E(\kappa) = \frac{L\kappa^2}{4\pi^2} + \frac{I\kappa^4}{24\pi^2} + \cdots, \quad (2.99)$$

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where
\[ L = 8\pi \int_0^\infty R(r)r^2 \, dr = 4\pi u^2 \lim_{r \to \infty} (r^3 f(r)) \quad (2.100) \]
is known as Saffman’s integral, which, under the assumption that \( f(r) \) is exponentially small for large \( r \), is zero. This implies that the energy spectrum function takes the form \( E(\kappa) \sim \kappa^4 \) at the large scales. This kind of spectrum is known as a Batchelor spectrum, or Batchelor-Proudman spectrum [12].

However, Birkhoff [16] showed that there is no reason to assume that \( f(r) \) and \( K(r) \) are exponentially small at large \( r \). Saffman used this to show that for \( L \neq 0 \) then the energy spectra could take the form \( E(\kappa) \sim \kappa^2 \) at the large scales, which is known as a Saffman-Birkhoff spectrum. The Saffman integral may not be zero, but it can be shown that it is an invariant, which implies that instead of (2.95)
\[ u^2 l^3 = \text{const}, \quad (2.101) \]
and instead of (2.96)
\[ u^2 \propto t^{-\frac{5}{2}}, \quad l \propto t^{\frac{7}{2}}. \quad (2.102) \]

There is still controversy whether turbulence has Batchelor or Saffman spectra, and Davidson [26] argues that the initial conditions are important. The Karman-Howarth equation can support evolution towards \( E(\kappa) \sim \kappa^4 \) at the large scales, but not to lower powers. However, if \( E(\kappa) \sim \kappa^2 \) to start with, it is self-perpetuating and so persists. A more detailed review of the arguments is given in Davidson [25]. Because of this issue, when simulating homogeneous isotropic turbulence, Davidson [26] recommends that the domain should be at least 20 times the size of the integral length scale, however, Pope [65] recommends that it should be at least 8 times the size.

**The final period of decay**

If the generation of large scales is not maintained and the turbulence is simply allowed to decay, then the Reynolds number continually decreases. In Batchelor turbulence, \( \text{Re} \sim t^{\frac{7}{2}} \), and in Saffman turbulence \( \text{Re} \sim t^{\frac{5}{2}} \). Therefore, viscous effects eventually dominate inertial effects, the flow relaminarises, and all of the small scales are removed from the flow.

If the non-linear effects are neglected in the Karman-Howarth equation, which are accounted for by \( K(r) \), then Batchelor and Townsend [13] showed that one solution of the equation is
\[ f(r) \sim \exp \left( -\frac{r^2}{8\nu t} \right), \quad (2.103) \]
\[ u^2 \sim t^{\frac{7}{2}}. \quad (2.104) \]

This is a good fit to the data [57], but occurs at very low Reynolds numbers.
2.3.4 Experimental Observations

There is a large amount of data in the literature about the statistics of turbulent structure in many different flows. Here, a small sample is presented to provide an overview of the basic statistics that can provide a benchmark against which the capability of the code can be compared.

A commonly quoted starting point for experimental observations of homogeneous turbulence is the grid turbulence work of Comte-Bellot and Corrsin. In their 1966 paper [23] they showed that by imposing a slight contraction (1.27:1) to the duct, the inherent anisotropy of grid turbulence could be reduced to within a few percent. They followed this up with another paper in 1970 [24] in which they measured correlation functions and energy spectra at a Taylor Reynolds number (2.83) up to Re_λ = 72. The longitudinal and lateral correlation functions were found to have forms as shown by the schematic in figure 2.2.

Structure function data on the centreline of an axisymmetric jet (Re_λ = 852) and in duct flow (Re_λ = 515) were compiled by Anselmet et al. [7]. High-order structure functions were measured, up to n = 14 in the duct flow, and n = 18 in the jet. An intermittency exponent was determined via the sixth-order moment, and was found to be µ = 0.2 ± 0.05. Power-law exponents did not support the linearity of the β-model, and better agreement was found with the log-normal model up to moments as high as 12. However, larger moments than this did not agree with either model.

Probability density functions of the velocity were shown to be near Gaussian. Similar plots of the first-order structure function revealed non-Gaussian distributions, which became increasingly more exponential at small scales.

A comprehensive review was performed by Saddoughi and Veeravalli in 1994 [74]. They used the 24.5×36.5
metre Full-Scale Aerodynamics Facility at NASA Ames Research Center to obtain a Reynolds number of \( \text{Re}_\lambda = 1450 \). Their preliminary aim was to test the local isotropy hypothesis of \( k^{41} \). They presented energy spectra with large inertial ranges that also had ‘bumps’ before the dissipation scales. Their comparison of the universal scaling for one dimensional longitudinal energy spectra in a broad range of turbulent flows has become one of the most famous plots in turbulence research (see figure 9 in [74]). They measured an inertial range constant, see (2.74), \( C = 1.5 \pm 0.1 \), and an exponential decay in the dissipation range, \( E(\kappa) \sim \exp(-\beta \kappa \eta) \), where \( \beta \approx 5.2 \). They also looked at second and third-order structure functions in the inertial range and pointed out that the lack of arbitrary constant in the form of the third-order structure function provides a method of calculating the dissipation rate (provided an inertial range exists).

### 2.3.5 Averaged Equations of Motion

The different methods of averaging outlined in section 2.3.1 can be applied to the Navier-Stokes equations. In section 2.4, further averaging methods will be introduced. Each method has its own objective, but in each case the inherent non-linearity of the Navier-Stokes equations gives rise to what is known as the ‘closure problem’ of turbulence. This problem will be demonstrated in the following analysis, where \( \langle \cdot \rangle \) will be used to denote a generic average, without assuming any properties of that average other than that it separates the flow variables into an averaged part and a fluctuating part, i.e. \( q(x, t) = \langle q(x, t) \rangle + q'(x, t) \), and \( \langle q' \rangle = 0 \), for a general quantity \( q \).

Averaging the incompressibility condition (2.3) gives

\[
\nabla \cdot (\langle \mathbf{u} \rangle + \mathbf{u}') = 0,
\n\tag{2.105}
\]

which when averaged shows that the average velocity is incompressible

\[
\nabla \cdot \langle \mathbf{u} \rangle = 0.
\n\tag{2.106}
\]

Subtracting this from (2.105) implies that the fluctuating part is also incompressible

\[
\nabla \cdot \mathbf{u}' = 0.
\n\tag{2.107}
\]

Rather than applying the averaging to the Navier-Stokes equations, introduce a generic quantity \( q_i \), which can represent velocity, temperature, tracer etc., for which the transport equation is

\[
\frac{\partial q_i}{\partial t} + (\langle \mathbf{u} \rangle \cdot \nabla) q_i = -\frac{\partial p_{q_i}}{\partial x_{q_i}} + \kappa_{q_i} \nabla^2 q_i + f_{q_i},
\]

where it is understood that the pressure gradient is only present in the corresponding momentum equation, and \( \kappa_{q_i} \) and \( f_{q_i} \) are the diffusivity constant and source term for \( q_i \).

Splitting each quantity into the averaged and fluctuating part gives

\[
\frac{\partial}{\partial t} \left( \langle q_i \rangle + q_i' \right) + (\langle \mathbf{u} \rangle + \mathbf{u}') \cdot \nabla (\langle q_i \rangle + q_i') = -\frac{\partial}{\partial x_{q_i}} \left( \langle p_{q_i} \rangle + p_{q_i}' \right) + \kappa_{q_i} \nabla^2 (\langle q_i \rangle + q_i') + (\langle f_{q_i} \rangle + f_{q_i}').
\]

\[
(2.109)
\]
When this equation is averaged,

\[
\frac{\partial \langle q_i \rangle}{\partial t} + \langle (\mathbf{u} \cdot \nabla) (q_i) \rangle + \langle (\mathbf{u}' \cdot \nabla) q_i' \rangle = - \frac{\partial \langle p_{ki} \rangle}{\partial x_i} + \kappa_i \nabla^2 \langle q_i \rangle + \langle f_{ki} \rangle,
\]

(2.110)

the non-linearity of the advection term gives rise to an extra term, which by (2.107) can be written as

\[
\langle (\mathbf{u}' \cdot \nabla) q_i' \rangle = \nabla \cdot \langle q_i' \mathbf{u}' \rangle = \frac{\partial}{\partial x_k} \langle q_i' u_k' \rangle = - \frac{\partial R_{ki}}{\partial x_k},
\]

(2.111)

where \( R_{ki} = - \langle u_k' q_i' \rangle \). For the special case where \( q_i = u_i \), \( R_{ij} = - \langle u_i' u_j' \rangle \) is known as a Reynolds stress, due to the resulting similarity with the viscous stresses. The general term \( R_{ki} \) is often referred to as a Reynolds stress for convenience, where it will be understood that the suffixes \( i, j, r \) and \( z \), respectively, will refer to velocities \( u_i, u_j, u_r \) and \( u_z \), respectively, and any other suffix will refer to that particular variable, in this case \( q_i \).

The similarity of the viscous and Reynolds stresses is particularly clear if the Navier-Stokes equations are written as

\[
\frac{\partial \langle u_i \rangle}{\partial t} + \langle (\mathbf{u} \cdot \nabla) (u_i) \rangle = - \frac{\partial \langle p \rangle}{\partial x_i} + \frac{\partial}{\partial x_j} \left( \langle \tau_{ij} \rangle + R_{ij} \right) + \langle f_i \rangle,
\]

(2.112)

where \( \langle \tau_{ij} \rangle \) is the averaged viscous stress. This is the first sign of the closure problem of turbulence. Averaging the non-linear term has given rise to the extra Reynolds stress term, a second-order correlation. As will become apparent, the transport equation for this second-order correlation can be derived, but involves third-order correlations. The third-order correlation transport equation involves fourth-order correlations, and so on. To close the system, assumptions have been made, and the \( n \)-th order correlation has to be modelled, for some \( n \). Turbulence models are discussed in section 2.3.6.

Subtracting (2.110) from (2.109) gives the transport equation for \( q_i' \)

\[
\frac{\partial q_i'}{\partial t} + \langle (\mathbf{u} \cdot \nabla) q_i' \rangle + \langle (\mathbf{u}' \cdot \nabla) q_i' \rangle - \langle \mathbf{u}' \cdot \nabla q_i' \rangle = - \frac{\partial p_{ki}'}{\partial x_k} + \kappa_i \nabla^2 q_i' + f_{ki}',
\]

(2.113)

which can similarly be written for \( q_j' \). Multiplying the \( q_i' \) equation by \( q_j' \), the \( q_j' \) equation by \( q_i' \), summing the result and taking the average gives

\[
\left( \frac{\partial}{\partial t} \langle q_i' q_j' \rangle + \langle q_i \frac{\partial q_j'}{\partial t} \rangle \right) + \left( \langle q_j' \frac{\partial q_i'}{\partial x_k} \rangle + \langle q_i' \frac{\partial q_j'}{\partial x_k} \rangle \right) + \left( \langle u_k' \frac{\partial q_i'}{\partial x_k} \rangle \right) + \left( \langle q_i' u_k' \frac{\partial q_j'}{\partial x_k} \rangle \right) + \left( \langle q_i' u_k' \frac{\partial q_j'}{\partial x_k} \rangle \right) = - \left( \langle \kappa_i \langle q_j' \nabla^2 q_i' \rangle + \kappa_q \langle q_i' \nabla^2 q_j' \rangle \rangle + \langle q_i' q_j' \rangle \right).
\]

(2.114)

With repeated uses of the chain rule and incompressibility, this simplifies to

\[
\left( \frac{\partial}{\partial t} \langle q_i' q_j' \rangle \right) + \left( \langle u_k' \frac{\partial q_i' q_j'}{\partial x_k} \rangle \right) + \left( \langle q_i' u_k' \frac{\partial q_j'}{\partial x_k} \rangle \right) + \left( \langle q_j' u_k' \frac{\partial q_i'}{\partial x_k} \rangle \right) = - \left( \langle \kappa_i \langle q_j' \nabla^2 q_i' \rangle + \kappa_q \langle q_i' \nabla^2 q_j' \rangle \rangle + \langle q_i' q_j' \rangle \right).
\]

(2.115)
and finally allows a transport equation for the general second-order correlation to be formulated

\[
\left( \frac{\partial}{\partial t} + (\mathbf{u}) \cdot \nabla \right) \langle u'_i u'_j \rangle = \left( \tau^{R}_{ki} \frac{\partial}{\partial x_k} + \tau^{R}_{kj} \frac{\partial}{\partial x_j} \right) \frac{\partial}{\partial x_k} \langle q'_i q'_k \rangle \\
- \left( \frac{\partial}{\partial x_k} \langle q'_i q'_k \rangle \right) + \left( \frac{\partial}{\partial x_k} \langle q'_i q'_k \rangle \right) - \left( \frac{\partial}{\partial x_k} \langle q'_i q'_k \rangle \right) + \left( \frac{\partial}{\partial x_k} \langle q'_i q'_k \rangle \right) + \left( \frac{\partial}{\partial x_k} \langle q'_i q'_k \rangle \right)
\]

\[
+ \left( \frac{\kappa_q + \kappa_q}{2} \right) \left( \nabla^2 \langle q'_i \rangle - 2 \left( \frac{\partial q'_i}{\partial x_k} \frac{\partial q'_i}{\partial x_k} \right) \right) + \left( \frac{\kappa_q - \kappa_q}{2} \right) \left( \nabla^2 \langle q'_i \rangle - 2 \left( \frac{\partial q'_i}{\partial x_k} \frac{\partial q'_i}{\partial x_k} \right) \right). \]

For the general velocity Reynolds stress, this simplifies to

\[
\left( \frac{\partial}{\partial t} + (\mathbf{u}) \cdot \nabla \right) \langle u'_i u'_j \rangle = \left( \tau^{R}_{ki} \frac{\partial}{\partial x_k} + \tau^{R}_{kj} \frac{\partial}{\partial x_j} \right) \frac{\partial}{\partial x_k} \langle u'_i u'_k \rangle \\
- \left( \frac{\partial}{\partial x_k} \langle p'_i u'_j \rangle + \frac{\partial}{\partial x_j} \langle p'_i u'_k \rangle - 2 \langle p'_i S'_{ij} \rangle \right) + \nu \left( \nabla^2 \langle u'_i u'_j \rangle - 2 \left( \frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k} \right) \right) + \left( \langle u'_i f'_j \rangle + \langle u'_j f'_i \rangle \right). \]

This demonstrates the closure problem outlined above. The evolution equations for these second-order terms involve third-order correlations, \( \partial_k \langle u'_i u'_j u'_k \rangle \) in (2.117). Turbulence models used to close the equations will be discussed in section 2.3.6.

**Axisymmetric time-averaged equations of motion**

For discussion of the round jet, it will be necessary to have these equations in axisymmetric form, and so the Reynolds stress transport equations are briefly derived. In this context, it is appropriate to define the temporal average and take advantage of azimuthal symmetry. Therefore, let the average of the Reynolds stress transport equations are briefly derived. In this context, it is appropriate to define the temporal average and take advantage of azimuthal symmetry. Therefore, let the average of \( q(r, \theta, z, t) \) be defined by

\[
\bar{q}(r, z) = \frac{1}{2\pi \Delta t} \int_{t_0}^{t_0 + \Delta t} \int_0^{2\pi} \bar{q}(r, \theta, z, t) \, d\theta \, dt,
\]

where \( \Delta t \) is the time period over which averaging is taken, assumed to be large compared with the time scales of the processes in the jet, e.g. the eddy turnover time. The averaged Navier-Stokes equations in cylindrical polar coordinates \( (r, \theta, z) \), under the Boussinesq approximation (see section 2.1) are

\[
(\bar{u} \cdot \nabla) \bar{u}_r = -\frac{\partial \bar{\rho}}{\partial r} + \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \bar{u}_r}{\partial r} \right) + \frac{\tau_{rr}^R}{r} \right) + \frac{\tau_{\theta\theta}^R}{r} + \nu \left( \nabla^2 \bar{u}_r - \frac{\bar{u}_r}{r^2} \right), \tag{2.119}
\]

\[
(\bar{u} \cdot \nabla) \bar{u}_z = -\frac{\partial \bar{\rho}}{\partial z} + \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \bar{u}_z}{\partial r} \right) + \frac{\tau_{zz}^R}{r} \right) + \nu \nabla^2 \bar{u}_z + \alpha_T \bar{T}. \tag{2.120}
\]

Following the same method as above, the Reynolds stress transport equations are

\[
(\bar{u} \cdot \nabla) \bar{u}'_r \bar{u}'_r = 2 \left( \frac{\tau_{rr}^R}{r} \frac{\partial \bar{u}_r}{\partial r} + \tau_{rr}^R \frac{\partial \bar{u}_r}{\partial r} \right) - \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \bar{u}'_r \bar{u}'_r \right) + \frac{\tau_{rr}^R}{r} \right) + \frac{\tau_{\theta\theta}^R}{r} + \nu \left( \nabla^2 \bar{u}'_r \bar{u}'_r - 2 \frac{\bar{u}'_r \bar{u}'_r}{r^2} \right), \tag{2.121}
\]

\[
(\bar{u} \cdot \nabla) \bar{u}'_z \bar{u}'_z = 2 \left( \frac{\tau_{zz}^R}{r} \frac{\partial \bar{u}_z}{\partial r} + \tau_{zz}^R \frac{\partial \bar{u}_z}{\partial r} \right) - \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \bar{u}'_z \bar{u}'_z \right) + \frac{\tau_{zz}^R}{r} \right) + \nu \left( \nabla^2 \bar{u}'_z \bar{u}'_z - 2 \frac{\bar{u}'_z \bar{u}'_z}{r^2} \right) + 2\alpha_T \bar{u}'_z \bar{T}, \tag{2.122}
\]

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(\bar{u} \cdot \nabla) \bar{u}' \bar{u}'_z = \left( r_{rr} \frac{\partial \bar{u}_z}{\partial r} + r_{rz} \left( \frac{\partial \bar{u}_r}{\partial r} + \frac{\partial \bar{u}_z}{\partial z} \right) + r_{zz} \frac{\partial \bar{u}_r}{\partial z} \right) - \left( \frac{1}{r} \frac{\partial}{\partial r} (ru'_r u'_z) + \frac{\partial}{\partial z} (u'_r u'_z) \right)

- 2 \left( \frac{\partial}{\partial r} p' u'_z + \frac{\partial}{\partial z} p' u'_r - p' \frac{\partial u'_r}{\partial r} - p' \frac{\partial u'_r}{\partial z} \right) + \frac{u'_r u'_z u'_r}{r}

+ \nu \left( \nabla^2 u'_r u'_z - 2 \left( \frac{\partial u'_r}{\partial r} \frac{\partial u'_z}{\partial r} + \frac{\partial u'_r}{\partial z} \frac{\partial u'_z}{\partial z} \right) - \frac{u'_r u'_z}{r^2} \right) + \alpha \bar{u}_z T'.

(2.123)

2.3.6 Turbulence Models

This section discusses various modelling techniques for closing the averaged equations of motions. The simplest models are considered first, followed by methods of increasing complexity, and finishing with the full Reynolds stress transport equations. A full review of modern techniques is well beyond the scope of this work, so only a brief summary is given of each method. Although turbulence models are not applied in this study, the following description gives context to later discussions and defines some necessary concepts.

**Boussinesq’s eddy viscosity hypothesis**

Building on the similarity between the Reynolds and viscous stresses, in 1877 Boussinesq proposed a shear-stress strain-rate relation for simple statistically one-dimensional flow as

$$
\tau_{xy} = \nu \tau \left( \frac{\partial \langle u_x \rangle}{\partial y} \right),
$$

(2.124)

where \( \nu_{r} \) is an eddy viscosity. This can be generalised to three-dimensional flows simply

$$
\tau_{ij} = \nu_{r} \left( \frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i} \right) - \frac{1}{3} \langle u'_k u'_k \rangle \delta_{ij}.
$$

(2.125)

The closure problem has been reduced to determining the eddy viscosity \( \nu_{r} \), where no assumptions have yet been made about spatial or temporal dependence. The simplest idea is to assume that the eddy viscosity is constant, but this seldom leads to realistic models.

**Prandtl’s mixing length model**

Prandtl [67] was the first to propose a model for \( \nu_{r} \) that depended on the flow, which was based on ideas from the kinetic theory of gases and the simple dimensional necessity

$$
\nu_{r} = l_m u_m
$$

(2.126)

where \( l_m \) is called the mixing length, and \( u_m \) is some (positive) measure of the velocity, both of which require estimates for closure. Prandtl suggested for statistically one-dimensional flows that \( u_m \propto l_m \left| \frac{\partial u}{\partial y} \right| \), and so

$$
\nu_{r} = l_m^2 \left| \frac{\partial \langle u_x \rangle}{\partial y} \right|,
$$

(2.127)
which leaves \( l_m \) to be determined empirically. A three-dimensional generalisation was proposed by Smagorinsky [80] based on the mean strain rate

\[
\nu_\tau = l_m^2 \sqrt{2 \langle S_{ij} \rangle \langle S_{ij} \rangle},
\]

(2.128)

but still relies on the specification of the mixing length \( l_m \). This simple, general turbulence model has been used in a variety of flows, for a review see Wilcox [92], for example.

**One-equation model**

The velocity scale, \( u_m \), can be estimated in terms of the turbulent kinetic energy (TKE), \( k = \frac{1}{2} \langle u'_i u'_i \rangle \), as \( u_m = c \sqrt{k} \), for some constant \( c \), as suggested independently by Kolmogorov [49] and Prandtl [68]. This can then be used in the mixing length model to give \( \nu_\tau = l_m c \sqrt{k} \), but requires estimates for \( k \) and \( l_m \). As before, \( l_m \) is specified from empirical data, but a transport equation for \( k \) can be formulated, hence the name one-equation model. The exact transport equation for \( k \) can be derived by summing the transport equations (2.117) for the diagonal components of the Reynolds stress tensor, which gives

\[
\left( \frac{\partial}{\partial t} + \langle \mathbf{u} \rangle \cdot \nabla \right) k = \langle u'_i u'_i \rangle \frac{\partial \langle u_i \rangle}{\partial x_j} + \nabla \cdot \left( \frac{\nu_\tau}{\sigma_k} \nabla k \right) - C_D \frac{k^{\frac{3}{2}}}{l_m}.
\]

(2.129)

The first term on the right-hand side is the production of TKE, which is the exact term, and naturally, Boussinesq’s equation is used for the Reynolds stress term. The second term is the diffusion of TKE, which has been modelled by a gradient-diffusion argument, and \( \sigma_k \) is known as the turbulent Prandtl number for TKE. The third term is dissipation, and has been modelled based on dimensional arguments, i.e. \( \varepsilon \sim \frac{k^2}{l_m} \), taking \( u \sim \sqrt{k} \) and \( l \sim l_m \). Again, a review can be found in Wilcox [92].

**Two-equation model**

The most famous two-equation model is the \( k-\varepsilon \) model. This is a straightforward extension of the one-equation model outlined above, which, instead of using an expression for the dissipation of TKE, a second equation is solved, a transport equation for \( \varepsilon \). Because two-equation models solve for two quantities, it is possible to derive expressions for \( \nu_\tau \), as well as length and time scales. Therefore, there is no need to specify quantities like the mixing length \( l_m \), and such two-equation models are said to be complete. In the case of the \( k-\varepsilon \) model, the eddy viscosity can be written as

\[
\nu_\tau = C_\mu \frac{k^2}{\varepsilon},
\]

(2.130)

where \( C_\mu \) is a model constant.

Although an exact equation for \( \varepsilon \) can be derived, it is seldom used; an empirical equation is usually preferred. The *standard* model is due to Jones and Launder [44], and can be written as

\[
\left( \frac{\partial}{\partial t} + \langle \mathbf{u} \rangle \cdot \nabla \right) \varepsilon = C_P \left( r_{ij} \langle S_{ij} \rangle \right) \varepsilon + \nabla \cdot \left( \frac{\nu_\tau}{\sigma_\varepsilon} \nabla \varepsilon \right) - C_\varepsilon \frac{\varepsilon^2}{k},
\]

(2.131)
Reynolds stress models

A more complete way to account for the interaction of the turbulence with the mean flow is to use the full Reynolds stress transport equations (2.117), which can be written as

$$\left( \frac{\partial}{\partial t} + \langle \mathbf{u} \rangle \cdot \nabla \right) \tau_{ij}^R = \mathcal{P}_{ij} + \mathcal{G}_{ij} + \phi_{ij} + \left( \mathcal{D}_{ij}^\tau + \mathcal{D}_{ij}^p + \mathcal{D}_{ij}^r \right) - \varepsilon_{ij}$$  \hspace{1cm} (2.132)

A complete description of this equation and the ways in which the terms are modelled is beyond the scope of this work, for a good starting reference, see Hanjalić and Jakirlić [37]. However, a brief summary of the terms is given here.

The term \( \mathcal{P}_{ij} = r_{ik} \frac{\partial \langle u_i \rangle}{\partial x_k} + r_{jk} \frac{\partial \langle u_j \rangle}{\partial x_k} \) represents turbulence production by mean flow deformation. Generation of turbulence can also be brought about by the interaction with the body forces \( \mathcal{G}_{ij} = \langle u_i' u_j' \rangle + \langle u_i' f_j' \rangle \).

Turbulent diffusion is represented by three terms \( \mathcal{D}_{ij} = \mathcal{D}_{ij}^\tau + \mathcal{D}_{ij}^p + \mathcal{D}_{ij}^r \), where \( \mathcal{D}_{ij}^\tau = - \left( \frac{\partial}{\partial x_k} \langle p' u_j' \rangle + \frac{\partial}{\partial x_k} \langle p' u_i' \rangle \right) \), are diffusion due to fluctuating velocity and pressure, and \( \mathcal{D}_{ij}^p = \nu \nabla^2 \langle u_i' u_j' \rangle \) represents molecular diffusion. Viscous destruction is captured by \( \varepsilon_{ij} = 2\nu \left( \frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k} \right) \). Finally, and usually the most important term for turbulence modelling, is the pressure-strain interaction \( \phi_{ij} = 2 \langle p' S_{ij} \rangle = \left\langle p' \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right\rangle \), which generally acts in such a way to make the turbulence more isotropic. The terms \( \mathcal{P}_{ij}, \mathcal{G}_{ij} \) and \( \mathcal{D}_{ij}^\nu \) are known quantities, but the remaining terms, \( \mathcal{D}_{ij}^\tau, \mathcal{D}_{ij}^p, \varepsilon_{ij} \) and \( \phi_{ij} \), need to be modelled. The way in which these terms are modelled is not considered in full here, see [37] for a review, but it is worth mentioning the \( \varepsilon_{ij} \) term. There are two approaches that can be taken. A transport equation can be solved, similar to (2.131). Alternatively, a simple explicit expression can be used, such as \( \varepsilon_{ij} = \frac{2}{3} \nu \left( \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \right) \delta_{ij} \), which assumes the small scales are locally isotropic.

The major drawback of using a Reynolds stress transport model is the shear size of the system. For a three-dimensional system, modelling just the velocity components, i.e. no scalar equations, and without a transport equation for \( \varepsilon \), there are ten equations to be solved. Including temperature, a scalar and a transport equation for \( \varepsilon \) increases the number of equations to twenty-one, without accounting for gradient-diffusion equations. However, the bonus is that the equations can overcome some of the major drawbacks of the simpler models. Even the \( k-\varepsilon \) model can perform badly in the presence of strong anisotropies.
2.4 Simulating Turbulent Flows

2.4.1 Numerical Approaches

Direct Numerical Simulation

Direct Numerical Simulation (DNS) is the simplest method in terms of formulation. The idea is to model the Navier-Stokes equations in their entirety without any extra modelling. The term DNS has come to take two meanings. The first, and the literal interpretation, is that the equations are simply solved. The second, more restrictive meaning, has the additional requirement that all the scales of motion (time and length) are well resolved. If this is possible, then the accuracy of such a method is unparalleled by simulation. However, the major drawback of DNS is the shear expense at high Reynolds numbers; the computational cost of a DNS simulation grows with Re$^3$. It is this cost that motivates alternative approaches.

DNS calculations place great demands on the accuracy of numerical methods, and so (pseudo-)spectral methods are often used. These are ideally suited to homogeneous turbulence in a periodic domain, and modifications have to be made to handle directions of inhomogeneity and boundaries. Spectral methods are not used exclusively for DNS simulations, it is of course possible to any of the methods outlined in section 2.2. A review can be found in Moin and Mahesh [56], and Pope [65] gives a brief summary of different flows and extensions.

Reynolds Averaged Navier Stokes Equations

One approach to circumvent the range of time and length scales required for DNS is to use Reynolds averaging to obtain the mean equations of motion, the so-called Reynolds Averaged Navier Stokes (RANS) equations, (2.112). The averaging takes advantage of statistical properties to reduce the computational expense. This method is particularly appropriate for statistically steady flows, but can also be used to take advantage of geometric symmetry. However the averaging is performed, the equations involve the unknown Reynolds stresses, which must be modelled to close the system. Any of the turbulence models outlined in section 2.3.6 is appropriate, which account for a small sample of available models.

Large Eddy Simulation

Large Eddy Simulation (LES) lies between DNS and RANS, as three-dimensional unsteady equations are solved, but only some of the small scale structure is resolved. The idea is to use a spatial filter operation to separate the large and small scales. This decomposition is analogous to Reynolds decomposition, but has important differences. Each flow variable is decomposed into a filtered component and a residual or
subgrid-scale (SGS) component. The Navier-Stokes equations are also filtered, and similar to Reynolds averaging the non-linear term gives rise to an extra term, the residual stress tensor or SGS stress tensor. Again, this term has to be modelled to close the system so that the filtered equations can be solved.

The general filtering operation, denoted here by an overbar, can be written as a convolution

\[
\bar{u}(x,t) = \int_V G(r,x)u(x-r,t)\,dV,
\]

where \( V \) is the entire domain, and the filter \( G \) has the restriction \( \int_V G(r,x)\,dV = 1 \). Two important differences between RANS and LES arise from this filtering operation. Firstly, in general, the filtered fluctuation is not zero, i.e. \( \bar{u} \neq 0 \). Since \( G \) is not necessarily homogeneous, differentiation and filtering do not necessarily commute. This can lead to extra terms in the averaged equations that require further modelling. The result is essentially the same as Reynolds averaging, the filtering operation and non-linear advection term lead to extra terms that require modelling to close the system. However, because large scale unsteady motion can be captured, LES is more appropriate than RANS for certain kinds of flow where such motions are important, and is computationally less demanding than DNS. A good introduction to LES can be found in Pope [65], and a review in Lesieur and Metais [54].

**Monotone Integrated Large Eddy Simulation**

Monotone Integrated Large Eddy Simulation (MILES) is a technique introduced by Boris et al. [18], motivated by the principles behind shock-capturing schemes used for compressible flows. The idea is that turbulence is characterised by high levels of fluctuating vorticity, and therefore sharp velocity gradients. It was described in section 2.2 that high-order monotone schemes are required to resolve such flow features correctly. The “convenient conspiracy”, as it was called by Oran and Boris [62], is that these high-order monotone schemes have an inherent truncation error that acts as a numerical diffusion, which can emulate the effects of physical viscosity, see Grinstein and Guirguis [36]. Using a finite-volume approach, the cell averaging discretisation of the data can be considered to be implicit filter. Therefore, these components combine to form a natural form of LES, and hence the name MILES. Furthermore, it was demonstrated by Ghosal [73] that the numerical error in LES codes can be of a similar order of magnitude to the SGS model, and mask its effect. It has also been shown by Fureby et al. [34] and Menon et al. [55] that in certain flows the gross scale features appear to be insensitive to the choice of SGS model, particularly if the spatial resolution is high enough. Porter et al. [66] showed that it is possible to recover energy spectra that fit the \( k^{-1}\) description. Fureby and Grinstein [33] applied a MILES approach to free shear flows, specifically transitional free jets. They demonstrated that the SGS models (both implicit and explicit) only affect the high wavenumber end of the inertial range and viscous range. It was also shown that the MILES approach did not perform worse than other SGS models tested, and was better if the resolution was fine enough for the SGS cutoff to be in the inertial range and so the majority of the energy was resolved. It was concluded that a MILES approach was suitable for moderately high Reynolds number free shear flows. A
2.4.2 Numerical Observations of Homogeneous Isotropic Turbulence

There are two main groups of work that tend to be used as starting points when simulating homogeneous isotropic turbulence. Vincent and Meneguzzi published two papers, [89] in 1991 and [90] in 1994. Jiménez and Wray also published two papers, [43] in 1993 with Saffman and Rogallo, and [42] in 1998.

Vincent and Meneguzzi used a pseudo-spectral method to perform a forced DNS with a resolution of $240^3$ at $R_\lambda = 150$. They showed that the predominant feature of a turbulent field was the organisation of vorticity into elongated thin tubes, or worm vortices. They found a power-law in the energy spectra for wavenumbers below 25 with an exponent slightly larger than the theoretical minus five-thirds. Probability density functions of velocity were near Gaussian while velocity derivatives and the first-order structure function at small scales were highly non-Gaussian. Power-law exponents of the structure functions were in agreement with the results obtained by Anselmet et. al. [7]. The intermittency factor, which was not particularly accurately determined, was found to be slightly lower at 0.13. In their second paper attention was focused on the worm vortices that were shown to form as vortex sheets roll up due to shear instabilities. It was proposed that this process was at least as important as vortex stretching in transferring energy from large to small scales.

Jiménez et. al. also used a spectral code, and looked at forced turbulence over a range of Taylor Reynolds numbers from 35 to 170. They essentially confirmed the results of Vincent and Meneguzzi, but they also used vortex characterising and tracking techniques to study their properties and statistics. Amongst other things they showed these vortices had radii of a few times the Kolmogorov scale, $\eta$, and lengths that were of the order of the integral scale, $l$. Table 2.1 shows the high-order moments of the velocity gradient PDFs at the different Reynolds numbers simulated.

<table>
<thead>
<tr>
<th>$Re_\lambda$</th>
<th>$M_3$</th>
<th>$M_4$</th>
<th>$M_5$</th>
<th>$M_6$</th>
<th>$M_4$</th>
<th>$M_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>35.1</td>
<td>-0.490</td>
<td>4.2</td>
<td>-6.5</td>
<td>40</td>
<td>5.7</td>
<td>90</td>
</tr>
<tr>
<td>64.1</td>
<td>-0.495</td>
<td>4.6</td>
<td>-8.0</td>
<td>55</td>
<td>6.1</td>
<td>110</td>
</tr>
<tr>
<td>94.1</td>
<td>-0.520</td>
<td>5.3</td>
<td>-10.0</td>
<td>80</td>
<td>7.6</td>
<td>200</td>
</tr>
<tr>
<td>168.1</td>
<td>-0.525</td>
<td>6.1</td>
<td>-12.0</td>
<td>125</td>
<td>9.4</td>
<td>370</td>
</tr>
</tbody>
</table>

Table 2.1: Moments of longitudinal and lateral velocity gradients taken from Jiménez et. al. [43].
2.5 The Code: IAMR

IAMR is an incompressible variable density Navier-Stokes solver, written at the Center for Computational Sciences and Engineering, Lawrence Berkeley National Laboratory. It uses a second-order projection method to solve the Navier-Stokes equations in conservation form, is capable of Adaptive Mesh Refinement (AMR) and is parallelised. A fractional-step scheme is used, specifically a two-step predictor-corrector method, an outline of which is presented below. The details of the AMR method and the development of the algorithm can be found in Almgren et. al. [5], and the references therein. No development of the algorithm was necessary for this study, and only minor modifications required to account for off-source heating. Therefore, only a brief summary of the algorithm is given here.

The algorithm is based around a hierarchy of Cartesian grids of uniform cell widths. Velocities, density and advected scalars are evaluated at cell centres (cell averages), denoted by \( u_{i,j,k}^n, \rho_{i,j,k}^n \) and \( c_{i,j,k}^n \), respectively, and pressure is staggered in time and evaluated on cell nodes (corners), and is denoted by \( p_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} \). Inflow, outflow, slip, no-slip and periodic boundary conditions can be used, as described in section 2.2.

The advection term is captured by evaluating intercell fluxes at each cell face \( F_{adv,i,j,k}^{n+\frac{1}{2}} = \bar{q}^{n+\frac{1}{2}} u_{adv,i,j,k}^{n+\frac{1}{2}} \), and each flow variable \( q \). This is achieved by extrapolating each flow variable to the cell face and half time step using a second-order Taylor expansion in space and time, replacing temporal with spatial derivatives using the equations of motion. This defines a left and right state at each face. An advective velocity is defined by upwinding these left and right states. An intermediate projection is used to enforce the divergence free constraint on this advective velocity, and this defines \( u_{adv,i,j,k}^{n+\frac{1}{2}} \). This advective velocity is then used to upwind the left and right states for each variable, including the velocities, to give \( \bar{q}^{n+\frac{1}{2}} \), completing the intercell flux. This intercell flux combines with any source terms and similar fluxes for the diffusive terms to give an update expression for each flow variable. This completes the predictor step. The velocity field obtained from this update is not necessarily divergence free, so an approximate projection step is used to complete the iteration.

Equations of motion

The discretised equations of motion that are solved are written

\[
\begin{align*}
\frac{u^* - u^n}{\Delta t} &= -\nabla \cdot F_{u,adv}^{n+\frac{1}{2}} + \frac{1}{\rho^{n+\frac{1}{2}}} \left( -\nabla p^{n-\frac{1}{2}} + \mu \nabla^2 \left( \frac{u^* + u^n}{2} \right) + H_u^{n+\frac{1}{2}} \right), \\
\frac{\rho^{n+1} - \rho^n}{\Delta t} &= -\nabla \cdot F_{\rho,adv}^{n+\frac{1}{2}}, \\
\frac{c^{n+1} - c^n}{\Delta t} &= -\nabla \cdot F_{c,adv}^{n+\frac{1}{2}} + \left( \kappa_c \nabla^2 \left( \frac{c^{n+1} + c^n}{2} \right) + H_c^{n+\frac{1}{2}} \right),
\end{align*}
\]

where \( u^* \) is the intermediate velocity field, \( \rho \) and \( c \) are density and advected scalar, respectively, and \( H_q \) denotes the source term for the general flow variable \( q \). The advection term is approximated using a
monotonicity-limited fourth-order slope approximation, see [22].

**Projection**

The projection is based on a Hodge decomposition, specifically every vector $\mathbf{u}$ can be written as $\mathbf{u} = \mathbf{u}_d + \nabla \phi$, where $\nabla \cdot \mathbf{u}_d = 0$. An approximate projection is used to solve a discrete Poisson equation for an appropriately defined $\phi$, which is used to enforce the divergence free constraint and update the pressure field.

To project the vector $\mathbf{u}^*$, the following discrete equation is solved for $\phi$

$$\mathcal{L}(\phi) = \mathcal{D}(\mathbf{u}^*),$$

(2.137)

where $\mathcal{L}$ is a discrete Laplacian, and $\mathcal{D}$ is a discrete divergence operator. Then the velocity and pressure can be updated as follows

$$\mathbf{u}^{n+1} = \mathbf{u}^* - \mathcal{G}(\phi),$$

(2.138)

$$p^{n+\frac{1}{2}} = p^{n-\frac{1}{2}} + \phi,$$

(2.139)

where $\mathcal{G}$ is a discrete gradient operator. The implementation is much more complicated than this, but this description summarises the general idea. The details can be found in [6].

**Adaptive mesh refinement**

Adaptive mesh refinement is achieved using a hierarchy of grids. Specifically, a base grid is defined, and then depending on user-defined gridding criteria, regions of the domain are flagged for refinement. These regions are then covered by grids at a higher resolution (either twice or four times the resolution), and the process is repeated for however many levels are specified. Each level of grids is then evolved with a time step appropriate for that level, i.e. if there is one level of refinement with a factor two, the refined grids are evolved with a time step that is half of the time step of the base grid. Data from adjoining grids at the same level or, in the absence of such a grid, the level below is used as boundary conditions for each grid, and the grids are synchronised as appropriate. Again, the implementation is much more complicated than this, but a full account is not appropriate here. The details can be found in [5].
This section discusses the phenomenology of turbulent buoyant jets and plumes, with and without off-source heating. All cases will consist of a maintained (statistically steady) source of momentum and/or buoyancy, angled in the positive vertical direction. A pure source of momentum is known as a jet, and a pure source of buoyancy (no momentum) is known as a plume. A source of both momentum and buoyancy is known as a forced plume or a buoyant jet. It is usual (and is the case here) to consider a jet or plume in an unbounded stationary fluid, which may or may not be stratified, although naturally occurring examples are rarely so simple.

This section is organised as follows. The traditional analysis of jets and plumes is summarised in section 2.6.1, followed by a description of second-order models, i.e. models that account, at least partially, for turbulent effects, in section 2.6.2. A summary of experimental results in section 2.6.3. The effects of off-source heating are discussed in section 2.6.4. Experimental and numerical results are summarised, along with the analysis of J. C. R. Hunt, which has been used to propose an explanation for the observations made in heated jets. Finally, a description of temporally evolving circular shear layers is given in section 2.6.5. This flow has been studied numerically in an attempt to draw analogies with spatially evolving jets, with significantly reduced computational expense.

### 2.6.1 Integral Models

The usual starting point for analytic theory of turbulent jets and plumes is the 1956 paper by Morton, Taylor and Turner [60], henceforth referred to as MTT. In this paper they set out the basic framework upon which many works have been based. They consider maintained point sources, although similar treatment can be applied to line sources without difficulty.

One of the key assumptions is that the rate of entrainment of fluid at the edge of the jet or plume is proportional to a characteristic velocity at that height, and is known as the entrainment assumption or entrainment hypothesis. This entrainment assumption was introduced by G. I. Taylor and has been used successfully in many different applications, as highlighted by Turner’s review paper [86].

In the case of stratified fluids, it is necessary to assume that the profiles of mean velocity and buoyancy are of similar form at all heights. In a homogeneous fluid, the entrainment and self-similar assumptions are interchangeable and imply each other. The last main assumption made in MTT is that the Boussinesq approximation (see section 2.1) is valid. The following description takes a similar approach to MTT, just with different notation.

Assuming that the flow is axisymmetric and statistically steady, cylindrical polars \((r, \theta, z)\) and the averaging defined by (2.118) are appropriate, and the average will be denoted by an overbar. Define fluxes
for mass \(\pi \rho_r \bar{Q}\), momentum \(\pi \rho_r \bar{M}\) and buoyancy \(\pi \rho_a \bar{B}\), where

\[
\bar{Q} = 2 \int_0^\infty \bar{u}_z r \, dr, \\
\bar{M} = 2 \int_0^\infty \bar{u}_z^2 r \, dr, \\
\bar{B} = 2 \int_0^\infty g \frac{\rho_a - \bar{\rho}}{\rho_r} \bar{u}_z r \, dr, \\
\bar{C} = 2 \int_0^\infty \bar{c} \bar{u}_z r \, dr,
\]

where \(\bar{u}_z(r, z)\) and \(\bar{\rho}(r, z)\) are the mean streamwise velocity and mean density, \(\rho_a(z)\) is the ambient density, \(\rho_r = \rho_a(0)\) is the constant reference density, usually taken to be the ambient density at the source, and \(g\) is acceleration due to gravity. The factors \(\pi \rho_r\) have been removed from the definitions of \(\bar{Q}, \bar{M}\) and \(\bar{B}\), as is customary, to simplify the resulting equations. These quantities are actually known as specific fluxes. The mass flux would be more accurately described as volume flux, but it seems more intuitive to discuss mass, momentum and buoyancy fluxes. Moreover, different authors use different constants in the definitions of the fluxes, but this choice is consistent with the treatment of Morton. Note that fluxes due to turbulence are neglected here, and will be discussed later. The use of an overbar on the fluxes is an abuse of notation, and denotes that these are the fluxes due to the mean flow, not the average.

For illustrative purposes, the self-similar profiles were taken in MTT to be ‘top-hat’ profiles, whereby the velocity and buoyancy are constant within the jet radius and zero outside. Other choices can be made, but simply changes the values of the constants resulting from integrations. In the top-hat case,

\[
\bar{Q} = b^2 \bar{u}_0, \\
\bar{M} = b^2 \bar{u}_0^2, \\
\bar{B} = b^2 \bar{u}_0 g \frac{\rho_a - \bar{\rho}}{\rho_r} = b^2 \bar{u}_0 g', \\
\bar{C} = b^2 \bar{u}_0 \bar{c}_0,
\]

where \(\bar{\rho}(z)\), \(\bar{u}_0(z)\) and \(\bar{c}_0(z)\) are the characteristic density, velocity and tracer concentration scales, respectively, \(g\) and \(g'\) are acceleration due to gravity and reduced gravity, respectively, and \(b(z)\) is the jet radius.

The system is then constructed by conservation of mass, momentum and buoyancy, as follows

\[
\frac{d\bar{Q}}{dz} = 2b \alpha_c \bar{u}_0, \\
\frac{d\bar{M}}{dz} = b^2 g \frac{\rho_a - \bar{\rho}}{\rho_r}, \\
\frac{d\bar{B}}{dz} = 2b \alpha_c \bar{u}_0 g' \frac{\rho_r - \rho_a}{\rho_r}, \\
\frac{d\bar{C}}{dz} = 0,
\]

where \(\alpha_c\) is the coefficient of entrainment, relating the rate of entrained fluid to the characteristic velocity, \(\bar{u}_0\), and it has been assumed that the Reynolds number is sufficiently high to neglect the viscous terms.
Throughout this work the ambient is taken to be unstratified, i.e. $\rho_a = \rho_r$. It follows that the right hand side of (2.150) is zero, and so $B$ becomes constant. The system can then be solved to give

$$\hat{Q} \propto z^\frac{3}{4}, \quad \tilde{M} \propto z^\frac{1}{4}, \quad \hat{B} = \text{const}, \quad \tilde{C} = \text{const}, \quad (2.152)$$

which lead directly to

$$b \propto z, \quad \bar{u}_0 \propto z^{-\frac{1}{2}}, \quad g\frac{\rho_r - \bar{\rho}_0}{\rho_r} \propto z^{\frac{3}{4}}, \quad \bar{c}_0 \propto z^\frac{5}{4}. \quad (2.153)$$

Similar treatment for a jet, i.e. $\hat{B} = 0$, leads to

$$b \propto z, \quad \bar{u}_0 \propto z^{-1}, \quad \bar{c}_0 \propto z^{-1}. \quad (2.154)$$

It is clear that both jets and plumes have straight sides ($b \propto z$), and so occupy conical regions. It was shown by \textit{MTT} that the constant of proportionality in the plume case is the same for all plumes, specifically $b = \frac{6\alpha_\varepsilon}{5}z$. Similarly, for jets $b = 2\alpha_\varepsilon z$. Both jets and plumes present a decay of streamwise velocity and buoyancy (scalar concentration in a jet). The velocity decay in plumes is slower than in jets due to the continual generation of momentum due to buoyancy. A knock-on effect of this is that the time scale in jets grows more rapidly with streamwise distance than in plumes. The dilution of the advected scalar is more rapid in a plume.

Morton [58], and later Morton and Middleton [59], extended the analysis of \textit{MTT} to the case where both momentum and buoyancy are supplied at the source. The resulting flow depends on the balance parameter

$$\Gamma = \frac{5\hat{B}\bar{Q}^2}{4\alpha_\varepsilon M^\frac{2}{3}}, \quad (2.155)$$

and the length scale

$$l_M = \frac{M^\frac{2}{3}}{\hat{B}^\frac{1}{2}}, \quad (2.156)$$

which has come to be known as the Morton length scale, or jet length. The balance parameter describes a class of flows, of which jets ($\Gamma = 0$) and plumes ($\Gamma = 1$) are special cases. When $\Gamma \in (0,1)$, the flow undergoes a jet-plume transition, where the momentum near the source dominates buoyancy effects, to a plume-like region in the far field, $z \gg l_M$, where the buoyancy plays the dominant role. Morton referred to these as forced plumes, but are also known as buoyant jets. Note that the far field is always plume-like, however little buoyancy is supplied at the source. Conversely, if $\Gamma > 1$, then there is a momentum deficit, and the flow undergoes an acceleration to balance the momentum and buoyancy. Again, the far field is always plume-like. Morton referred to these as lazy plumes. In each case the balance parameter tends to the limiting value of 1, i.e. $\Gamma = 1$ is a stable equilibrium, and $\Gamma = 0$ is unstable. Also note that it can be shown that

$$\frac{db}{dz} = 2\alpha_\varepsilon \left(1 - \frac{2}{5}\Gamma\right), \quad (2.157)$$

which leads to the special cases for jets ($b = 2\alpha_\varepsilon z$ for $\Gamma = 0$) and plumes ($b = \frac{6\alpha_\varepsilon}{5}z$ for $\Gamma = 1$), but also shows that for $\Gamma > \frac{5}{2}$ the plume radius can actually decrease.
An interesting question is whether the entrainment coefficient is universal. Values reported in the literature support the argument that it is higher in plumes than in jets. Based on Gaussian profiles, as opposed to top-hat, Fischer et al. [29] proposed values of 0.0535 and 0.0833 for jets and plumes, respectively. Papanicolaou and List [64] report values in close agreement, 0.0545 and 0.0875.

In the case of a buoyant jet, attempts have been made to match the value for the entrainment coefficient somehow. Based on the work of Priestley and Ball [70] and [69], using conservation of energy, and the local Richardson number the following relation was proposed

\[
\alpha_e = \alpha_j + (\alpha_p - \alpha_j) \left( \frac{R_i}{R_{lp}} \right),
\]

where \(\alpha_p\) and \(\alpha_j\) are the plume and jet entrainment coefficients, respectively, and \(R_i = \frac{\nabla^2 \beta}{M^2}\) defines an appropriate Richardson number (see section 2.1), and the suffices \(l\) and \(p\) denote local and plume values, respectively. It is interesting to note the similarity between this Richardson number and Morton’s balance parameter, particularly since the ratio of the two is a multiple of the entrainment coefficient. Another suggested form relating the entrainment coefficient to the Richardson number, see Fischer et al. [29],

\[
\alpha_e = \alpha_j \exp \left( \frac{R_i}{R_{lp}} \ln \left( \frac{\alpha_p}{\alpha_j} \right) \right),
\]

The comparison by Wang [91] shows little difference between the two functions and the experimental measurements, but the former is adopted in their model.

### 2.6.2 Second-order Models

This section considers models that begin to account for the effect of the turbulence on the flow. Under the assumptions that the Reynolds number is sufficiently high that viscous effects can be neglected, that the flow is statistically steady and axisymmetric, and that the Boussinesq approximation is valid, then the equations of motion (2.12)-(2.16), averaged according to (2.118), become

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \bar{u}_r \right) + \frac{\partial}{\partial z} \left( \bar{u}_z \right) = 0,
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \bar{u}_r \bar{u}_r \right) + \frac{\partial}{\partial z} \left( \bar{u}_z \bar{u}_r \right) = \frac{\partial \bar{p}}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \tau^R_{rr} \right) + \frac{\partial}{\partial z} \left( \tau^R_{rz} \right) - \frac{\tau^R_{rT}}{r},
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \bar{u}_r \bar{u}_z \right) + \frac{\partial}{\partial z} \left( \bar{u}_z \bar{u}_z \right) = \frac{\partial \bar{p}}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \tau^R_{rz} \right) + \frac{\partial}{\partial z} \left( \tau^R_{zz} \right) + \alpha_T \bar{T},
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \bar{u}_r \bar{c} \right) + \frac{\partial}{\partial z} \left( \bar{u}_z \bar{c} \right) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \tau^R_{rc} \right) + \frac{\partial}{\partial z} \left( \tau^R_{zc} \right),
\]

for radial and vertical average velocities \(\bar{u}_r\) and \(\bar{u}_z\), average temperature above ambient \(\bar{T}\), average tracer \(\bar{c}\), Reynolds stresses \(\tau^R_{ij} \approx -\frac{\partial Q}{\partial r^2}\), \(\alpha_T = g \beta_T (\beta_T\ is\ the\ coefficient\ of\ volumetric\ thermal\ expansion)\), and \(\alpha_e\) is a coefficient that describes the effect of volumetric heating.

\[1^{Note}\ that\ Fischer\ defines\ the\ Richardson\ number\ as\ the\ square\ root\ of\ the\ definition\ given\ here\ (the\ reciprocal\ of\ which\ is\ often\ known\ as\ a\ Froude\ number),\ which\ highlights\ the\ lack\ of\ convention\ in\ the\ literature\ with\ Richardson\ numbers.\]
Several different approaches can be taken at this stage to deal with the second-order correlations. Three common approaches are outlined briefly below.

A common step at this stage is to neglect pressure variations and incorporate the streamwise turbulent intensity in to the gradient on the left-hand side

\[
\frac{\partial}{\partial r} (r\bar{u}_r \bar{u}_z) + \frac{\partial}{\partial z} (r\bar{u}_z \bar{u}_z - r\tau_{rz}^R) = \frac{\partial}{\partial r} (r\tau_{rz}^R) + \alpha_T \bar{T}_r, \tag{2.165}
\]

Alternatively, according to Hussein et. al. [40] or Shabbir and George [78], the pressure gradient term can be approximated by

\[
\frac{\partial \bar{p}}{\partial z} \approx \frac{\partial}{\partial z} (\tau_{rr}^R) \tag{2.166}
\]

which modifies the momentum equation thus

\[
\frac{\partial}{\partial r} (r\bar{u}_r \bar{u}_z) + \frac{\partial}{\partial z} (r\bar{u}_z \bar{u}_z - r(\tau_{zz}^R - \tau_{rr}^R)) = \frac{\partial}{\partial r} (r\tau_{rz}^R) + \alpha_T \bar{T}_r, \tag{2.167}
\]

and similarly for the temperature equation.

Another approach is to assume that the boundary layer approximation is appropriate. This involves neglecting the advection terms in the radial momentum equation and assuming that streamwise gradients of Reynolds stresses are small compared with the radial gradients. Then, integrating the radial momentum equation radially from \( r \) to infinity gives

\[
\bar{p} - \bar{p}_\infty = \tau_{rr}^R - \int_r^\infty \frac{\tau_{rr}^R - \tau_{\theta\theta}^R}{r} \, dr, \tag{2.168}
\]

which implies that the pressure term in the streamwise momentum equation consists purely of streamwise gradients of Reynolds stress and so may be neglected. The equations then reduce to

\[
\frac{\partial}{\partial r} (r\bar{u}_z) = 0, \tag{2.169}
\]

\[
\frac{\partial}{\partial r} (r\bar{u}_r \bar{u}_z) + \frac{\partial}{\partial z} (r\bar{u}_z \bar{u}_z) = \frac{\partial}{\partial r} (r\tau_{rz}^R) + \alpha_T \bar{T}_r, \tag{2.170}
\]

\[
\frac{\partial}{\partial r} (r\bar{u}_r \bar{T}) + \frac{\partial}{\partial z} (r\bar{u}_z \bar{T}) = \frac{\partial}{\partial r} (r\tau_{rT}^R) + \alpha_c \bar{c}r, \tag{2.171}
\]

\[
\frac{\partial}{\partial r} (r\bar{u}_r \bar{c}) + \frac{\partial}{\partial z} (r\bar{u}_z \bar{c}) = \frac{\partial}{\partial r} (r\tau_{rc}^R). \tag{2.172}
\]

This simple approach will be assumed henceforth for this consideration of second-order models.

Assuming self-similar profiles the radial and streamwise velocities, temperature and tracer profiles can be written

\[
\bar{u}_r(\eta, z) = \bar{u}_0(z) g(\eta), \tag{2.173}
\]

\[
\bar{u}_z(\eta, z) = \bar{u}_0(z) f(\eta), \tag{2.174}
\]

\[
\bar{T}(\eta, z) = \bar{T}_0(z) \theta(\eta), \tag{2.175}
\]

\[
\bar{c}(\eta, z) = \bar{c}_0(z) \psi(\eta), \tag{2.176}
\]
where $\eta = \frac{r}{b(z)}$ is the radial coordinate normalised by a local length scale $b(z)$, usually taken to be a measure of the jet half-width.

Using these profiles, the equations can be integrated with respect to $r$ from zero to infinity

$$
\frac{d}{dz} \left( b^2 \tilde{u}_0 \right) \int_0^\infty f \eta \, d\eta = b \tilde{u}_0 \lim_{\eta \to \infty} (-\eta \eta),
$$

(2.177)

$$
\frac{d}{dz} \left( b^2 \tilde{u}_0 \right) \int_0^\infty f^2 \eta \, d\eta = \alpha_T b^2 \tilde{u}_0 \int_0^\infty \theta \eta \, d\eta,
$$

(2.178)

$$
\frac{d}{dz} \left( b^2 \tilde{u}_0 \tilde{T}_0 \right) \int_0^\infty f \theta \eta \, d\eta = \alpha_c b^2 \tilde{c}_0 \int_0^\infty \psi \eta \, d\eta,
$$

(2.179)

$$
\frac{d}{dz} \left( b^2 \tilde{u}_0 \tilde{c}_0 \right) \int_0^\infty f \psi \eta \, d\eta = 0,
$$

(2.180)

assuming that the Reynolds stresses vanish at infinity. The quantity $\lim_{\eta \to \infty} (-\eta \eta)$ describes the entrainment of fluid, which can be used to define an entrainment coefficient $\alpha_e = \left( \frac{2}{f(z) \eta} \right)^{1/2} \lim_{\eta \to \infty} (-\eta \eta)$, which must be a constant. This is a consequence of assuming that the solutions are self-similar.

By introducing a generic flow variable $q$, the symmetry in the equations can be exploited. Specifically, (2.170)-(2.172) can be written, in non-conservative form, as

$$
\tilde{u}_r \frac{\partial \tilde{q}}{\partial r} + \tilde{u}_z \frac{\partial \tilde{q}}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \tilde{q} \right) + \alpha_s \tilde{\alpha},
$$

(2.181)

where $s$ is the generic source variable corresponding to the $q$ equation. Similar treatment of (2.174)-(2.176) allows the self-similar profiles to be written as:

$$
\tilde{q}(\eta, z) = \tilde{q}_0(z) \chi_q(\eta).
$$

(2.182)

This can be differentiated with respect to $\eta$ and $z$ to give respectively:

$$
\frac{\partial \tilde{q}}{\partial \eta} = \frac{\tilde{q}_0}{b} \chi_q',
$$

(2.183)

$$
\frac{\partial \tilde{q}}{\partial z} = \frac{d\tilde{q}_0}{dz} \chi_q - \frac{\tilde{q}_0}{b} \frac{db}{dz} \eta \chi_q',
$$

(2.184)

where primes denote differentiation with respect to $\eta$.

The integral equations can be written generically as

$$
\frac{d}{dz} \left( b^2 \tilde{u}_0 \tilde{q}_0 \right) \int_0^\infty f \chi_q \eta \, d\eta = \alpha_s b^2 \tilde{s}_0 \int_0^\infty \chi_s \eta \, d\eta.
$$

(2.185)

Using the self-similar profiles and derivatives, (2.169) can be solved for $\tilde{u}_r$ in terms of $\tilde{u}_z$ and $f(\eta)$:

$$
\tilde{u}_r = -\frac{1}{r} \int_0^r \frac{\partial \tilde{u}_r}{\partial z} \, dr
$$

$$
= -\frac{b}{\eta} \int_0^\eta \left( \frac{d\tilde{u}_0}{dz} f - \tilde{u}_0 \frac{db}{dz} \eta f' \right) \, d\eta
$$

$$
= -\frac{b}{\eta} \frac{d\tilde{u}_0}{dz} \int_0^\eta f \eta \, d\eta + \tilde{u}_0 \frac{db}{dz} \int_0^\eta \left( \eta^2 f' \right) \, d\eta
$$

$$
= \left( \tilde{u}_0 \frac{db}{dz} \right) \eta f - \left( b \frac{d\tilde{u}_0}{dz} + 2 \tilde{u}_0 \frac{db}{dz} \right) \frac{1}{\eta} \int_0^\eta f \eta \, d\eta
$$

$$
= \left( \tilde{u}_0 \frac{db}{dz} \right) \eta f - (2\alpha_c \tilde{u}_0) \frac{1}{\eta} \int_0^\eta f \eta \, d\eta.
$$

(2.186)
Note that by (2.173)

\[ g(\eta) = \left( \frac{db}{dz} \right) \eta f - (2\alpha_z) \frac{1}{\eta} \int_{0}^{\eta} f \eta \, d\eta, \quad (2.187) \]

which means that together similarity and incompressibility (\( \nabla \cdot \mathbf{u} = 0 \)) imply that not only is the entrainment rate constant, but also that the jet/plume has straight sides, with \( \frac{db}{dz} \propto \alpha_z \).

Now, assume that the normalisation factor for the general Reynolds stress, \( \tau_{\text{eq}}^R \), is \( \bar{u}_0 \bar{q}_0 \). Further, assume the one dimensional version of Boussinesq’s eddy-viscosity equation, (2.124), with an eddy-viscosity that is dependent on both spatial variables, \( \nu^2(\eta, z) = \bar{v}_0^2(z) \nu^2_0(\eta) \). This results in a general Reynolds stress of the form

\[ \tau_{\text{eq}}^R(\eta, z) = \bar{u}_0(z) \bar{q}_0(z) \nu^2_0(z) \nu^2_0(\eta) \frac{d}{d\eta} \chi_0(\eta). \quad (2.188) \]

Substituting (2.183), (2.184), (2.186) and (2.188) into (2.181) gives

\[
\left( \left( \frac{\bar{u}_0}{\bar{q}_0} \frac{db}{dz} \right) \eta f - (2\alpha_z \bar{u}_0) \frac{1}{\eta} \int_{0}^{\eta} f \eta \, d\eta \left( \frac{\bar{q}_0}{\bar{b}_0} \chi_0 \right) \right) + (\bar{u}_0 f) \left( \frac{\bar{q}_0}{\bar{b}_0} \frac{db}{dz} \chi_0 - \frac{\bar{q}_0}{\bar{b}_0} \frac{db}{dz} \eta \chi_0' \right) = \frac{b}{\bar{q}_0 \nu_0^3} \frac{d\bar{q}_0}{dz} (\eta \nu^2_0 \chi_0') + \alpha_s \bar{s}_0 \chi_0. \quad (2.189) \]

The first and last terms on the left-hand side cancel, and by multiplying through by \( \frac{b_0}{\bar{u}_0 \bar{q}_0 \nu_0^3} \) yields

\[
\left\{ \frac{b}{\bar{u}_0 \nu_0^3} \frac{d\bar{u}_0}{dz} \right\} (\eta f \chi_0) - \left\{ \frac{2\alpha_z b}{\bar{u}_0 \nu_0^3} \right\} \left( \chi_0' \right) \frac{d\eta}{d\eta} = (\eta \nu^2_0 \chi_0')' + \left\{ \frac{\alpha_s b \bar{s}_0}{\bar{u}_0 \bar{q}_0 \nu_0^3} \right\} (\eta \chi_0). \quad (2.190) \]

Specifically, the streamwise momentum equation is now

\[
\left\{ \frac{b}{\bar{u}_0 \nu_0^3} \frac{d\bar{u}_0}{dz} \right\} (\eta f^2) - \left\{ \frac{2\alpha_z b}{\bar{u}_0 \nu_0^3} \right\} \left( f' \right) \frac{d\eta}{d\eta} = (\eta \nu^2_0 f')' + \left\{ \frac{\alpha_s b \bar{T}_0}{\bar{u}_0 \nu_0^3} \right\} (\eta \theta). \quad (2.191) \]

In the simple case of constant eddy-viscosity, i.e. independent of both spatial coordinates, the equations can be solved and the functional forms of \( f(\eta) \) and \( g(\eta) \) obtained. This solution was first obtained for a jet in 1933 by Schlichting [75], see also [76], with an approach similar to the following. By introducing \( F(\eta) \) such that

\[
\frac{db}{dz} f(\eta) = \frac{\nu^2_0 F'(\eta)}{\eta}, \quad (2.192) \]

the streamwise momentum equation reduces to the ODE

\[
\left( \frac{F''}{\eta} - \frac{F'}{\eta} \right)' + \frac{F'^2}{\eta} + F \left( \frac{F''}{\eta} - \frac{F'}{\eta} \right) = 0. \quad (2.193) \]

After a little algebra, repeated integration and conveniently defining \( b(z) \) such that when \( r = b(z) \) the velocity is a quarter that of the centreline velocity, then this can be solved to give

\[
\bar{u}_z(\eta, z) = \frac{\bar{u}_0(z)}{(1 + \eta^2)^{\frac{3}{2}}}, \quad (2.194) \]

and \( \frac{db}{dz} = 8 \nu^2_r \). The radial velocity and Reynolds stress term can be subsequently derived and are found to be

\[
\bar{u}_r = \frac{1}{2} \alpha_s \eta \bar{u}_0 \frac{1 - \eta^2}{(1 + \eta^2)^2}, \quad (2.195) \]

\[
\tau_{rz}^R = -\frac{1}{2} \alpha_s \eta \bar{u}_0^2 \frac{1}{(1 + \eta^2)^3}. \quad (2.196) \]

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Experimental data suggests that this form of the streamwise velocity is a close match near the axis, but is a significant overestimation at the jet edge.

Now suppose that the eddy viscosity coefficient can be a function of the radial coordinate, and consider jet flow. Assume that the form of the streamwise velocity, which experimental data suggests is close to Gaussian, \( f(\eta) = e^{-\eta^2} \), then the radial velocity and eddy-viscosity can be derived from (2.186) and (2.191) as

\[
\frac{\bar{u}_r}{\bar{u}_0} = \alpha_\varepsilon \left( \frac{1 + 2\eta^2}{\eta} e^{-\eta^2} - 1 \right), \quad \nu_\tau = \frac{\alpha_\varepsilon}{2} \left( 1 - e^{-\eta^2} \right), \quad (2.197)
\]

from which the Reynolds stress can be deduced

\[
\frac{-\tau_{rz}}{\bar{u}_0^2} = \alpha_\varepsilon \left( \frac{1 - e^{-\eta^2}}{\eta} \right) e^{-\eta^2}. \quad (2.198)
\]

Similar treatment of a plume yields

\[
\frac{\bar{u}_r}{\bar{u}_0} = \alpha_\varepsilon \left( \frac{1 + 6\eta^2}{\eta} e^{-\eta^2} - 1 \right), \quad \nu_\tau = \frac{3\alpha_\varepsilon}{10} \left( 1 - e^{-\eta^2} \right), \quad \frac{-\tau_{rz}}{\bar{u}_0^2} = \frac{3\alpha_\varepsilon}{5} \left( \frac{1 - e^{-\eta^2}}{\eta} \right) e^{-\eta^2}. \quad (2.199)
\]

The expression for the eddy viscosity allow a simple deduction about the important of the turbulence compared with viscous effects. At the axis, assuming a value for the entrainment coefficient of \( \alpha_\varepsilon = 0.05 \), then the axial value of the eddy viscosity is approximately 0.025, which corresponds to a Reynolds number of 40.

### 2.6.3 Experimental Observations

The straight-sidedness of jets and plumes, and the decay rates of velocity and scalar concentrations has been confirmed by many authors, in both liquid- and gas-phase experiments. A recent example of detailed jet measurements can be found in Hussein, Capp and George [40]. Measurements of plumes, and indeed forced plumes, can be found in Papanicolaou and List [64], Shabbir and George [78], and more recently Wang and Law [91], who compiled a table comparing parameters in different jet and plume experiments.

Using experiments of gaseous buoyant plumes, Rouse, Yih and Humphreys [72] pointed out that the widths of the velocity and buoyancy profiles are not necessary equal. This has also been shown to be the case in jets, where there seems to be agreement that the width of a passive scalar advected with the jet is always wider. Plume investigators are yet to agree which profile is wider, or provide an explanation why both results are reported, and probably depends critically on experimental conditions; Papanicolaou and List [64] claim that the contrary results were measured too close to the source. There is a comparison in Wang and Law [91], tables 2 and 3. Morton [58] took account of the difference in profile widths (which was omitted in (2.155) above), although assumed it to be constant. This seems to be a fair assumption in either the pure jet or pure plume cases, but has been shown to vary in the mixed case, e.g. Wang and Law [91].
Indeed, Wang and Law [91] measured centreline decay in a jet with the parameters $k_{jw}$ and $k_{jc}$ for velocity and tracer, respectively, and $k_{pw}$ and $k_{pc}$ in a plume. For a jet, these parameters were defined by the equations

$$\frac{\bar{u}_s}{u_0} = \frac{1}{k_{jw}} \frac{z}{D}, \quad \text{and} \quad \frac{\bar{c}_s}{c_0} = \frac{1}{k_{jc}} \frac{z}{D}, \quad (2.200)$$

where $D$ is the nozzle diameter and the suffix $s$ denotes source conditions. For a plume, the corresponding equations are

$$\sqrt{\frac{M_s}{z u_0}} = \frac{1}{k_{pw}} \left( \frac{z}{D} \right)^{-\frac{3}{5}}, \quad \text{and} \quad g' = k_{pc} B_0^2 \frac{z}{\eta_{pc}} \frac{z}{\eta_{pc}}, \quad (2.201)$$

where $\bar{M}_0$ and $\bar{B}_0$ are the mean momentum and buoyancy fluxes, respectively, at the source. By assuming Gaussian profiles, velocity and scalar widths were defined such that

$$\frac{\bar{q}}{q_0} = \exp \left( -\left( \frac{r}{\eta q z} \right)^2 \right), \quad (2.202)$$

for a general flow variable $q$. Values for the constants were compiled from different authors, and were given in the following ranges: $k_{jw} = 5.8-6.71$, $\eta_{jw} = 0.103-0.115$, $k_{jc} = 4.96-5.37$, $\eta_{jc} = 0.126-0.136$, $k_{pw} = 3.4-4.7$, $\eta_{pw} = 0.100-0.135$, $k_{pc} = 9.1-11.3$, and $\eta_{pc} = 0.109-0.125$.

### 2.6.4 Off-source Heating

There are various sources in the literature that describe changes in entrainment behaviour resulting from different causes. Ricou and Spalding [71] injected flammable gases into a stagnant ambient and reported a decrease in entrainment rate of up to 30% when the injected gas burned than when it did not. Hermanson and Dimotakis [38] observed a strong reduction in spread rate and entrainment in mixing layers with heat release.

Observations of cumulus clouds have also raised questions about the effects of heat release on entrainment behaviour. Clouds form due the condensation of water vapour carried in atmospheric plumes. This condensation has an associated latent heat release, which introduces buoyancy away from the source of the plume. Scorer’s book [77] contains a photograph depicting tall, almost vertically-sided, clouds over the Caribbean. This is indicative of a decrease in lateral entrainment, and Paluch [63] reports that it is reduced almost to zero. Squires and Turner [82] and Warner [41] have shown that plume models based on a constant entrainment assumption cannot predict both the height of rise and dilution of these clouds.

**Hunt’s Proposal**

J. C. R. Hunt [39] presented the following analysis in a discussion of atmospheric jets and plumes. As will be described shortly, this analysis has been applied by several authors to laboratory experiments analogous to the latent heat release in atmospheric plumes. Hunt’s proposal is presented first, along with a minor modification that makes a fundamental difference to the interpretation of the analysis. The idea
is that by accounting for the streamwise momentum flux due to the turbulence, an increase in the relative turbulent intensity could lead to a decrease in mass flux.

Refining the definition of specific momentum flux in (2.145) to account for the turbulence leads to

\[
M = 2 \int_0^\infty \overline{u_z^2} r \, dr
\]

\[
= 2 \int_0^\infty \left( \overline{u_z^2 + u_z^2} \right) r \, dr
\]

\[
= \bar{M} + M',
\] (2.203)

where \( \bar{M} \) and \( M' \) are the mean and turbulent momentum fluxes. Assuming \( Q = b^2 \bar{u}_0 \) and \( M = b^2 \bar{u}_0^2 \), then the mass flux can be related to the mean momentum flux

\[
Q^2 = b^2 \bar{M}.
\] (2.204)

Using (2.203), the momentum equation can be written as

\[
\frac{dM}{dz} = F \quad \Rightarrow \quad \frac{d\bar{M}}{dz} = F - \frac{dM'}{dz},
\] (2.205)

for a general forcing \( F = \int_0^\infty \bar{f}_z r \, dr \), where \( \bar{f}_z \) is the mean force acting on the fluid in the streamwise direction. Integrating this between two planes \( z_1 \) and \( z_2 \) gives

\[
M_2 = \lambda M_1, \quad \text{where} \quad \lambda = 1 + \left( \frac{1}{M_1} \int_{z_1}^{z_2} F \, dz \right) - \left( \frac{M_2 - M_1'}{M_1} \right).
\] (2.206)

Using (2.204), it is then possible to relate the change in mass flux through these two planes in terms of \( \lambda \)

\[
\frac{Q_2}{Q_1} = \frac{b_2}{b_1} \sqrt{\frac{M_2}{M_1}} = \frac{b_2}{b_1} \sqrt{\lambda}.
\] (2.207)

From this, Hunt deduced that any positive forcing, \( F \), will lead to an increase in entrainment, whereas a tendency for the jet to break up into individual eddies, so that the relative turbulent intensity increases, will result in a decrease in entrainment. However, this second deduction assumes that the momentum flux is constant. In the flow that is of interest in this dissertation, the momentum flux cannot necessarily be considered to be constant. Therefore, the deduction needs to be modified slightly. The result is that the mass flux can decrease if there is a relative increase in turbulent intensity, rather than an increase in relative turbulent intensity. This subtle change of wording is better described mathematically, and involves considering the term

\[
\frac{1}{M} \frac{dM'}{dz},
\] instead of \( \frac{d}{dz} \left( \frac{M'}{M} \right) \). (2.208)

The two expressions are clearly equivalent if the mean momentum flux is constant, but may differ when it is not.

An alternative way of looking at the same analysis is to solve for the entrainment coefficient in the mass conservation equation

\[
\frac{dQ}{dz} = 2\alpha_e \sqrt{\bar{M}},
\] (2.209)
where \( u_e \) is the entrainment velocity, and leads to

\[
\alpha_e = \frac{1}{2\sqrt{M}} \frac{dQ}{dz} = \frac{1}{2} \frac{db}{dz} + \frac{b}{4\sqrt{M}} \frac{dM}{dz} = \frac{1}{2} \frac{db}{dz} + \frac{b}{4} \left( \frac{F}{M} - \frac{1}{M} \frac{dM'}{dz} \right),
\]

where it is clear which derivative should be considered to assess the change in entrainment velocity due to the turbulent momentum flux. It should be borne in mind that this approach simply absorbs any change in self-similar structure into the entrainment coefficient, and should be interpreted carefully.

**Results of Narasimha**

In order to investigate changes in entrainment behaviour due to heat release, Bhat, Narasimha and Arakeri [15] pioneered a novel laboratory experiment with off-source heating analogous to that in clouds. By using an acidic jet injected into a deionised ambient, they were able to selectively heat the jet using a series of electrodes; the acidic jet fluid conducts and so is heated, whereas the non-conducting ambient is not. Four results papers followed. Elavarasan et al. [28] and Bhat and Narasimha [14] looked at off-source heating in jets. Venkatakrishnan et al. [88] and Venkatakrishnan, Bhat and Narasimha [87] performed experiments in plumes. These papers are henceforth referred to as EBNP, BN1, VBN and VBN, respectively.

The main conclusions from EBNP and BN1 are summarised as follows:

- Heating significantly changes the structure and dynamics; large scale eddies are disrupted.
- The mean streamwise velocity remains Gaussian throughout.
- The mean streamwise velocity decay is arrested, and accelerates at sufficiently high heating rates.
- The jet spread rate is reduced in the upper part of the HIZ and just above it.
- The mass flux at first increases more rapidly than in the unheated jet, but is nearly constant further downstream.
- The RMS streamwise fluctuations remain almost constant throughout the HIZ, and so the normalised values decrease. It was concluded that the original proposal of J. C. R. Hunt was not verified by this data.
- The shape of the turbulent velocity distribution is modified by heating and is not recovered rapidly in the post-HIZ zone.

In BN1 the mass flux is considered in more detail. A comparison is made between the measured mass flux and the mass flux derived by assuming a constant entrainment coefficient. The former is significantly lower. It is concluded that “the assumption of constant entrainment coefficient is not at all realistic in the HIZ, and indeed that it suffers a drastic reduction at high values of the heating parameter”. This
is, of course, in contrast to the intuitive idea that the entrainment coefficient would increase towards a plume-like value.

The scalar and velocity widths are also contrasted in BN1. It is observed that although the visual thinning is reflected in the scalar width, an increase in the velocity with occurs. This behaviour is taken into account when the integral model is compared with the experimental data. It is shown that by assuming a change in the ratio of velocity and scalar widths, the velocity decay is more accurately predicted than assuming the ratio to be constant. However, there is still disagreement with the experimental data. The conclusion was that the width ratio has to be taken into account but that further work was required before a convincing model could be formulated. It is strange that given this conclusion, the mass flux data was not reassessed. Since velocities at the edge of the jet are difficult to measure, the profiles are assumed to be self-similar and the mass flux derived from the centreline velocity and the jet width. However, the scalar width was chosen, possibly because it is easier to measure, and naturally results in an underestimation of the mass flux.

Heated jets and plumes were compared in VBPN, where it was observed that plumes are more strongly affected with heat addition that jets. They also noted that a transition to plumes did not occur in the post-HIZ for their experimental configuration. The heated plume results of VBN are similar to those summarised above for heated jets. It was concluded that the heating rate was sufficient to disrupt the eddy structure but insufficient to significantly affect the mass flux.

The dimensionless measure of heating rate is the Richardson number, see section 2.1. It is simply the ratio of buoyancy to inertia. This can be done in terms of local values, as in VBN, who defined

$$\text{Ri} = \frac{\Delta \rho \ g \ b}{\rho_0 \ u^2}. \quad (2.211)$$

This can also be written in terms of the heat addition rate, $Q_H$ (Watts), using the specific heat of water $c_p$. This was the case in EBNP, who used

$$\text{Ri} = \beta_T \frac{Q_H}{\rho c_p} \frac{1}{b_0 u_b^3}, \quad (2.212)$$

where $\beta_T$ is the volumetric thermal expansion coefficient introduced in section 2.1, and the suffix $b$ denotes the bottom of the HIZ.

BN1 used basic jet properties to define an analogous heating rate in terms of nozzle values that can be evaluated a priori. Since $b$ scales with $z$, and $u$ scales with $z^{-1}$ then, setting $z = \frac{x}{d}$ for the number of nozzle diameters from the source to the bottom of the HIZ, gives

$$G = \beta_T g \frac{Q_H \ z^2}{\rho c_p} \frac{1}{d^3 \ u_0^3}. \quad (2.213)$$

In the experiments of BN1 $G$ was varied from 0.25 to 4.7. Note the use of $G$ rather than $\text{Ri}$ here. Although not mentioned directly, is a Grashof number, which was said in BN1 to be analogous to the bulk Richardson number.
Narasimha, Saxena and Kailas [61] examined the structure of heated plumes using wavelet analysis. They found that heating led to enhanced mixing activity away from the centreline.

Basu and Narasimha [8], henceforth referred to as BN2, performed DNS simulations of an analogous temporally evolving flow with off-source heating. The details will be described in the section 2.6.5. They applied a variety of heating profiles and heating rates and observed that the width of the jet could increase or decrease depending on the way in which heating was applied. However, in each case a decrease in normalised fluctuations was reported. Siddhartha et. al. [79] applied wavelet analysis to the numerical data and found a higher tendency for large structures to telescope into their downstream neighbour than in unheated jets.

**Results of Agrawal and Prasad**

More recently, Agrawal and Prasad have investigated the heated jet, both experimentally, using the similar approach to BN1, but also using DNS of the spatially evolving case.

Agrawal, Sreenivas and Prasad [4], referred to as ASP, used particle image velocimetry (PIV) to measure both the streamwise and the radial velocity profiles. Moreover, they used thermochromic liquid crystals (TLC) to measure the temperature profile.

Using laser induced fluorescence (LIF) they confirmed the dramatic effect of heating on the structure of the jet, and the reduction in scalar width. They observed that the disruption of the eddy structure occurs only in the upper part of the HIZ, and postulate that this disruption might partly be the reason for the observed decrease in spread rate. Data were presented at $G = 1$, where the same dimensionless heating parameter as BN1 was used, see 2.213.

Velocity measurements in the unheated case showed self-similar collapse to a Gaussian streamwise velocity. The radial velocities presented significant scatter, due to their small magnitude, but did not disagree with the theoretical profile. In the heated jet ($G = 4.3$) the streamwise velocities are reported to change profile from Gaussian, in the first half of the HIZ, to what ASP term a “flat-top Gaussian”, in the post-HIZ, although this change is not clear in the presented data. It was proposed that the profile is $(1 + B\eta^2) \exp(-\eta^2)$, for $B > 0$, instead of simply $\exp(-\eta^2)$. The radial velocities changed from a normal jet profile below the HIZ, to one that presented inflow at all radial locations in the lower-HIZ, and in the upper half of the HIZ between these two states, the results presented are almost uniformly zero.

The centreline streamwise velocity at $G = 4.3$ presented an excessive decay in the first half of the HIZ, i.e. it decayed more than an unheated jet. In the upper half of the HIZ the velocity decay arrests and becomes almost constant, and then accelerates in the post-HIZ. This was accompanied by a growth in the velocity width in the lower-HIZ, which became nearly constant in the upper half of the HIZ and in the post-HIZ. This is in contrast to the scalar width, which decreased.
Mass and momentum fluxes are then considered. The momentum flux increases due to heating, as expected. The mass flux increases in the lower-HIZ, above which it becomes nearly constant. It is argued that above the lower-HIZ, the mass flux is underestimated because the streamwise velocity has a flat-top Gaussian distribution. The following argument was used to reason that a flat-top Gaussian distribution with the same centreline value and width has a greater mass flux. Let $f(\eta) = (1 + B\eta^2) \exp(-\eta^2)$ and $g(\eta) = \exp(-\eta^2)$ be the flat-top Gaussian and ordinary Gaussian distributions. Then the ratio of the mass fluxes is $M_f / M_g = 1 + B$. However, this reasoning should be interpreted carefully. When $B = 1$, the $e^{-1}$ point that defines the width occurs at $\eta = 1.46$, and so is much wider than the corresponding Gaussian profile (which is $\eta = 1$). Since the experiments measured the width as the $e^{-1}$ point, the mass fluxes should be compared when this occurs at the same radial position. Therefore, the corresponding Gaussian distribution should be $f(\eta) = \exp\left(-\left(\frac{\eta}{1.46}\right)^2\right)$. The resulting mass flux ratio is $M_f / M_g = 0.93$, and not 2, as in the original estimate. In fact, only for values of $B$ greater than approximately 1.8 (and less than zero) is the mass flux greater than the corresponding Gaussian profile with the same $e^{-1}$ point.

Temperature measurements are presented at $G = 13.5$. Around the half-way point of the HIZ the temperature profile presents an off-centre maximum (or double-bump profile, as ASP call it). In the upper half of the HIZ the temperature profile appears to be close to a flat-top Gaussian, and does not have an off-centre maximum. It is proposed that the double-bump profile arises due to a competition between the profiles of the acid concentration and the streamwise velocity. The temperature increase is reported to be proportional to the concentration of the acid, and so a greater temperature rise would be expected at the axis, where the concentration is higher. However, the fluid is moving at greater speed in the centre, so the residence time in the HIZ is lower.

No second-order correlation data are presented in ASP.

Direct numerical simulation of the heated jet was conducted by Agrawal, Boersma and Prasad [1] using a modified version of the numerical code of Boersma [17]. The viscous form of the equations outlined in section 2.1 were used. The Reynolds number was 1000, to match approximately the value used in their experiments. The Prandtl and Schmidt numbers were set to unity to reduce computational expense.

The simulations were conducted on a spherical grid so that the domain is shaped similarly to the conical jet structure expected. The lateral edge of the domain was chosen to be close to the spread rate of a jet, angled at $\pi/40$, so that the distance between the edges of the jet and domain remains roughly equal for all axial locations.

A second-order, finite-volume, TVD approach was taken, with a pressure-correction method to enforce incompressibility. The resolution was $270 \times 80 \times 48$ in the $r$, $\theta$, $\phi$ directions, respectively. This corresponds to 43 diameters in the downstream direction, and 8 diameters in the radial direction. Heating was applied between 23 and 31 diameters.
To show that the resolution was sufficient the following argument was used. Energy dissipation was estimated as 
\[ \varepsilon = \frac{1}{2} \frac{\bar{u}^3}{z-z_0}, \]
and the Kolmogorov length scale as 
\[ \eta = \left( \frac{\varepsilon}{\nu} \right)^{1/4}. \]
At heights 20, 30 and 40 nozzle diameters, the Kolmogorov length scale was estimated to be 0.13, 0.09 and 0.09 diameters, respectively. The grid spacing (in each direction) at the centreline were (0.17,0.07,0.015), (0.21,0.08,0.017) and (0.26,0.09,0.02), at each height, respectively. So since these were off the same order, and with unit Prandtl and Schmidt numbers, it was concluded that the resolution was sufficient for DNS. No attention was given to the numerical error inherent in such a scheme.

The inflow was forced with a sinusoidal perturbation, with 2% amplitude, close to the natural frequency of the jet. Outflow was specified at the upper boundary, and traction-free lateral boundaries were used. As in the experiments, the jet was allowed to reach similarity before heating was applied.

The dimensionless heating parameter was calculated in terms of local values as (2.212), where \( Q_H \) was evaluated \emph{a posteriori} by integrating
\[
\frac{Q_H}{\rho c_p} = \int_V \frac{\partial T}{\partial t} dV = \alpha_c \int_V \bar{c} dV,
\]
where the integrating volume was the HIZ. The resulting parameter was
\[
\text{Ri}_* = \frac{2\pi \alpha T \alpha_c}{b_n \bar{u}_z^2} \int_{z_0}^{\infty} \int_0^{\infty} \bar{c} r dr \, dz.
\]
A higher heating rate was used (\( \text{Ri}_* = 12 \), compared with \( \text{Ri}_* = 0.3 \) in ASP) so that the jet achieved stationarity more quickly.

The scalar concentration dropped more sharply in the HIZ than in a cold jet, and there was a visual thinning. Heating arrested the centreline streamwise velocity decay, and then caused it to accelerate. The maximum centreline temperature occurred within the HIZ, and decayed above it. This is likely to be due to such a strong heating rate.

Profiles of radial velocities showed that the heating leads to a pronounced change, whereby an inflow was experienced at all radial locations. Heating was reported to change the streamwise velocity profile from Gaussian to double-peaked Gaussian. This behaviour was also observed in the concentration profile. The off-centre maximum is present in the temperature profiles. The reason for these profiles is attributed to the competing effects of tracer concentration and residence time in the HIZ.

The scalar and velocity widths grew approximately linearly in the pre-HIZ with a ratio of about 1.2, although there was some scatter in the data. The scalar width is reported to “bulge” just before the HIZ, which may be argued to be within the scatter of the data, but agrees with experimental data of BN1. A dramatic reduction in both the scalar and velocity widths is observed very soon after heating begins. The reduction in scalar width is much greater than the velocity width, which results in a scalar width that is actually lower than the velocity width. Near the top of the domain the scalar width becomes wider than the velocity width. It is concluded that this is due to the concentration becoming an active scalar
in the HIZ and a passive scalar in the post-HIZ. This reasoning is surely questionable because the active temperature width also becomes wider than the velocity width the top of the domain.

Only centreline values for the fluctuations are presented, radial profiles are not. The streamwise velocity, tracer and temperature fluctuations increase in the HIZ, with a slight decrease in the streamwise velocity fluctuation initially. All normalised centreline fluctuations are higher in the post-HIZ than in the pre-HIZ.

The results were published collectively in Agrawal and Prasad [3], where experimental data was presented for both radial profiles and centreline evolution of normalised fluctuating quantities. In the lower-HIZ, centreline values of the streamwise velocity fluctuations were approximately 60% higher than in the unheated case, decreased with height, but always remained higher than the unheated jet. The radial profiles are similar to the unheated jet (maximum at the centre and decreasing towards the edge), but at the top of the HIZ, the profile had changed shape and is reported to be almost constant across the range measured.

2.6.5 A Temporally Evolving Analogue

Due to the range of time and length scales involved in spatially evolving jets, it is much easier to consider simulating a temporally evolving analogue. In essence, the streamwise and time axes are swapped. In a spatially evolving jet, statistics evolve along the streamwise axis and averaging is performed in time. In the temporal framework, the evolution is in time with averages taken in the streamwise direction. This is achieved by working in a (triply-)periodic box, which is ideal for spectral simulations. The initial conditions consist of a cylinder of fluid moving in the streamwise direction, and the remainder of the domain is stationary. The result is a circular shear layer. This shear layer undergoes Kelvin-Helmholtz instabilities, similarly to a planar shear layer, and becomes fully three-dimensional after secondary instabilities have occurred. The shear surface continues to develop and spreads outwards, which is why the analogue is drawn with a spatially evolving jet. However, in an incompressible regime, periodicity in the streamwise direction implies that the average radial velocity is uniformly zero. Therefore, entrainment, in the sense of the spatially evolving jet, does not occur; the turbulence interface simply spreads laterally. Furthermore, the scaling laws, to be derived, are different from a spatially evolving jet. For these reasons, this flow will be referred to as a temporally evolving circular shear layer, and not a temporal jet. Since the lateral boundary conditions are usually also periodic, the flow is in fact an infinite array of shear layers. However, an advantage of zero average radial velocity is that there is no competition between adjacent structures to entrain fluid, and it is only when the shear layer reaches the edge of the domain that its progress is self-impeded. It should be emphasised that zero average radial velocity is not a consequence of lateral periodicity, but streamwise periodicity and incompressibility.

Basu and Narasimha [8], referred to as BN2, simulated this analogue flow with a spectral method to
investigate the response to off-source heating. In the temporal setting, this corresponds to injecting heat over a restricted period of time. Before the results are considered, a theoretical description is given.

### Theoretical Description

As usual, let $t$ denote time, and $(r, \theta, z)$ define cylindrical polar coordinates. For this flow, averages are again denoted by overbars and calculated using the integral

$$\bar{q}(r,t) = \frac{1}{2\pi L} \int_0^L \int_0^{2\pi} q(r,\theta,z,t) \, d\theta \, dz$$

(2.216)

for a generic quantity $q(r,\theta,z,t)$, and where $L$ denotes the size of the (cubic) domain. The integral only makes sense for values of $r$ less than $L/2$, i.e. the cylinder within the cube. Dynamics outside this cylinder are not important for the evolution of the shear layer, and are ignored.

Velocity and length scales can be arbitrarily defined similar to the spatially evolving jet, but for later convenience $\bar{u}_z$ is followed here. It was observed in that the centreline quantities showed unduly large variations to be used to derive characteristic scales. Therefore, integral velocity and length scales were defined

$$\tilde{u}_z(t) = \frac{\int u_z^2 r \, dr}{\int u_z r \, dr},$$

(2.217)

$$\tilde{b}(t) = \frac{\int u_z r \, dr}{\sqrt{\int u_z^2 r \, dr}}.$$  

(2.218)

A characteristic integral turbulence intensity, $\tilde{u}_z'(t)$, was defined “in the same way as $\tilde{u}_z(t)$ except that $u_z$ is replaced by $u_z'$”. What is meant by this is not exactly clear, and since the denominator of (2.217) would involve the average of the fluctuation, which is of course zero, the definition of $\tilde{u}_z'$ does not make sense. When the evolution of this quantity was plotted it was normalised by $\tilde{u}_z$, so the definition should not be normalised. For this reason, the RMS velocity fluctuation was first defined

$$\tilde{u}_z'^2 = u_z'^2 = (u_z - \bar{u}_z)^2,$$

(2.219)

and the integral turbulence intensity was then defined in the same way as $\tilde{u}_z$ with $u_z$ replaced by $u_z'$

$$\tilde{u}_z'(t) = \frac{\int u_z'^2 r \, dr}{\int u_z'^2 r \, dr}.$$  

(2.220)

In a similar way to spatially evolving jets, dimensional arguments can be used to derive the scaling laws for the temporal evolution. Neglecting viscous effects, the only characteristic scales are $\tilde{u}_z$ and $\tilde{b}$, the only quantity with the units of time is $\frac{\tilde{b}}{\tilde{u}_z}$. Therefore, $\frac{\tilde{b}}{\tilde{u}_z} \propto t$. With streamwise periodicity and incompressibility, the mass flux must be constant, and therefore so must $\tilde{b}^2 \tilde{u}_z$. Therefore, the scalings for this flow can be derived as

$$\tilde{b}(t) \propto t^{\frac{4}{3}}, \quad \tilde{u}_z(t) \propto t^{\frac{2}{3}},$$

(2.221)

for an appropriately defined time origin, and are hence quite different from the spatially evolving jet. In fact, the evolution resembles an axisymmetric wake.
In the absence of body forces and if the Reynolds number is assumed to be sufficiently high that viscous effects can be neglected, the average streamwise momentum equation can be written as

$$\frac{\partial \bar{u}_z}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \tau_{r_z} \right). \quad (2.222)$$

Note that the boundary layer approximation is not required. Now, if Boussinesq’s eddy-viscosity equation is used, this becomes an axisymmetric diffusion equation

$$\frac{\partial \bar{u}_z}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \nu_r \frac{\partial \bar{u}_z}{\partial r} \right), \quad (2.223)$$

where $\nu_r$ is the eddy-viscosity, which may be time and/or radially dependent. Adopting a similar approach to the spatially evolving jets, assume that the velocity and eddy-viscosity can be written

$$\bar{u}_z(r, t) = \bar{u}_z(t) f(\eta), \quad \nu_r(r, t) = \nu_0(t) g(\eta), \quad (2.224)$$

where $\eta = \frac{z}{fb(t)}$. Substituting into (2.223) gives

$$\frac{d\bar{u}_z}{dt} f(\eta) - \frac{\bar{u}_z}{b} \frac{db}{dt} \left( \eta \frac{df}{d\eta} \right) = \frac{\bar{u}_z \nu_0}{b^2} \frac{d}{d\eta} \left( \eta \frac{df}{d\eta} \right). \quad (2.225)$$

Since mass flux is constant,

$$\frac{\bar{u}_z}{b} \frac{db}{dt} = \frac{1}{2} \frac{d\bar{u}_z}{dt}, \quad (2.226)$$

which allows the left-hand side to be rearranged to give

$$\left[ \frac{\tilde{b}^2}{\nu_0 \bar{u}_z} \frac{d\tilde{u}_z}{dt} \right] \left( f + \frac{1}{2} \eta \frac{df}{d\eta} \right) = \frac{1}{\eta} \frac{d}{d\eta} \left( \eta g \frac{df}{d\eta} \right). \quad (2.227)$$

Since the term in square brackets is the only time-dependent term, it must be constant, $-\kappa$, say. Therefore, the eddy-viscosity can be written as

$$\nu_0 = -\frac{\tilde{b}^2}{\kappa \bar{u}_z} \frac{d\bar{u}_z}{dt}, \quad (2.228)$$

which, with (2.218) and (2.217), implies the eddy-viscosity decays with time to the one-third power.

Observing that

$$\left( f + \frac{1}{2} \eta \frac{df}{d\eta} \right) = \frac{1}{\eta} \frac{d}{d\eta} \left( \frac{1}{2} \eta^2 f \right), \quad (2.229)$$

then (2.227) becomes

$$-\frac{1}{\eta} \frac{d}{d\eta} \left( \frac{1}{2} \kappa \eta^2 f \right) = \frac{1}{\eta} \frac{d}{d\eta} \left( \eta g \frac{df}{d\eta} \right). \quad (2.230)$$

If the eddy viscosity is assumed to be independent of $\eta$, this equation can be integrated, without loss of generality taking $g = 1$, to derive a functional form for $f(\eta)$. The only solution that is bounded as the axis is $f(\eta) \propto \exp \left( -\kappa \left( \frac{\eta}{b} \right)^2 \right)$, so assuming a constant eddy viscosity gives rise to a Gaussian streamwise velocity. Note that the same functional form is obtained for the laminar case, in which case two dimensionless groups can be formed $\tilde{\tau}_{r_z}$ and $\text{Re} = \frac{\tilde{u}_z \tilde{b}}{\nu_0}$, and so the restriction $\frac{\tilde{b}}{\tilde{u}_z} \propto t$ is no longer valid. The resulting scaling laws are $\tilde{b} \propto t^{\frac{1}{2}}$ and $\tilde{u}_z \propto t^{-1}$. 

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Retaining the spatial dependence of \( g(\eta) \), and integrating (2.230) with respect to \( \eta \) allows the equation to be solved for \( g(\eta) \)

\[
g(\eta) = -\frac{1}{2} \frac{\kappa}{\eta} \left( \frac{df}{d\eta} \right)^{-1} (\eta^2 f + \alpha),
\]

where \( \alpha \) is the constant of integration. An expression for the Reynolds stress is thus derived in terms of the streamwise velocity distribution

\[
\frac{\tau_{xz}^R}{\bar{u}_z} = \nu_0 \left( \frac{df}{d\eta} \right) \left( \eta^2 f + \alpha \right),
\]

(2.232)

\[
= \frac{1}{2} \frac{\bar{b}}{\bar{u}_z^2} \frac{d\bar{u}_z}{dt} \left( \eta^2 f + \alpha \right).
\]

(2.233)

So if \( \bar{b} = 3Bt^{\frac{1}{3}} \), \( \bar{u}_z = Ut^{\frac{1}{2}} \), and \( \alpha = 0 \) so that \( \tau_{xz}^R \) is bounded at the axis, then

\[
\frac{\tau_{xz}^R}{\bar{u}_z} = -\frac{B}{3} \eta^2 f(\eta),
\]

(2.234)

which is clearly independent of time. Therefore, if the assumptions of self-similarity and Boussinesq’s eddy viscosity equation are valid, then the Reynolds stress is also self-similar and can be related to the mean streamwise velocity by (2.234).

**Numerical Results**

Basu and Narasimha [8] took advantage of the triply-periodic domain and simulated this flow with a spectral code. The incompressible Navier-Stokes equations, under the Boussinesq approximation, were solved, which in non-dimensional form are

\[
\nabla \cdot \mathbf{u} = 0,
\]

\[
\left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \begin{pmatrix} \mathbf{u} \\ T \end{pmatrix} = \begin{pmatrix} -\nabla P + \text{Re}^{-1} \nabla^2 \mathbf{u} + GTe_z \\ \text{Re}^{-1} \text{Pr}^{-1} \nabla^2 T + Qs(r) \end{pmatrix},
\]

(2.235)

(2.236)

where \( \mathbf{u} \), \( P \) and \( T \) are the velocity, pressure and temperature fields, \( e_z \) is a unit vector in the streamwise direction, \( \text{Re} \) and \( \text{Pr} \) are the Reynolds and Prandtl numbers, \( G \), and \( Q \) are dimensionless heating parameters, and \( s(r) \) is a shape function that controls how the heating is added.

The flow evolved as outlined above, with the early stages similar to a traditional planar shear layer. Kelvin-Helmholtz instabilities led to vortex roll-up, which was followed by vortex pairing and a transition to three-dimensionality. After this transition, self-similar behaviour was recovered with the integral length and velocity scales following power-law evolution close to the theoretically predicted values. The streamwise velocity presented a self-similar collapse, but still seemed to be evolving near the axis at late times. The streamwise velocity fluctuations also presented a self-similar collapse, but there was more variation than in the streamwise velocity.

Heating was added using the axisymmetric source term, \( s(r) \), in the temperature equation (2.236). The dimensionless heating parameter, corresponding to the Richardson number of the spatially evolving heated
jets, was defined to be $G^* = GQ$. Three different heating rates were examined, $G^* = 0.01, 0.04$ and 0.1. Three different profiles were examined, which were similar to the computed mean streamwise velocity profiles in the unheated case at three different times.

Heat injection was naturally seen to accelerate the fluid, but depending on the heating profile both an increase or a decrease in the width could be produced. In all cases the normalised turbulent intensity was observed to decrease. It was found that the strongest effects were produced by higher amounts of heat addition and narrow heating profiles.
Chapter 3

Homogeneous Isotropic Turbulence
3.1 Introduction

Before complicated inhomogeneous and/or anisotropic flows are simulated, it is necessary to investigate how the code handles homogeneous isotropic flow, and the extent to which turbulent statistics can be recovered without the use of an explicit subgrid scale turbulence model. An overview of turbulent statistics observed in various experiments and numerical studies of homogeneous isotropic turbulence is described in sections 2.3.4 and 2.4.2, respectively, and is summarised below:

- The flow is organised into vortex tubes with radii of order of the Kolmogorov length scale, $\eta$, and lengths of order of the integral scale, $l$.

- Longitudinal and lateral velocity correlation functions have forms as shown in figure 2.2.

- At scales small compared with the integral scale, the energy spectrum collapses when normalised by the Kolmogorov scales:
  - Inertial range: $E \sim (\kappa \eta)^{-5/3}$
  - Dissipation range: $E \sim \exp(-\beta \kappa \eta)$, Saddoughi and Veeravalli [74] have estimated that $\beta$ is approximately 5.2.

- Velocity PDFs are near Gaussian.

- Velocity gradient PDFs are non-Gaussian and Reynolds number dependent.

- Moments of velocity gradient PDFs are Reynolds number dependent.

- If the flow is decaying then the kinetic energy decays with power-laws with exponents that depend on the Saffman integral, $L$, see (2.102):
  - Batchelor turbulence: $L = 0$, the velocity scale decays with $t^{-10/7}$ and the length scale grows with $t^{2/7}$.
  - Saffman turbulence: $L \neq 0$, $t^{-5/7}$ and $t^{2/7}$.

This summary outlines a benchmark that will be used to evaluate the MILES approach with this code, and the way in which the code replicates each of the above points will be considered.

3.1.1 Initial Conditions

All of the simulations in this chapter have been conducted in a triply-periodic cubic domain, of size $L = 1$. The flow was initiated with a random velocity field where the bulk of the energy was at the large scales fitted to a spectrum chosen without making assumptions about the structure of the flow. This was
achieved by generating white noise in spectral space, scaling it to the prescribed spectrum and mapping it to physical space with a discrete fast Fourier transform. The spectrum used was of the form

$$E(\kappa) = \kappa^n \exp \left( \frac{n}{m} \left( 1 - \left( \frac{\kappa}{\kappa_0} \right)^m \right) \right),$$

(3.1)

to take the form $\kappa^n$ at the large scales, peak at $\kappa_0$, and decay as $\exp(-\kappa^m)$. A random distribution of eddies of size $\frac{2\pi L}{\kappa_0}$ corresponds to $n = 4$ and $m = 2$, see Davidson [26].

After initialisation, the flow can be allowed to evolve freely (decay) or maintained with a source term. In the case of decaying turbulence in a real-world viscous fluid, the energy will cascade to small scales and be dissipated, so that at large times the flow becomes uniformly stationary, i.e. all small scales are removed from the flow. To maintain the turbulence, a forcing term can be used to generate large scale flow. In spectral simulations this is often achieved by using an anti-diffusive term over a limited range of wavenumbers. However, such a term is not simple to implement in a finite-volume code. Therefore, the term used in the forced simulations took the form

$$\mathbf{F}_\lambda(x, y, z) = \begin{pmatrix}
\sin \left( \frac{2\pi x}{\lambda} \right) 
\sin \left( \frac{2\pi y}{\lambda} \right) \\
\sin \left( \frac{2\pi y}{\lambda} \right) 
\sin \left( \frac{2\pi z}{\lambda} \right) \\
\sin \left( \frac{2\pi z}{\lambda} \right) 
\sin \left( \frac{2\pi x}{\lambda} \right)
\end{pmatrix},$$

(3.2)

where $\lambda$ is the wavelength of the forcing. This forcing term is constant in time and so not applicable to viscous calculations, as the velocity would eventually become aligned with the forcing term. A more comprehensive forcing term that includes time dependence can be used to overcome this problem, but quickly becomes computationally expensive, and has not been necessary for the current investigation.

### 3.1.2 Overview

High resolution calculations ($N_x \geq 256$) are computationally expensive, so the bulk of this investigation has been conducted at moderate resolutions ($N_x = 128$). The basic evolution of the flow is first explained in section 3.2. The effect of various parameters are considered in the sections that follow. The form of the initial spectrum is investigated in section 3.3, both in terms of the large scale form $\kappa^n$, and initial peak $\kappa_0$. Forced simulations are described in section 3.4. Simulations at different resolutions are conducted in section 3.5. The effect of viscosity is then examined in section 3.6, and is reconciled with the MILES approach. A summary of the chapter is given in section 3.7.
3.2 Flow Evolution

To establish how the initial conditions develop, consider an unforced simulation at $128^3$ with initial spectrum parameters $n = 4$, $m = 2$ and $k_0 = 4$. This initial energy spectrum is shown in figure 3.1. The exponential decay ensures that the energy containing scales are large, with very little energy at the small scales. The levelling off of the spectrum at high wavenumbers is due to machine error in the discrete Fourier transform. This truncation occurs when the energy is thirty orders of magnitude below that at the energy containing scales, and will not have any effect on the simulations.

A three-dimensional rendering of the initial magnitude of vorticity is shown in figure 3.2(a). Notice there are no discernible vortex tubes, just small blobs of vorticity. Figure 3.2(b) is the same type of rendering for a very early developing flow field, $t \approx 0.1$. The vorticity is arranged in long thin tubes, fitting the Vincent and Meneguzzi description. Figure 3.2(c) corresponds to a later time, $t \approx 20$; if the vorticity is normalised by the peak enstrophy at that time, there is qualitatively little difference between any such image. The vortex tubes are much smaller than in the developing field, both in length and radius.

Figure 3.3(a) shows the evolution of a measure of the isotropy of the flow, specifically $\frac{u_i - u_j}{u_i + u_j}$, for each pair of RMS velocities $u_i$ and $u_j$, so zero is isotropic. The values remain within around 0.09, although there is a large amount of variation. This is simply due to the fluctuating nature of this problem, and should not influence the results substantially.

Figure 3.3(b) shows the evolution of total kinetic energy and enstrophy. Initially the energy remains close to a constant, and then around $t = 0.15$ begins to decay. At this time the growing enstrophy reaches a peak, and then also decays. This occurs because the initial flow field is sufficiently smooth that the numerical truncation error is small. Therefore, the numerics closely approximate the incompressible Euler equations, and so energy is conserved. As the vorticity field evolves into the characteristic tube structures, the energy cascade begins, leading to a growth in enstrophy. With the true Euler equations, it

![Figure 3.1: Initial energy spectrum. Note the order of magnitude of the decay. The levelling off at high wavenumbers is due to machine error in the discrete Fourier transform.](image)
Figure 3.2: Three-dimensional renderings of the magnitude of vorticity at three different times. Each plot has been normalised by the peak vorticity at that time.
Figure 3.3: (a) Evolution of the isotropy $\frac{u_i - u_j}{u_i + u_j}$, for each $i$ and $j$. (b) Evolution of the total kinetic energy and enstrophy, normalised by the corresponding values at $t = 0$.

is argued that the enstrophy will tend to infinity in a finite time, see Lesieur [53] for example. However, as the velocity gradients increase, numerical diffusion becomes important and the flow begins to decay. This backs up the idea that the numerical diffusion only acts at the small scales.

This is where monotonicity is important. If the code wasn’t monotonicity preserving, then spurious oscillations would be seen in the regions of high velocity gradients. This generation of small scales, which would form structures of their own and adversely affect the dynamics. Figure 3.4 shows two slices of the magnitude of vorticity at $t \approx 2$, and compares the effect of the slope limiter used in the estimation of the intercell flux for the non-linear advection term. It is clear that when the limiter is turned off, the small scale structures are smaller than when the limiter is turned on. Since the vorticity field is so densely packed, it is impossible to distinguish between the real flow features and spurious structures generated by numerical oscillations. These oscillations are bound to be present, as it was shown in section 2.2 that numerical oscillations are introduced in regions of steep gradients, which are present throughout the domain in this calculation. Therefore, by using the limiter, any numerical errors will have a diffusive effect rather than generating unwanted small scale structures.

On the whole, in figure 3.3(b), the power-law exponent of the decay in total kinetic energy, although not constant, can be seen to be closer to the theoretical value for Batchelor turbulence ($E \sim t^{-\frac{10}{7}}$), as shown by the solid black line, than Saffman turbulence ($E \sim t^{-\frac{6}{7}}$), shown by the dashed black line.

The integral length scale, as defined in (2.61) integrating the longitudinal velocity correlation function, is shown in figure 3.5(a). At early times the length scale decreases. This corresponds to the onset of the energy cascade, where the energy is conserved and vorticity is becoming arranged into vortex tubes. The growth of the length scale that follows is initially closer to the predicted power-law for Saffman turbulence than Batchelor turbulence. The rate of growth appears to reduce at around $t = 4$ to below the two theoretical values. The growth then levels off after $t = 20$. It is likely that the evolution of
the length scale is being affected by long-range correlation effects. The flow is evolving in a restricted domain and as the length scale grows, the influence of the periodic boundary conditions becomes more important.

Evidence for this is provided by the velocity correlation functions, figures 3.5(b,c). From times $t = 0.1$ to $t = 0.2$, when the enstrophy peaks and the integral length scale attains its minimum, denoted by the red curves, the velocities are correlated over short distances. As the flow evolves through times $t = 0.7$ to $t = 1.3$, the curves are shown in green, the correlation functions gradually increase to the blue curves, which correspond to times between $t = 4$ and $t = 10$. At late times, $t > 10$, shown in purple, the velocities are correlated across the complete range of scales available in the periodic domain. This is where the slight anisotropy in the flow is visible, but each velocity component presents the same overall behaviour.

The onset of the energy cascade and late-time saturation is reflected in the energy spectra. Spectra at early times are shown in figure 3.6(a). The initial spectrum is shown in green, where it is clear that the energy containing scales are large. As the flow develops and the cascade begins, the energy is transferred from large scales down to the smaller scales. During this transitional stage there is a hint of an inertial range with a minus five-thirds power-law decay, but this is only over a very short range of wavenumbers and is not present after the peak in enstrophy. The small scales appear to be tending to a limiting form, unlike the large scales, which are continually evolving.

The small scale limit of the spectra can be demonstrated by plotting the spectra on a semilogarithmic scale. Figure 3.6(b) shows such spectra over the entire duration of the calculation. This plot has been normalised by the dissipation rate and Kolmogorov length scale. The dissipation rate is evaluated.
Figure 3.5: (a) Evolution of the integral length scale. (b,c) Velocity correlation functions: (b) longitudinal, (c) lateral. Components of velocities denoted by symbols: $x$ by crosses, $y$ by pluses, and $z$ by diamonds. Colours denote time: red $t \approx 0.1-0.2$, green $t \approx 0.7-1.3$, blue $t \approx 4-10$, and purple $t > 10$.

by a simple centred finite-difference approximation to the rate of change of total kinetic energy. The Kolmogorov length scale is estimated as follows. On an unnormalised plot, each spectrum presents exponential decay over a range of wavenumbers. The gradient of each was approximated by a linear fit between wavenumbers 20 and 48. Then, if it is assumed that this decay takes the same form as reported by Saddoughi and Veeravalli [74], $E \sim \exp(-5.2\kappa\eta)$, then the Kolmogorov length scale can be determined. This is possibly not the most ideal way to evaluate the Kolmogorov length scale as it assumes a fully resolved dissipation range at a sufficiently high Reynolds number. Henceforth, it will be more accurately described as the effective Kolmogorov length scale. The evolution of the effective Kolmogorov length scale is shown in figure 3.6(c). There is a small growth during the development stage of the flow, but at late times it remains approximately constant around 3.1 cell widths. Normalising by the dissipation rate and effective Kolmogorov length scale, the energy spectra plotted on a semilogarithmic scale, figure 3.6(b), collapse onto a self-similar curve at the small scales. There is some evolution at the large scales, but even this becomes small at late times, shown by the overlapping blue and purple curves.

The details of the large scale development can be seen in the energy spectra shown in figure 3.6(d), which
Figure 3.6: (a) Logarithmic plot of kinetic energy spectra (not normalised) during the development stage, showing the cascade of energy from large to small scales. (b) Semilogarithmic plot of kinetic energy spectra normalised by dissipation rate and effective Kolmogorov length scale. (c) Estimation of the Kolmogorov length scale based on the exponential range of the energy spectra at large wavenumbers, normalised by the cell width of the computational grid. (d) Logarithmic plot of kinetic energy spectra normalised by total kinetic dissipation rate and effective Kolmogorov length scale. The colours are described in the text, and are the same as figures 3.5(b,c).

include the later times. The normalisation is the same as figure 3.6(b). The evolution at large scales is clearer here. However, the initial development is distorted by the normalisation, figure 3.6(a) is more representative of this. After this initial development stage however, the large scales appear to follow a power-law decay, which is smaller than the theoretical minus five-thirds predicted in the inertial range, but the power does increase slightly with time. Again, it is likely that correlation effects and the lack of separation of scales is the cause for the disagreement.

The structure of the turbulence can be further analysed by considering the probability density functions of the velocities and the longitudinal and lateral velocity gradients, figure 3.7. Each plot has been normalised by the corresponding standard deviation, and a Gaussian profile with unit standard deviation is shown in black for comparison. The velocities are clearly close to Gaussian in distribution, whereas
Figure 3.7: Probability density functions normalised to unit standard deviation: (a) velocity, (b) longitudinal velocity gradient, and (c) lateral velocity gradient. The black curve in each plot is a Gaussian distribution with unit standard deviation.

the velocity gradients are not. The asymmetry (skewness) in the longitudinal gradients observed in the literature is present here. The lateral gradients appear to have exponential wings, as expected. The moments of these gradients are shown in figure 3.8. The magnitude of each moment increases until $t = 20$, corresponding to the time when the length scale levels off, whereafter it remains approximately constant. The longitudinal gradients have third- to sixth-order moments of approximately $-0.55$, $3.9$, $-5.5$ and $32$, respectively, and the lateral gradients have fourth- and sixth-order moments of approximately $5$ and $60$, respectively. Comparison with the values in Jiménez et. al. [43] shows that the skewness factor has a greater magnitude than expected ($-0.49$ to $-0.525$). The higher moments are all smaller than the $Re_\lambda = 35$ DNS simulation presented in [43], which were $4.2$, $-6.5$ and $40$, respectively, for the longitudinal moments and $5.7$ and $90$ for the transverse moments. This suggests that the Reynolds number of this flow is quite small, $Re_\lambda \approx 20-30$

This raises the question, what is the effective Reynolds number? This question is in essence, what is the effective viscosity? Since the dissipation rate can be calculated, an estimate for the effective viscosity can be derived from the effective Kolmogorov length scale using the relation $\nu = (\varepsilon \eta^4)^{\frac{1}{4}}$, and is shown in
Figure 3.8: Moments derived from PDFs of longitudinal and lateral velocity gradients: (a) Longitudinal gradient skewness; (b) Longitudinal gradient flatness; (d) Longitudinal gradient super-skewness; (e) Longitudinal gradient super-flatness; (c) Lateral gradient flatness; (f) Lateral gradient super-flatness. Colours as figures 3.7(b,c).

The corresponding implied effective Reynolds number in figure 3.9(b). As the dissipation rate is decreasing and the effective Kolmogorov length scale is approximately constant, the effective viscosity is constantly adjusting to accommodate. More specifically, if the rate that energy is reaching the fixed effective dissipation scale is decreasing, then the effective viscosity required to dissipate it also decreases. This leads to an increase in the effective Reynolds number, at least until the integral length scale becomes constant, whence the Reynolds number also becomes constant. In figure 3.9(b), a second curve is plotted in green. This Reynolds number has been calculated assuming an arbitrary constant viscosity of $\nu = 10^{-3}$. It is in no way representative of this flow, but highlights the difference between the evolution of a real-world fluid and the simulation.

In the absence of a well defined viscosity, and assuming the fundamental property of the fluid that determines the destruction of kinetic energy is the Kolmogorov length scale, then the parameters that describe the flow are the RMS velocity, the integral length scale and the Kolmogorov length scale. This means that only one parameter involves time and so it is not possible to form a dimensionless group that involves velocity. Essentially, the analogous Reynolds number is the ratio of the integral length scale to the cell width (i.e. resolution). The velocity simply determines the time scale (i.e. the time step). This
can be demonstrated mathematically by rewriting the velocity scale in terms of the dissipation rate using the relation \( \varepsilon \sim \frac{u^3}{l} \). Then using the same relation to determine the effective viscosity above, the Reynolds number can be written
\[
Re \sim \left( \frac{\varepsilon_l}{\varepsilon \eta^4} \right)^{1/3} \sim \left( \frac{l}{\eta} \right)^{4/3},
\]
which does not include a quantity with units involving time, although the length scale \( l \) is time-dependent. Since \( \eta \) is determined by the cell width and the numerical algorithm, and at late times \( l \) is determined by the size of the domain, then the Reynolds number simply scales with the resolution to the four-thirds power. In terms of computational expense, the length of a simulation grows with the resolution to the power four, which implies the length of simulation grows with the Reynolds number to the power three. High Reynolds number calculations become expensive very rapidly.

**Summary**

A simulation has been run to establish the flow evolution of homogeneous isotropic turbulence in a triply-periodic cube at a moderate resolution. The flow was initiated with the bulk of the energy at the large scales, without assuming any properties of a turbulent fluid. The flow rapidly develops the characteristic tube-like worm vortices reported in DNS calculations. During this time, the total kinetic energy is conserved and the enstrophy grows as the development of small scale structures increase the velocity gradients.

After this development stage, the total kinetic energy appears to present a power-law decay that is close to the theoretical value predicted for Batchelor turbulence, but this is not conclusive. Furthermore, the integral length scale appears to present power-law growth that is closer to the predicted value for Saffman turbulence, but again, this is not conclusive. The flow appears to be strongly affected by
periodicity effects. Evidence for this is provided by the velocity correlation functions, which show that the velocities are correlated across the complete range of scales. This is a problem with homogeneous isotropic turbulence at this resolution, compounded by a relatively large initial length scale.

The energy spectra present an exponential decay at the small scales, which is characteristic of a dissipation range, \[74\]. The presence of such a dissipation range restricts the range of length scales that can be captured at this resolution. As the integral length scale becomes comparable with the size of the domain, the correlation functions indicate that the periodic boundary conditions begin to play a dominant role at the large scales. The result is a lack of scale separation to allow the development of the expected inertial range with a minus five-thirds decay. A power-law decay is indeed observed, but has a much smaller exponent. Even without a fully developed inertial range, the flow is divided into large scales, where inertial effects are dominant, and a small scale dissipation range, with an inviscid cascade of energy between them. In this way, the ideas behind \(k41\) apply, but in a subtly different way, to be discussed shortly. The PDFs and moments of the velocity and velocity gradients suggest that the structure of the resolved flow field is similar to that reported in the literature \[43\], albeit at a Taylor Reynolds number around 20.

One fundamental difference is particularly apparent - there is no final period of decay. In a real-world viscous fluid, a decaying flow field like the one simulated would eventually have all of the small scales removed, the flow would relaminarise and the velocity would tend uniformly to zero, with no structure. This is not what occurs in this simulation. Unlike the case of non-zero molecular viscosity, not all of the small scales are removed from the system. Although the velocity field decays, the decay is not uniform; small scale velocity gradients remain even at late time, and the flow field has structure. This difference is purely a consequence of the MILES approach.

In a real-world high Reynolds number viscous fluid, energy is transferred from large to small scales by non-linear, inviscid interactions. This cascade, and therefore the large scales, controls the dissipation rate, which is therefore independent of viscosity. For a given dissipation rate, the energy cascades to smaller and smaller scales until viscosity becomes important, thus determining the Kolmogorov length scale. As the velocity decays and the flux of energy to smaller scales decreases, the dissipation rate also decreases, resulting in a change in the Kolmogorov length, specifically growth, because the viscosity is fixed. In this MILES simulation it would appear that this is not the case. The inviscid cascade from large to small scales follows essentially the same ideas as \(k41\), because the large scales are well resolved and the dynamics are well represented by the high-order discretisation. The lack of resolution in the simulations presented here prevents scale separation, and so the correlation of the velocity field results in a power-law decay that is different from the theoretical value. It is the small scale energy destruction that is significantly different from a real-world fluid. Because the algorithm has been carefully constructed to be monotonicity preserving, there are no spurious oscillations that can lead to the development of unphysical small scale
structures. However, since the energy dissipation is due to the numerical algorithm, the length scale at which energy is removed from the system - the effective Kolmogorov length scale - is determined by the grid scale. The dissipation rate decreases as the velocity decays, but the Kolmogorov length scale remains unchanged, and does not increase as it would in a real-world fluid. This means that, although there is insufficient scale separation at the resolutions presented here for an inertial range with the theoretical power law decay, there is always some degree of scale separation, over which an energy cascade can occur, and maintain the small scale structure of the flow. It is this scale separation that allows statistics indicative of turbulent flow to be recovered at this low effective Reynolds number.

This synopsis implies that the fluid being simulated differs subtly from a real-world viscous fluid. The fundamental property of the fluid that determines how kinetic energy is removed from the system appears not to be viscosity, but a fixed length scale that is determined by the grid resolution and the numerical algorithm. This is not a surprising results for a MILES simulation, but should be borne in mind when analysing other flows.
3.3 Effect of Initial Spectrum

In this section the way in which the flow is initiated is investigated to see if the resulting turbulence is affected in any way. The two parameters involved are the initial energy peak wavenumber, $k_0$, and the exponent of the energy spectrum at large scales, $n$.

**Peak Wavenumber**

Simulations were run at the same resolution as the previous section, $N_x = 128^3$, with initial spectrum parameters $n = 4$ and $m = 2$, as before. A simulation was run for each of the values $k_0 = 1, 2, 4, 8$ and 16. These initial spectra are shown in figure 3.10(a). Simulations were run until time $t = 10$ for the lowest two wavenumbers, and to $t = 50$ for the higher three, for reasons that will become evident.

Figure 3.10(b) shows the evolution of the total kinetic energy in each simulation. It is clear that when the initial peak wavelength is larger, the total kinetic energy is conserved for longer. This is because it takes longer for the energy to cascade to the small scales, where numerical diffusion begins to have an effect. The theoretical exponents of energy decay are shown by the black lines. When the initial peak has a smaller wavelength there is closer agreement with the Batchelor exponent (the solid black line). This gives weight to the argument that periodicity is having a significant effect on the dynamics. When the peak wavelength is small compared with the size of the domain there is space into which the flow can evolve and decay freely.

In the initial developing stage of the flow, the greater time taken for the small scales to develop allows a greater relative growth in enstrophy, figure 3.10(c). Each case, however, eventually reaches a peak and then decays, fitting with the standard evolution established in the previous section.

The periodicity effects are reflected in the evolution of the integral length scale, figure 3.10(d). The simulations initiated with $k_0 = 4$ and 8 appear to present a short period where the growth in the length scale is close to that expected for Saffman turbulence. The simulation initiated with $k_0 = 16$ appears closer to that expected for Batchelor turbulence. This again supports the suggestion that the periodicity is affecting the large scale evolution. The case where the energy containing scales are smallest appears to be evolving in a way consistent with Batchelor turbulence, and vice versa. It would appear that there is a limiting length scale. It can also be seen that the length scales in the simulations with the initial peak at the three largest values have reached this limiting value by $t = 10$, yet the others are still evolving, which is why these simulations have been run for a longer period.

The late-time statistics, however, remain largely unchanged. Although individual realisations are quite different, the structure of the turbulence is comparable. Figure 3.10(e) shows the Kolmogorov length scale, derived as above, for the five simulations. There is reasonable agreement between all simulations.
Figure 3.10: Comparison of the turbulent statistics when the flow is initialised with different peak wavenumbers. (a) Initial kinetic energy spectra. (b) Evolution of the total kinetic energy. (c) Evolution of the total enstrophy. (d) Evolution of the integral length scale. (e) Evolution of the effective Kolmogorov length scale. (f) Kinetic energy spectra at $t = 10$. 
and all results are close to three cell widths.

The energy spectra at the end of each calculation are shown in figure 3.10(f), normalised by the energy dissipation rate (approximated by a finite-difference of the total kinetic energy) and the effective Kolmogorov length scale. There is very close agreement in the small scales between different initial conditions, and only small variations at the large scales.

Table 3.1 shows the moments of the longitudinal and lateral velocity gradients. There is a slight variation in each case, but the values are close to those highlighted in the previous section.

**Large Scale Exponent**

A further simulation was run using a different initial spectrum to investigate the effect of the large scale exponent. Specifically, a comparison is made of the initial spectrum parameter $n$ with the values 2 and 4, with $m = 2$ and $\kappa_0 = 16$, to allow for large scale evolution.

The general evolution and late-time statistics are not significantly different from each other, and are not discussed. As with the different peak wavenumber, the initial conditions are eventually forgotten. The only significant difference between the two simulations is in the evolution of the large scales at early times. Figure 3.11(a,b) shows the evolution of the early-time energy spectra for the two cases, note that these have not been normalised. The spectra for the $\kappa^2$ case appear not change a great deal at the large scales, the small scales decrease and the peak moves gradually to the larger wavenumbers. In the $\kappa^4$ case, the small scales evolve in the same way, but the large scales appear to evolve to a smaller exponent at very early times; the dotted line denotes $\kappa^3$ growth, which matches better than either $\kappa^4$ or $\kappa^2$, shown as solid and dashed lines, respectively. This does not fit with the literature, e.g. [26], which suggests that a $\kappa^4$ spectrum should persist.

Table 3.1: Moments of longitudinal and lateral velocity gradients for different peak wavenumbers.

<table>
<thead>
<tr>
<th>Peak $\kappa_0$</th>
<th>Long. $M_3$</th>
<th>$M_4$</th>
<th>$M_5$</th>
<th>$M_6$</th>
<th>Lat. $M_4$</th>
<th>$M_6$</th>
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</tr>
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</tr>
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<td>55</td>
</tr>
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<td>3.8</td>
<td>-5.3</td>
<td>30</td>
<td>4.6</td>
<td>48</td>
</tr>
</tbody>
</table>
Figure 3.11: Comparison of early-time kinetic energy spectra (not normalised) initiated with different large scale distribution. (a) $\kappa^2$. (b) $\kappa^4$.

**Summary**

The effect of the initial spectrum has been investigated, both the initial peak wavenumber and the large scale exponent. The initial conditions have been shown to affect only the initial stages of the flow, and the late-time statistics remain largely unchanged. By initiating the flow so that the energy containing scales are larger results in a greater time required before the total kinetic energy begins to decay. This implies that the transfer of energy from large to small scales is captured inviscidly, and that the dissipation due to the numerical truncation error only acts at the small scales. At this resolution, there is insufficient scale separation for an inertial range to develop that is independent of the large scales and posses a minus five-thirds decay.
This section is concerned with the effect of maintaining the turbulence using a forcing term. The source term (3.2) has been used throughout, and all simulations were conducted at \( N_x = 128^3 \). Four values of the parameter \( \lambda \) have been used, specifically \( L, \frac{L}{2}, \frac{L}{4} \) and \( \frac{L}{8} \). These will be referred to in terms of the wavenumbers \( n_\lambda = 1, 2, 4 \) and \( 8 \), for convenience. The flow field was initiated using the same spectrum parameters as the original decaying case, \( n = 4, m = 2 \) and \( \kappa_0 = 8 \). Simulations were run until time \( t = 10 \).

The total kinetic energy is shown in figure 3.12(a). After the initial developing stage the energy settles down to an approximately constant state. The value of this energy depends on the wavenumber of the forcing, roughly \( \frac{0.4}{n_\lambda} \). The enstrophy is shown in figure 3.12(b). Similarly, after the initial development, the enstrophy tends to a constant, which decreases as the forcing wavenumber increases.

The integral length scale also settles down to a constant value after the initial development, figure 3.12(c). Naturally, a higher forcing wavelength results in a larger integral length.

![Figure 3.12: Comparison of evolution of integral quantities at difference forcing lengths. (a) Evolution of the total kinetic energy. (b) Evolution of the total enstrophy. (c) Evolution of the integral length scale.](image-url)
Figure 3.13: Comparison of three-dimensional renderings of the magnitude of vorticity with different forcing lengths. Top left \( n_\lambda = 1 \), top right \( n_\lambda = 2 \), bottom left \( n_\lambda = 4 \) and bottom right \( n_\lambda = 8 \).

Figure 3.13 shows three-dimensional visualisations of the magnitude of vorticity, each plot has been normalised by the corresponding peak value. The large scale forcing promotes more extensive vortex tubes, and the resulting vorticity field appears more coherent than the unforced case, see figure 3.2(c). The small scale forcing leads to much smaller vortex tubes that appear to be more densely distributed than in the other cases.

As with the previous simulations, the energy spectra present exponential decay at the small scales. The effective Kolmogorov length scale has been approximated in the same way, and is shown in figure 3.14(a). It would appear that the closer the forcing length is to the small scales, the smaller the effective Kolmogorov length scale is. This implies that the small scales can be influenced by the large scales, but the variation is still within 10% of three cell widths.

The dissipation rate has been evaluated as follows. If it is assumed that the total kinetic energy is
Figure 3.14: Comparison of turbulent statistics at different forcing lengths. (a) Evolution of the effective Kolmogorov length scale. (b) Evolution of the dissipation rate. (c) Kinetic energy spectra at $t = 10$. (d) Evolution of the effective Reynolds number.

constant, then the energy dissipated is the same as the energy being supplied, which can be determined by evaluating the integral

$$\varepsilon = \int_V \mathbf{F}_\lambda \cdot \mathbf{u} \, dV,$$

where $V$ is the entire domain. The evolution in each case is shown in figure 3.14(b). A longer forcing length appears to be injecting more energy, and is associated with a larger variation about the mean.

Figure 3.14(c) shows the energy spectrum for each simulation at time $t = 10$, normalised by the dissipation rate and the effective Kolmogorov length scale. The spectra not only collapse at the small scales, as expected, but also appear to lie along the same power-law for wavenumbers greater than the forcing. This power-law has an exponent that is much closer to minus five-thirds than in the unforced cases, but it is still slightly smaller. These results fit with the ideas behind $k41$. It would appear that forcing the large scales reduces the periodicity effects observed in the unforced case. This injection of large scale energy provides a source for the decay process, and because the length scale is fixed and greater than the scale at which energy is dissipated, an inertial range can develop that is not overly affected by the larger scales in the flow. Because the dynamics of this inertial range are not influenced by the large scale
Table 3.2 shows the moments of the velocity gradient PDFs. It is clear that the longer forcing wavelength leads to moments with greater magnitudes, which is also indicative of higher Reynolds numbers as suggested above. The moments for $n_{\lambda} = 1$ are close to those presented in Jiménez at a Taylor Reynolds number of $Re_{\lambda} = 35.1$.

Summary

The main effect of injecting energy at the large scales using this forcing term is that a statistically steady state is reached. The flow begins to develop as in the unforced case, and becomes organised into tubular vortices. The size and density of these vortices depend on the wavelength of the forcing. A longer forcing wavelength gives rise to longer and more sparse vortex tubes. The second effect of large scale generation is that the periodicity effects appear to be dominated by the forcing term. This allows an inertial range to be observed with a power-law decay that is much closer to the theoretical minus five-thirds than in the unforced case. Again, the results presented are consistent with the ideas behind K41.

generation or small scale dissipation, the code captures a power-law decay close to minus five-thirds. The effective Reynolds number has been calculated using the RMS velocity, the integral length scale and the effective viscosity, which has been estimated in the same way as the previous sections using the dissipation rate. The evolution is shown in figure 3.14(d). Unsurprisingly, the longer forcing wavelength leads to an increased effective Reynolds number - the integral length scale is determined by the forcing length scale, and since the effective Kolmogorov length scale is constant and the velocity can be scaled out then the only thing that can affect the Reynolds number is the forcing wavelength. Note that the highest Reynolds number here is higher than that in the unforced case by about 20%.

Table 3.2: Moments of longitudinal and lateral velocity gradients with different forcing wavelengths.

<table>
<thead>
<tr>
<th>$n_{\lambda}$</th>
<th>Long.</th>
<th>Lat.</th>
</tr>
</thead>
<tbody>
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<td></td>
<td>$M_3$</td>
<td>$M_4$</td>
</tr>
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</tr>
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</tr>
<tr>
<td>8</td>
<td>-0.3</td>
<td>3.3</td>
</tr>
</tbody>
</table>
3.5 Effect of Resolution

This section investigates whether changing the resolution can change the separation of scales, and considers the effect on the resulting statistics.

**Decaying turbulence**

A simulation was run for each of the resolutions $32^3$, $64^3$, $128^3$ and $256^3$, with initial condition parameters $n = 4$, $m = 2$ and $k_0 = 8$, until time $t = 10$.

Figure 3.15(a) shows the evolution of the total kinetic energy at each resolution. In the developing stage of the evolution, a similar effect to that observed with different values of the initial peak wavenumber, $k_0$, is observed. The low resolution simulations conserve energy for a very short period of time, if at all. At higher resolutions, energy is conserved for longer. The different resolutions do not present power-law exponents that are in agreement with each other. The two higher resolutions seem to decay more rapidly when the decay begins, but the decay rate slows at later times. There is too much variation to draw definite conclusions. It is curious why the lower resolution decay rate appears to be slightly lower, so that by time $t = 10$ there is less energy in the system at the higher resolutions. It would be natural to assume that the lower resolutions would be more dissipative. The cause for the different decay rates at different resolutions is likely to be related to the correlation effects described in previous sections. Because the large scales are dominated by periodicity effects, the transfer of energy for large to small scales are affected by the resolution.

The peak enstrophy is higher at the higher resolutions, figure 3.15(b). As in the previous sections, energy and enstrophy decay only once there are small scale structures on which numerical diffusion has an effect. With the higher resolution simulations, it takes longer for these small scales to be generated by the cascade process, and the higher resolution can support higher gradients so the growth in enstrophy is greater.

The integral length scale is shown in figure 3.15(c). There is no clear agreement with either of the theoretical growth rates, and after the initial development stage, there is little evidence of any kind of trend in the results.

The effective Kolmogorov length scale has been estimated in the same way as the previous sections, and is shown in figure 3.15(d). The trend is for a smaller normalised Kolmogorov length scale with a lower resolution, however, all values are close to three cell widths. This is an interesting, if expected, result. The dissipation of energy is solely due to the truncation error, and so relative the cell width, the dissipation occurs at the same length scale.

Figure 3.15(e) shows the energy spectra for each resolution, normalised by Kolmogorov length scale and
Figure 3.15: Comparison of flow evolution and turbulent statistics of decaying turbulence at different resolutions. (a) Evolution of total kinetic energy. (b) Evolution of total enstrophy. (c) Evolution of integral length scale. (d) Kinetic energy spectra, $t = 10$. (e) Evolution of integral length scale. (f) Evolution of effective Reynolds number.
Table 3.3: Moments of velocity gradients at different resolutions for decaying turbulence.

<table>
<thead>
<tr>
<th>Res.</th>
<th>Long.</th>
<th>Lat.</th>
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<tbody>
<tr>
<td>N</td>
<td>$M_3$</td>
<td>$M_4$</td>
</tr>
<tr>
<td>32</td>
<td>-0.44</td>
<td>3.2</td>
</tr>
<tr>
<td>64</td>
<td>-0.48</td>
<td>3.4</td>
</tr>
<tr>
<td>128</td>
<td>-0.53</td>
<td>3.7</td>
</tr>
<tr>
<td>256</td>
<td>-0.56</td>
<td>3.95</td>
</tr>
</tbody>
</table>

dissipation rate, which has been evaluated by the gradient of the total kinetic energy. As usual, the small scales collapse onto the same dissipation range. The large scales are comparable, but there is a slight increase in the magnitude of the power-law decay in the inertial range with increasing resolution. The higher resolution appears to get closer to the minus five-thirds decay than the lower resolutions, but is still smaller.

The effective Reynolds number has been calculated in the same way as section 3.2, and is shown in figure 3.15(f). The jumps in the data are due to numerical errors in the approximation of the dissipation rate, and should be ignored. Naturally, the higher resolutions lead to higher Reynolds numbers. The increase fits with the scaling in (3.3), as doubling the resolution leads to an increase in Reynolds number of factor $2^{\frac{4}{3}} \approx 2.5$.

Table 3.3 shows the moments of the velocity gradient PDFs. There is a clear increase in magnitude of each moment with increase in resolution, but comparison with the moments presented in Jiménez et. al. [43] suggests that even the highest resolution still appears to have moments indicative of a Taylor Reynolds number of around $Re_\lambda = 30$.

**Maintained turbulence**

Simulations were also run at different resolutions that were maintained by the forcing term (3.2), with $\lambda = 2$. The total kinetic energy is shown in figure 3.16(a), where the estimation of the dissipation rate, evaluated by (3.4), is also plotted. In each simulation there is an initial development period, after which, the energy becomes constant. It is interesting that the rate at which energy is dissipated is independent of resolution, but not wholly unexpected as the large scales determine the dissipation rate. The forcing term injects energy at the large scales, which is transferred to small scales by the non-linear advection term and the numerical diffusion dissipates however much energy reaches the effective Kolmogorov length scale.

Normalised energy spectra are shown in figure 3.16(b). The collapse of the spectra is clear, both at large and at small scales. The decay is again close to minus five-thirds, but is slightly lower.
Figure 3.16: (a) Comparison of energy spectra at different resolutions with forcing, $t=10$. (b) Comparison of energy spectra at different resolutions with forcing, $t=10$. (c) Comparison of Reynolds number at different resolutions with forcing.

The effective Reynolds number is shown in figure 3.16(c). Again, it can be seen that doubling the resolution increases the effective Reynolds number by a factor of approximately 2.5. From the results investigating the effect of the forcing wavenumber, the lowest wavenumber effective Reynolds number is approximately 74, so it can be argued that a Reynolds number of approximately 185 can be achieved at $256^3$ by maintaining the turbulence with a forcing parameter of $n_\lambda = 1$.

Table 3.4 shows the moments of the velocity gradient PDFs. As expected, the forcing leads to moments with higher magnitudes, which are now closer to the moments in [43] at a Taylor Reynolds number of $Re_\lambda = 35$.

**Summary**

The effect of resolution is simple. An increase in resolution leads to a greater separation of scales, which allows higher Reynolds numbers to be achieved. The relation between Reynolds number and resolution, (3.3), has been confirmed.
An important point to stress is that the effective Kolmogorov length scale remains approximately three cell widths, independent of resolution. This is a natural result from taking a MILES approach, the dissipation is due to the algorithm and therefore the computational grid. The discretisation of the terms in the equations of motion does not have any knowledge of the number of cells in the domain, and so the action of the truncation error will be local, small scale and independent of resolution. Furthermore, the independence of the dissipation rate from the small scales is highlighted here. The maintained simulations at different resolutions are forced by the same forcing term, and so the injection of energy is the same. In each case, the steady state energy level is the same. Therefore, each simulation is dissipating the same amount of energy, which can be seen by evaluating the integral (3.4), and shown in figure 3.16. So the dissipation rate is controlled by the large scales, the energy cascades to small scales, where it is implicitly dissipated at an effective Kolmogorov length scale of approximately three cell widths, all independent of resolution.

Table 3.4: Moments of velocity gradients at different resolutions for maintained turbulence.

<table>
<thead>
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<th>Res.</th>
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<th>Lat.</th>
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<tr>
<td>N</td>
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<tr>
<td>32</td>
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</tr>
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<td>-0.44</td>
<td>3.55</td>
</tr>
<tr>
<td>128</td>
<td>-0.52</td>
<td>3.8</td>
</tr>
<tr>
<td>256</td>
<td>-0.57</td>
<td>4.1</td>
</tr>
</tbody>
</table>
This section looks at the effect of viscosity, and compares viscous simulations at one resolution with inviscid simulations at lower resolutions. The question being addressed is whether MILES simulations can be related to viscous simulations at higher resolutions.

The viscous simulations were run at a resolution of $128^3$ with viscosities $\nu = 10^{-4}$, $10^{-3}$ and $10^{-2}$. Inviscid simulations were run at resolutions $128^3$, $64^3$ and $32^3$. To initiate the velocity field, an inviscid simulation was run at $128^3$, with initial spectrum parameters $n = 4$, $m = 2$ and $\kappa_0 = 4$. At time $t = 0.26$, which corresponds to just after the peak in enstrophy, the calculation was stopped, and this velocity field was used as the initial conditions for the investigation. This was done to reduce the effect of the initial organisation, which would be clearly different in each simulation. Before this velocity field could be used at the lower resolutions, it was simply averaged onto the lower resolution grids. Simulations were subsequently run until $t = 5$, where $t = 0$ corresponds to the restart point. The lowest viscosity simulation was expected to be almost certainly underresolved, at least initially.

Figure 3.17(a) shows the evolution of the total kinetic energy. The inviscid simulations present the same behaviour as in section 3.5, the kinetic energy is conserved for longer at the higher resolutions but the decay is steeper.

The viscous simulations present a more intuitive trend - the energy decay begins sooner with higher viscosity. The least viscous simulation is not greatly different from the inviscid case. The $\nu = 10^{-3}$ case begins to decay sooner, but the decay rate appears to be close to the least viscous and inviscid calculations, i.e. close to the Batchelor exponent of $t^{-10/7}$. The most viscous case decays very quickly and does not present any kind of power-law.

The evolution of the integral length scale is shown in figure 3.17(b). Again, the inviscid simulations follow the evolution established in section 3.5. In the viscous simulations, the growth in length scale begins sooner with higher viscosities, but all appear to be close the Saffman exponent $t^{3/5}$. An interesting point to notice is that the viscous simulations attain much higher integral length scales than can be obtained in the inviscid calculations. This is due to the enforced separation of scales that arises due to a fixed Kolmogorov length scale. The small scales cannot be removed in the inviscid calculation, so there is no final period of decay, which is what leads to the large length scales in the viscous simulations.

Figure 3.17(c) shows the effective Kolmogorov length scale. The inviscid cases have been estimated as usual. The viscous cases evolve differently because the small scales structures are removed from the flow, which means that the dissipation range is not restricted to the highest wavenumbers. In the two least viscous cases, there appears to be appear an exponential region in the energy spectra (not shown), but no such range exists in the most viscous case. The gradient of the exponential range has been estimated manually, and then this estimate has then been used to approximate the effective Kolmogorov length.
scale in the same way as in the inviscid cases. The resulting trend is very simple. The inviscid cases remain close to three cell widths, as expected. The viscous estimates are seen to grow to much greater lengths. With greater viscosity, the effective Kolmogorov length scale growth is greater. The growth of the effective Kolmogorov length scale highlights the difference between a “miles fluid” and a viscous fluid. As the kinetic energy decays, the rate at which energy is transferred to smaller scales decreases, and so the energy is dissipated at a length scale that increases with time.

The (effective) Reynolds number is shown in figure 3.17(d). For the inviscid calculations, the same estimation technique has been used as in the previous sections. The viscous cases have been calculated using both the inviscid estimating technique (shown in magenta and to be referred to as the effective Reynolds number) and also using the value of viscosity specified for the calculation (shown in cyan and to be referred to as the viscous Reynolds number). The ratio of the two estimates is shown in yellow. The effective Reynolds numbers for the inviscid cases evolve as established in section 3.5. The viscous Reynolds numbers (cyan) present a different evolution - there is a decrease with time rather than an
increase. This behaviour again highlights the difference between a “miles fluid” and a viscous fluid, but is the expected behaviour of a viscous fluid - because the velocity scale is decaying more rapidly than the length scale is increasing, and the viscosity is constant. However, the effective Reynolds number for the viscous cases (magenta) present different behaviour to each other. The intermediate viscous case presents the same trend as the viscous Reynolds number, whereas the least viscous case presents similar behaviour to the inviscid runs. This is likely to be because the intermediate viscous case is not affected by the numerical error, whereas the least viscous case is affected by it. So the least viscous case has energy dissipated by both viscous effects and the numerical error. However, the intermediate viscous run appears to be independent of the numerical error, hence it can be said to be well resolved. This is confirmed by considering the ratio of the two estimates of the Reynolds numbers (yellow), specifically the viscous Reynolds number divided by the effective Reynolds number. In fact, this ratio is equivalent to the ratio of the actual viscosity and the effective viscosity estimate. In the intermediate case, the ratio is approximately a constant (12.3±0.3). If the numerical error was affecting the dissipation then this would not be constant, as is found to be the case in the least viscous run. This constant can be assumed to be the constant required in the relation for the effective viscosity in terms of the dissipation rate and effective Kolmogorov length scale, which can now be written as \( \nu = \alpha (\varepsilon \eta^4) \), where \( \alpha \approx 0.08 \). This implies that the effective viscosity estimates in the previous sections have been too large (and therefore the Reynolds number have been too small) by a multiplicative factor of around 12. This estimate is consistent with the trend presented by the ratio of the Reynolds numbers for the least viscous case. Initially, the calculation is not well resolved so the ratio is too large, i.e. too viscous because the numerical and viscous effects are combined. However, as the calculation becomes better resolved, the ratio tends towards the same constant attained in the intermediate viscous simulation.

**Summary**

The effect of viscosity has been considered in an attempt to relate MILES simulations to viscous simulations at a higher resolution. This is not as straightforward as it immediately sounds, for one simple reason. In a simulation using implicit diffusion, the effective viscosity is constantly changing, as established in section 3.2. What is being simulated is subtly different from a real-world viscous fluid, which in the context of homogeneous decaying turbulence leads to completely different outcomes. The effect of viscosity is intuitive. Higher viscosities lead to lower Reynolds numbers, the small scales are removed from the flow more rapidly, and the flow decays uniformly to zero more quickly. The fixed viscosity allows the Kolmogorov length scale to grow, which does not occur in a MILES simulation.

If the viscosity is too high, then the small scale structure is removed too quickly for the flow to be of interest. If the viscosity is too low, then the viscous effects can be dominated by the numerical error, and definitive conclusions cannot be drawn. However, at the intermediate viscosity simulated, the effective
viscosity estimated from the resulting turbulent statistics was found to be a constant multiple of the actual viscosity specified in the simulation. Coupled with the effective Kolmogorov length scale substantially larger than the three cell widths associated with the numerical diffusion, this simulation was deemed to be well resolved, i.e. independent of the numerical diffusion. The multiplicative factor can be used to estimate the effective viscosity in the inviscid case depending on the dissipation rate and cell width (assuming that the effective Kolmogorov length scale is approximately three cell widths), specifically \( \nu = \alpha' \varepsilon^{\frac{1}{3}} (dx)^{\frac{4}{3}} \), where \( \alpha' \approx 0.35 \). In a sense, this estimate has removed the assumptions necessary for estimating the effective Kolmogorov length scale by absorbing them into the constant \( \alpha' \). This estimate should be used tentatively, it is likely to be appropriate only for homogeneous isotropic turbulence, and has been derived from a single simulation at a low Reynolds number. However, it does give a good estimate of the order of magnitude of the effective viscosity.
A number of simulations of homogeneous isotropic turbulence have been run to investigate the extent to which turbulent structure and statistics can be recovered using IAMR inviscidly, and without the use of an explicit turbulence model. Various factors have been investigated, specifically the effects of the initial conditions, resolution, large scale generation and viscosity.

The flow field was found to arrange itself rapidly into the characteristic vortex tubes identified by Vincent and Meneguzzi [89], [90]. At late times, the visualisations of the vorticity field presented much smaller vortex tubes, which had radii of a few cell widths, but were shorter in extent than the integral length scale.

During the early times, if the energy containing scales were large, the total kinetic energy was observed to be conserved. This demonstrates that the inertial effects are handled inviscidly by the code. Only once the energy cascade has reached the small scales does the numerical diffusion begin to have an effect.

At the resolutions tested ($N_x \leq 256$), it was found that the velocity field became correlated across the complete range of scales available. This lack of separation of scales proved to be restrictive, such that the evolution of the total kinetic energy and integral length scale do not lend clear support for either Batchelor or Saffman turbulence. Indeed both types of evolution were observed, often the kinetic energy was seen to decay close to the Batchelor rate, and the length scale grow close to the Saffman rate. Higher resolution simulations are required to achieve a domain size to integral length scale ratio sufficient to investigate this problem, e.g. see Kaneda’s immense $N_x = 4096$ simulations [46].

Interesting results were found in the kinetic energy wavenumber spectra. In all of the inviscid calculations, the decay at high wavenumbers was found to be close to exponential. This observation is consistent with the large wind tunnel experiments of Saddoughi and Veeravalli [74], who found an exponential decay of the form $E(\kappa) \sim \exp(-5.2\kappa\eta)$. Assuming the same coefficient, this was used to estimate an effective Kolmogorov length scale, $\eta$. In each case, this effective Kolmogorov length scale was found to be close to three cell widths. Importantly, this was found to be independent of resolution, and with only small variations with other factors, such as forcing wavelength.

By normalising by the dissipation rate and the effective Kolmogorov length scale, the energy spectra were seen to collapse onto the same curve at the small scales. As should be expected, different forms were found at the large scales, depending on the details of the flow. In the decaying cases, a power-law decay was observed with an exponent that was smaller than the theoretical minus five-thirds predicted by K41. This is likely to be due to the lack of separation of scales, because of the periodicity and long range correlation, such that the inertial range\(^1\) was not independent of the large scales. However, when the turbulence was maintained by large scale flow generation, much closer agreement with the minus

\(^1\) The term ‘inertial range’ is retained because of the independence from diffusive effects.
five-thirds decay was found, particularly at higher resolutions. By injecting energy at the large scales, the effects of periodicity and long range correlations became less important. Because of this, the evolution was more consistent with the assumptions of $k_{41}$, and so an inertial range with a decay close to minus five-thirds was observed.

Probability density functions were used to examine the turbulence structure. As expected, the PDFs of the velocity field were found to be close to Gaussian, and the PDFs of the velocity gradients were not. The longitudinal gradients were found to have an asymmetry (skewness), and the lateral gradients appeared to have exponential wings, as reported in the literature. As expected, the moments of the gradients were found to depend on the Reynolds number. As higher Reynolds numbers were attained, the moments were observed to increase in magnitude. The moments suggested the Reynolds numbers were lower than the spectral DNS calculations at equivalent resolutions reported by Jiménez et al. [43], [42]. This is not unexpected as spectral codes are much better suited to these kinds of calculations.

Estimates of the effective viscosity were made by using the relation $\nu \sim (\epsilon \eta ^{4})^{\frac{1}{3}}$, as the dissipation rate could be measured (by the rate of change of total kinetic energy in the decaying case, and by the rate of injection of energy in the maintained case), and the effective Kolmogorov length scale could be estimated (by the dissipation range spectra). It was possible to determine an estimate for the scaling factor in this relation by taking the same approach with a well-resolved viscous simulation, in the sense that the evolution was not affected by the numerical error. It was found that the effective viscosity varied by a multiplicative constant from the actual viscosity in that simulation, which was taken to be the required constant in the relation. In fact, the estimate for the effective Kolmogorov length scale can be absorbed into this constant. If it is assumed the effective Kolmogorov length scale is a constant multiple of the cell width, then the effective viscosity can be written as $\nu = \alpha' \epsilon^{\frac{1}{3}} (dx)^{\frac{4}{3}}$, where it was found that $\alpha' \approx 0.35$.

Higher Reynolds numbers were attained when the turbulence was maintained by large scale flow generation and, naturally, higher resolutions. Using the estimate for the effective viscosity above, a Taylor microscale can be estimated, from which a Taylor Reynolds number can be found. These estimates turn out to be larger than the moments of the velocity gradients suggest. For example, the moments for the maintained simulation with a resolution of $N_x = 256$ were very close to those given in Jiménez [43] at a Taylor Reynolds number of 35. However, the estimate using the above relations turns out to be approximately 250. Even if the scaling of the effective viscosity is dropped, this estimate is still approximately 70.

The most important difference between a real-world viscous fluid and a “MILES fluid” is the fundamental property of the fluid that determines the destruction of kinetic energy at the small scales. For a real-world viscous fluid, this property is viscosity, which for the purposes of this discussion shall be treated as a constant. For a MILES fluid, it is the fixed effective Kolmogorov length scale. Essentially, in a real-world fluid the viscosity determines a length scale (the Kolmogorov length scale) at which kinetic
energy destruction occurs, given a dissipation rate determined by the large scales. Therefore, as a flow evolves, the dissipation rate can change, which results in a change in Kolmogorov length scale. This cannot happen in a MILES fluid. The large scale evolution leads to a transfer of energy to small scales, and this energy is dissipated when it reaches the effective Kolmogorov length scale, regardless of the transfer rate. The consequence of the fixed effective Kolmogorov length scale is that there is always a separation of scales, such that an inertial range is always present. This means that the flow appears to be turbulent even at low Reynolds numbers.

Overall, the flow fields and evolution that have been obtained are consistent with both the ideas behind $K41$ and the results reported in the literature. The flow is arranged into vortex tubes. The structure of the flow is consistent, both in the PDFs of the velocity and the velocity gradients, and in the kinetic energy wavenumber spectra. There is an inviscid transfer of energy from large to small scales, where it is dissipated. The fluid appears to be subtly different from a real-world viscous fluid, in the sense that the fundamental property that determines the small scale dissipation of kinetic energy is a fixed effective Kolmogorov length scale, as opposed to a viscosity. It is important to note that this difference only affects the small scales, but it can also lead to subtly different outcomes. The primary example here is that there is no final period of decay in the decaying simulations. However, in the cases where the turbulence is maintained by large scale flow generation the resulting flow is more consistent with a real-world viscous fluid. Ultimately, it is the large scale flow generation by the forcing term that dominates the flow, and because the code handles the large scale flow correctly, the results are more in line with what is expected.

In essence, the MILES approach is well suited to high Reynolds number regimes where the viscous effects are not dominant, and there is large scale flow generation. Therefore, buoyancy or shear driven flows are particularly appropriate, such as the Rayleigh-Taylor instability or free shear flows i.e. jets and plumes. In addition, the statistically steady nature of jets and plumes will be beneficial as there is a continual supply of fluid from the source. In conclusion, the results presented in this chapter support the use of this code for the proposed investigation.
Chapter 4

Temporally Evolving Circular Shear Layers
This chapter is concerned with temporally evolving circular shear layers. The work of Basu and Narasimha [8], referred to henceforth as BN2, is followed as closely as possible. The study provides further validation of the code’s performance, and opportunity is provided to test the adaptive mesh refinement in the context of free shear flow. The timing benefits are also demonstrated. Finally, consideration is given to the implicit approach to be used for spatially evolving jets, as simple tests are significantly easier to perform in this framework.

Section 4.2 describes the flow configuration. The flow evolution, scaling laws and self-similarity are examined in 4.3. Section 4.4 presents the response due to heating. Section 4.6 demonstrates the capability and benefits of the AMR. The study in BN2 was a DNS, and so a viscosity was specified. The same has been done here, but section 4.5 looks at the effect of viscosity and the MILES approach at different resolutions, in a similar fashion to section 3.6 in the homogeneous turbulence chapter. Section 4.7 closes this chapter with a summary and discussion of the results.

4.2 Initial Conditions

The Navier-Stokes equations were solved in three dimensions on a Cartesian grid, axisymmetry was only prescribed in the initial conditions. Source terms to account for heating were included as described in section 2.6.5 by equation (2.236). A triply-periodic cubic domain was used throughout.

The initial conditions were chosen to match BN2 as closely as possible. The following specification is exactly as presented in BN2, except where highlighted. The streamwise velocity field was initiated using a hyperbolic tangent to produce a circular shear layer with momentum thickness \( \theta_m \)

\[
 u_z = \frac{1}{2} \left( 1 - \tanh \left( \frac{r - r_0}{2\theta_m} \right) \right), \tag{4.1}
\]

where \( r_0 \) is the initial radius. A perturbation was added to the radial velocity, taking the form

\[
 u_r = f(r) \left( \sum_{n=1}^{3} a_n \sin \left( \frac{2\pi z}{\lambda_n} + \phi_n \right) + a_\theta \sum_{m=1}^{16} \sin (m\theta + \psi_m) \right), \tag{4.2}
\]

where \( f(r) \) restricts the perturbation to the region of shear

\[
 f(r) = \exp \left( -2 \left( \frac{r - r_0}{\delta} \right)^2 \right). \tag{4.3}
\]

Denoting the box size by \( L \), the initial diameter of the shear layer was \( d_0 = 2r_0 = \frac{L}{4} \), and the thickness of the perturbation fit, \( f(r) \), was \( \delta = \frac{L}{16} \). The three longitudinal modes had wavelengths \( \lambda_1 = L, \lambda_2 = \frac{L}{2} \) and \( \lambda_3 = \frac{L}{4} \), which were in phase with \( \phi_n = 0 \). The amplitudes were \( a_1 = 0.01, a_2 = 0.02 \) and \( a_3 = 0.03 \). The azimuthal perturbations all had the amplitude \( a_\theta = 0.003 \), with randomly chosen phases.
Values for the box size and the shear thickness were not specified in $\text{BN}2$. The initial values for the integral scales $\tilde{u}_z$ and $\tilde{b}$, as defined in (2.217) and (2.218), respectively, can be ascertained from the evolution plots. This yields approximate values of 0.88 and 0.98, respectively. The initial profile can be integrated to derive exact values for $\tilde{u}_z$ and $\tilde{b}$. Since $\tilde{u}_z$ is independent of $L$, a value for $\theta_m$ can be derived. Taking $\theta_m = \frac{L}{128}$ gives $\tilde{u}_z = 0.877$. To match the value of $\tilde{b}$ requires $L$ to take the curious value of approximately 11. However, the plot of the initial normalised streamwise velocity in $\text{BN}2$ puts the shear interface at approximately $\frac{\xi}{b} = 0.5$. With $\tilde{b}$ close to one, this implies that $L$ should take a value closer to 4. If it is assumed that the integrals involved in (2.217) and (2.218) have factors of $2\pi$, then the former remains unchanged, but the latter definition for $\tilde{b}$ inherits a factor $\sqrt{2\pi}$. By setting $L = 4$, so that the initial diameter $d_0 = 1$, then the initial value of $\tilde{b}$ is approximately 0.89, and the initial shear interface occurs at $\frac{\xi}{b} = 0.5$. These values were deemed sensible and best match the available data. The size of domain was set to $L = 4$, and the shear thickness to $\theta_m = \frac{L}{128}$.

The viscosity was set so that the Reynolds number matched $\text{BN}2$, $\nu = 6.25 \times 10^{-4}$, and similarly the Prandtl number was $\text{Pr} = \frac{\nu}{\kappa} = 7$. If it is assumed that the velocity and diameter are of the same order, then a Reynolds number of 1600 in an aqueous shear layer corresponds to a diameter of approximately 4cm. Even in a spectral code, $128^3$ is unlikely to be sufficient to capture the range of scales in a flow at a Reynolds number of 1600. The relation $\text{Re} = 2^{\frac{1}{3}} \left( \frac{L}{\xi} \right)^{\frac{1}{2}}$ suggests that $l \approx 600\eta$ at a Reynolds number of 1600. However, the flow in question is not dominated by viscous effects, and so the underresolved nature of the simulations is not likely to modify the observed results substantially. This will be investigated further in section 4.5.

The details of the heating will be given in section 4.4.
4.3 Flow Evolution

To compare the evolution with BN2, a simulation was run at a resolution of $128^3$, without AMR. Figure 4.1 shows the development of the flow through different visualisations of the tracer and magnitude of vorticity fields. The top two rows are horizontal slices of tracer (top) and magnitude of vorticity (second row). The third and fourth rows are vertical slices through the symmetry axis showing tracer and magnitude of vorticity, respectively. The bottom row is a three-dimensional rendering of the magnitude of vorticity. Left to right, the columns correspond approximately to times $t = 4, 10$ and $16$.

Flow Evolution

The longitudinal perturbation leads to the roll-up of four large circular vortices at around $t = 4$, figure 4.1(a). These structures undergo a pairing due to the subharmonic perturbation, and a transition to three-dimensionality, due to the azimuthal perturbation, follows in rapid succession around $t = 10$, figure 4.1(b). Note at this stage the core is still undiluted. After this time, the flow consists of a roughly cylindrical volume of small scale vorticity that spreads laterally as the flow evolves, figure 4.1(c).

Large Scale Evolution

Figure 4.2(a) shows the evolution of the integral scales. The velocity and length scales are scaled so that the self-similar decay, outlined in section 2.6.5, appears linear in this plot. The turbulent intensity has been normalised by the velocity scale, and multiplied by 10 to appear on the same plot. Specifically, the quantities $\tilde{u}_z^2$, $\tilde{b}_3$ and $10 \tilde{u}_z'$ are plotted, which are defined by (2.217), (2.218) and (2.220), respectively. After approximately $t = 10$ the velocity and length scales are close to linear, and the turbulent intensity has reached a nearly constant level. There appears to be a slight departure from linearity around $t = 26$.

Comparing with the evolution of BN2, the general flow is qualitatively the same. However, there appears to be a discrepancy in the time scale. The similarity period in this simulation begins shortly after $t = 10$, whereas in BN2 the same region appears to begin around $t = 20$. The values for the integral velocity and length scales at $t = 40$ in BN2 are comparable with the values in the simulation presented here at $t = 30$. The value that the turbulence intensity levels off at around 0.3 in BN2, which is comparable with the value here before the growth observed from around $t = 26$, where the value increases from around 0.3 to 0.38. It is possible that there is a time-shift of approximately 10 time units, or a multiplicative time scaling difference of around $\sqrt{2}$, but it is not clear which, if either. A second anomaly is the initial value of the turbulence intensity, which is close to zero in BN2 and about 0.35 in the simulation presented here. A difference in the initial conditions, specifically in the perturbation, can lead to the observed time scale discrepancy; a greater perturbation will cause the shear layer to break down more quickly. However, the amplitudes of the initial conditions are matched exactly to those specified in BN2. The only difference can
Figure 4.1: Two- and three-dimensional renderings of the magnitude of vorticity and tracer at three different times. A complete description is given in the text.
be the phases, which were shown in BN2 to make little difference, a result that has been confirmed with the present code. The different values for the turbulence intensity may be due to the differing definitions. It was explained in section 2.6.5 that the definition of the turbulence intensity scale did not make sense as presented in BN2, and so the assumed definition may be different to what was actually used in BN2. The difficulty in determining the box size may provide an alternative explanation. In particular, if the two definitions of $\bar{b}$, with and without the factor $\sqrt{2\pi} \approx 2.5$, have been used interchangeably then time scales that differ by a factor of 2.5 can result.

There is no obvious solution to this disagreement, and a remedy has not been found. The unfortunate consequence is that exact quantitative comparisons are difficult. The least problematic solution is to make allowances for the discrepancy in time scale, and bear this in mind when making comparisons. This way, the magnitudes of velocity and length scales are directly comparable.

**Averaged Profiles**

Radial profiles of the average streamwise velocity are presented in figure 4.2(b). The hyperbolic tangent initial condition is shown in yellow. By $t = 7$, in red, the shape of the velocity profile has changed but the centreline value has not decayed. This decay has begun by $t = 14$, in green, where the profile is seen to be clearly wider than the earlier plots. The velocity decay and growth in width continues at later times, shown in blue and black. These profiles have been normalised by the local velocity scale, $\bar{u}_z$, and length scale, $\bar{b}$, and are plotted in figure 4.2(c). The suggestion of similarity after $t = 10$ is confirmed here as there is a good collapse in these profiles, specifically the green, blue and black profiles. The later time, however, appears to show further evolution near the axis. This plot compares very well with the similar plot in BN2, taking account for the difference in time scale. The collapse to similarity and late-time evolution near the axis is observed in both studies. In BN2, this variation near the axis is attributed to a lack of computational data points for adequate averaging. However, a two-dimensional surface plot of the streamwise velocity averaged only in the streamwise direction (not shown) displays a pronounced bump around the axis, and so is very much a feature of the velocity field and not an averaging anomaly. It is, in fact, more likely to be due to streamwise periodicity. A Gaussian profile is shown in cyan for comparison, specifically $\bar{u}_z(\eta) = 1.75 e^{-\eta^2}$. This is somewhat arbitrary, but matches the green and blue profiles closely for small and moderate radial positions, and the only difference is at large radial positions where the Gaussian profile is larger. This means that the streamwise velocities do not agree completely with the Gaussian profile suggested from a constant eddy-viscosity analysis, as outlined in section 2.6.5.

**Turbulence Structure**

Figure 4.2(d) shows normalised radial profiles of streamwise velocity fluctuations. The initial perturbation can be seen clearly, and is restricted to the edge as prescribed. This, however, is substantially larger than
Figure 4.2: (a) Evolution of the integral velocity, length and turbulent intensity scales. The velocity and length scales have been scaled such that self-similar decay appears linear, i.e. \( \tilde{u} \), \( \tilde{b} \). The turbulent intensity normalised by the velocity scale, and scaled up to be shown on the same plot, i.e. \( 10 \tilde{f} \). (b) Radial profiles of averaged streamwise velocity. (c) Radial profiles of averaged normalised streamwise velocity. (d) Averaged streamwise velocity fluctuations, normalised by local integral velocity and length scales. (e) Radial profiles of averaged normalised Reynolds stress. The theoretical profile (2.234) is shown in cyan. (f) Radial profiles of the eddy viscosity coefficient, estimated by dividing the Reynolds stress by a numerical approximation to the radial gradient of the average streamwise velocity.
the profile presented in BN2, which is very small indeed. This lends support to the argument that the initial conditions are responsible for the time scale discrepancy, but as previously stated, the initial conditions here are exactly the same as specified in BN2. As the flow develops the fluctuations grow, and at later times, the profiles are roughly self-similar, but the collapse is far from perfect. This may be because self-similarity is not attained, or because there are insufficient data points, at a resolution of $128^3$, for the statistics to converge. Comparison with BN2 shows the same degree of self-similar collapse and scatter in the data.

Radial profiles of the normalised Reynolds stress are shown in figure 4.2(e). The behaviour is similar to the turbulent intensity above. As the flow develops, the Reynolds stress grows, and the late-time profiles are roughly similar. There is less convergence in these second-order correlations due to the limited number of computational cells over which averaging can be performed. These data were not presented in BN2.

In section 2.6.5, the expression (2.234) was derived for the Reynolds stress in a circular shear layer assuming self-similarity and Boussinesq’s eddy viscosity equation, (2.124). The factors $B$ and $U$ in the coefficient have been estimated by linear fits to the evolution of $\tilde{b}$ and $\tilde{u}_z$, respectively, between times $t = 14$ and 26. Assuming the Gaussian profile that was used in the streamwise velocity plot (and results from constant eddy viscosity analysis), the derived Reynolds stress profile is shown in cyan, and is not inconsistent with the data. The growth near the axis is close and the position and magnitude of the peak is roughly the same. The decay at larger radial positions appears to be slower than the simulated profiles, but this is consistent with the mismatch in the streamwise velocity profiles, and reiterates that the data do not agree with the constant eddy viscosity analysis. However, by using the exact streamwise velocity profiles, Reynolds stress profiles have also been evaluated according to (2.234). These are shown by the two-tone curves in figure 4.2(e), half cyan and the other half the colour corresponding to the respective time. It can be seen that these profiles are all very similar, are consistent with the data, and are in better agreement than the Gaussian profile. This lends support to the use of Boussinesq’s eddy viscosity hypothesis for this flow, provided the eddy viscosity is spatially varying.

Figure 4.2(f) shows profiles of the eddy viscosity coefficient. This has been estimated by dividing the Reynolds stress by a (second-order centred) numerical approximation to the radial derivative of the average streamwise velocity. Profiles from $t \approx 13$ to 40 are shown in cyan. Individual times used in the previous plots have been highlighted by the respective colour. There is a large amount of scatter in the data. This is because this estimate of the eddy viscosity is the ratio of a second-order correlation to a numerical derivative, both of which are close to zero at the axis and at large radial position, and a resolution of $128^3$ is not really sufficient for either of these quantities to have converged statistically. However, there is a suggestion of a self-similar collapse at moderate radial positions, $\eta \approx 0.35$ to 0.85. This lends further support for the suggestion that Boussinesq’s eddy viscosity hypothesis with a spatially dependent eddy viscosity coefficient is viable for this flow.
Summary

An unrefined simulation has been run to establish the flow evolution of the temporally evolving circular shear layer. The initial conditions of BN2 have been matched as closely as possible. This has not been successful due to contradictory information presented in BN2. The discrepancies notwithstanding, the resulting simulations are not inconsistent with each other. The flow evolution follows the same basic stages. There is a roll-up of circular vortices that undergo secondary instabilities and small scale three-dimensional structures develop. The integral velocity and length scales appear to follow the expected power-laws, and there is reasonable self-similar collapse in the radial profiles of the streamwise velocity and velocity fluctuations. The degree of self-similar collapse in both the simulation presented here and in BN2 is close. The radial profiles of both average streamwise velocity and turbulent intensity are certainly comparable, even to the extent of recovering the same centreline evolution in the streamwise velocity at late times. In addition to the comparisons with BN2, the Reynolds stress and eddy viscosity coefficient have been considered. There is self-similar collapse in the eddy viscosity coefficient, which lends support for the use of Boussinesq’s eddy viscosity hypothesis with a spatially dependent coefficient. However, it could be argued that if there is self-similar collapse in both the Reynolds stress and the streamwise velocity (and therefore its derivative), then the ratio of two self-similar quantities is bound to be self-similar. It would be interesting to extend this analysis to investigate whether the functional form of the derived eddy viscosity coefficient can be related to properties of the flow, e.g. the turbulent intensity, so that a predictive model can be formulated for this kind of flow. Although there are many differences between such a flow and spatially evolving free shear flows, this framework is much simpler and may prove to be enlightening for construction of such models in the more complicated situations.
4.4 The Response to Heating

The objective of the BN2 study was to investigate the response to cloud-like off-source heating. In the temporal framework, heating is added between two times once the flow has attained self-similarity. The effect of two parameters was investigated, specifically the effect of the heating rate and the functional form of the heating term. The same approach has been taken here, three heating rates and three profiles have been investigated. Taking account for the difference in time scale determined in the previous section, heating was applied between times \( t = 22 \) and 24, rather than \( t = 32 \) and 34 as in BN2.

The exact profile and widths of the heating profile, \( s(r) \), that were used in BN2, were not specified. The profiles were said to be similar those of the streamwise velocity at times \( t = 20, 32 \) and 40, and the intermediate value was used for the heating rate comparison. Here, \( s(r) \) was defined to have unit integral over the whole profile and a Gaussian distribution

\[
s(r) = \frac{2}{r_s^2} \exp \left( -\left( \frac{r}{r_s} \right)^2 \right),
\]

where \( r_s \) determines the profile width. The width values used below are \( r_s = 0.5, 0.75 \) and 1.0.

In each simulation, the calculation was restarted from a check point file that was output from the unheated case at \( t = 21 \). There will be no difference in the calculations until heating is turned on at \( t = 22 \), so there is no point rerunning the whole calculation from scratch.

**Effect of Heating Rate**

The results of heating at three different rates are compared in this section. The profile used had length scale \( r_s = 0.75 \). The values of the heating rate were \( G^* = 0.01, 0.04 \) and 0.1, referred to as low, medium and high, respectively, and are the same values used in BN2.

The effect of heating on the streamwise velocity is shown in figure 4.3(a), where \( \tilde{u}_z^2 \) is plotted. Note that only times \( t = 20 \) to 30 are plotted, which along with the scaling highlights the linearity of the unheated (blue) case. The vertical red lines denote the period of heating. In each case, the heating results in an arrest of the streamwise velocity decay. The arrest is stronger with a higher heating rate, and the velocity actually accelerates at the greatest heating rate. This behaviour is qualitatively the same as BN2, but the response is slightly less pronounced here. This is likely to be due to differences in the heating profile used, which is unavoidable.

The integrated length scales are plotted in figure 4.3(b). With this heating profile, the width of each heated case is narrower than the unheated case. A stronger heating results in an attenuation of the growth rate, although the width still increases with time. The behaviour is very similar to that observed in BN2.
Figure 4.3: Comparison of the flow evolution and radial profiles under different heating rates. (a) Integrated velocity scale, $\tilde{u}_z$. (b) Integrated length scale, $\tilde{b}^3$. (c) Integrated normalised turbulent intensity, $c u'_z e u_z$. (d) Radial profiles of normalised streamwise velocity. (e) Radial profiles of normalised streamwise velocity fluctuation. (f) Radial profiles of normalised Reynolds stress.
The evolution of the normalised integrated turbulent intensity scale is shown in figure 4.3(c). Heating appears to reduce the level of turbulence intensity, and the reduction is greater with higher heating rates. Again, this behaviour is close to that observed in BN2.

The normalised streamwise velocity profiles at $t = 30$ are shown in figure 4.3(d). There is very little difference, if any, with heating rate. Similar behaviour was observed in BN2, but there was a slight decrease in the values very close to the axis with heating. Again, this may be due to the subtle differences in the way the heating is applied.

Radial profiles of the normalised streamwise velocity fluctuations, $\bar{u}^2$, and normalised Reynolds stresses are shown in figures 4.3(e,f). Heating leads to a decrease in both types of profile, which is greater with higher heating rates. This behaviour of the streamwise fluctuations is the same as that presented in BN2, where the Reynolds stresses are not presented.

**Effect of Heating Profile**

This section compares the effect of the width of the heating profile. The widths were taken to have the values 0.5, 0.75 and 1.00, referred to as small, medium and large. The heating rate was kept constant at $G^* = 0.04$, so the total buoyancy added is the same in each case, but the temperature rise will be different because the volume over which heat is injected is different.

Figure 4.4(a) shows the evolution of the average streamwise velocity, with the usual scaling, $\bar{u}_x$. As expected, the heating leads to an increase in the velocity scale. The narrow profile leads to the greatest response. This is significantly different to what was found in BN2, where very little difference was observed between the three heated cases. This may be due to the uncertainty determining the heating profiles used, or the time discrepancy established in section 4.3.

The evolution of the characteristic width is shown in figure 4.4(b). The medium width profile decreases, as before, the narrow profile produces a more pronounced decrease in width, but the large profile leads to an increase. This is the same behaviour observed in BN2.

The normalised turbulent intensity, shown in figure 4.4(c), decreases in all cases. The response is more pronounced with the narrow heating profile, which is seen to recover at late times. This behaviour is again different to that presented in BN2, although the unnormalised behaviour (not shown) is comparable. This suggests that the source of the disagreement is the streamwise velocity scale used in the normalisation, which was seen in figure 4.4(a) to be significantly different to BN2.

The change in normalised streamwise velocity profile is restricted to a small increase near the axis, figure 4.4(d). The narrow heating profile leads to an increase near the centreline, whereas the wider profiles make little difference. These results are consistent with the observations of BN2.
Figure 4.4: Comparison of the flow evolution and radial profiles under different heating profiles. (a) Integrated velocity scale, $\overline{u_z^2}$. (b) Integrated length scale, $\overline{b^3}$. (c) Integrated normalised turbulent intensity, $\frac{\overline{u_z' u_z}}{\overline{u_z^2}}$. (d) Radial profiles of normalised streamwise velocity. (e) Radial profiles of normalised streamwise velocity fluctuation. (f) Radial profiles of normalised Reynolds stress.
Figure 4.4(e) shows the radial profiles of streamwise velocity fluctuations. The wider two profiles produce little change in the shape of the profile, but lead to a decrease in magnitude. The narrow profile brings about a more pronounced change. Although there is a decrease at moderate and large radial positions as with the other two cases, there is an increase in the intensity near to the axis. The behaviour presented in BN2 is a little different, but the narrow heating profile does result in higher levels at small to moderate radial positions ($\eta < 0.5$).

Figure 4.4(f) shows the normalised Reynolds stresses. A similar trend to the streamwise fluctuations is present here. The widest heating profile leads to a small decrease in magnitude and a narrowing of the Reynolds stress profile. The intermediate profile leads to a decrease in magnitude. The narrowest profile results in a normalised Reynolds stress lower than the widest case but compared with the intermediate case, it is enhanced for radial positions $\eta < 0.4$. Again, these data are not presented in BN2.

**Summary**

A series of simulations have been run to look at the response due to heating, and to compare the results with BN2. The disparate time scales and vague specification of the heating profile used has made direct comparison difficult, but within the limitations outlined, the results compare reasonably well.

As expected, the heating leads to an arrest of the streamwise velocity decay, and can lead to an acceleration if the heating is strong enough. As was found to be the case in BN2, the evolution of the characteristic length scales presents significantly different behaviour depending on the width of the heating profile used. A narrow profile leads to a decrease in the growth rate of the integrated length scale, whereas a wide profile can lead to an increase. The response of the averaged quantities, including the second-order correlations, are in good qualitative agreement with BN2. The only real difference between the results is found in the streamwise velocity evolution using different heating widths. In BN2, it was found that the streamwise velocity depended solely on the heating rate $G^*$, although only five simulations were presented. This disagreement it most likely to be due the differing time scales and the details of the heating profiles.
4.5 Effect of Viscosity

The simulations in the previous section have all been run with a viscosity matching the Reynolds number of \( \text{Re} = 1600 \). This section explores the effect of viscosity on the development of the shear layer and the turbulence structure. The aim of the investigation is to assess the suitability of the MILES approach in a high Reynolds number free shear flow framework.

The same approach was taken as for the homogeneous isotropic viscous investigation in section 3.6. Inviscid calculations were run at three different resolutions \( N_x = 32, 64 \) and 128, and calculations were run at the highest resolution \( (N_x = 128) \) with three different viscosities \( \nu = 10^{-2}, 10^{-3} \) and \( 10^{-4} \). An inviscid calculation was run at \( N_x = 128 \) using the same initial conditions as for the unheated case, until time \( t = 13 \). This flow field was used as the initial conditions for each simulation, averaging down onto lower resolutions where necessary, and subsequently run until \( t = 15 \). This means that the initial development of the shear layer is the same for each simulation and does not cloud the results.

Figures 4.5(a,b) show the evolution of the integrated velocity and length scales, with the usual scalings. The two plots are remarkably similar and present exactly the same trends. It is clear that the least viscous simulation is almost identical to the inviscid simulation at the same resolution \( N_x = 128 \). This is likely to be because the viscous effects are dominated by the truncation error inherent in the scheme. It would appear that for the first half of the calculation at \( N_x = 64 \) closely follows the previous two simulations, but then deviates - the velocity decays at a quicker rate accompanied by an increase in spread rate. Decreasing the resolution further \( (N_x = 32) \) appears to lead to a slower decay in the velocity scale, and a lower spread rate . This appears to be matched by an increase in viscosity - the \( \nu = 10^{-3} \) and \( N_x = 32 \) simulations appear to follow a similar evolution. Increasing the viscosity further leads to a dramatic change in behaviour, the velocity decays much more rapidly and the spread rate is higher than in any of the other cases.

The dramatic difference in behaviour of the most viscous case is highlighted by the evolution of the integrated turbulent intensity, figure 4.5(c). In this case, the viscosity removes the small scale motions very quickly. This calculation is clearly in a different regime to the other five. Indeed, visualisations of the flow variables for this case (not shown) are smooth in nature, whereas the other calculations present a highly fluctuating flow fields. As the viscosity is reduced, the turbulent intensity grows, up to the intensity shown by the dark blue curve. It is not clear why the two inviscid cases at the lower resolutions have a lower intensity for the first half of the calculation and a higher intensity for the second half.

The greatest insight into the behaviour observed can be found in the radial profiles of the Reynolds stresses, shown in figure 4.5(d) at time \( t \approx 5 \). Also shown in this plot (in magenta) are the viscous stresses that would result at the three different viscosities used in the investigation. These have been calculated based on the inviscid streamwise velocity field at \( N_x = 128 \), i.e. \( \nu \frac{\partial \bar{u}}{\partial r} \) has been plotted for
Figure 4.5: Comparison of the flow evolution at different viscosities and resolutions. (a) Integrated velocity scale, with the usual scaling $\tilde{u}^{\frac{2}{3}}$. (b) Integrated length scale, with the usual scaling $\tilde{b}^3$. (c) Integrated normalised turbulent intensity, $\tilde{u}'^{2}/\tilde{u}^{2}$. (d) Radial profiles of normalised Reynolds stresses and viscous stresses, $t \approx 5$.

$\nu = 10^{-4}, 10^{-3}$ and $10^{-2}$. The highly fluctuating nature of these profiles results from taking the numerical (radial) derivative. The essential point of note is that the viscous stresses in the most viscous case are comparable with the Reynolds stresses at the highest resolutions. Therefore, the simulations at lower viscosities have viscous stresses that are orders of magnitude smaller. This means that the dynamics of the large scale flow is dominated by the turbulence and not the effects of viscosity. However, the viscosity does not only affect the mean velocity, but also the Reynolds stresses. This is why in the most viscous case, the behaviour is very different - the turbulence is destroyed and so the mean velocity is dominated by the viscous stresses. At the intermediate viscosity, the effect of viscosity is important but does not dominate the evolution. The viscous action on the turbulence leads to a smaller Reynolds stress, which appears to be matched by the lowest resolution, at least initially.

These results lend support to the use of the MILES approach for free shear flows where the Reynolds number is sufficiently high that the viscous stresses are dominated by the turbulent Reynolds stresses.
This section is concerned with investigating the use of adaptive mesh refinement (AMR). The investigation is twofold. Firstly, static grid tests have been performed to calculate the benefits in terms of computational expense. These can be performed in any flow, but this is the most appropriate section to include the results. Secondly, the unheated circular shear layer was run in a number of configurations. All simulations were at the same effective resolution, but different base grids and refinement ratios were used. In each case, the gridding was dynamically arranged to follow the evolution of the shear layer. The results were then compared with the unrefined case to assess the accuracy of the AMR.

Timing Tests

Static grids were hardwired to cover a constant prescribed proportion of the domain. The time taken to calculate 100 fine resolution time-steps was then measured. The control was run at 128$^3$. One test was run with a base grid of 64$^3$ with one level of refinement factor 2. Two other tests were run with a base grid of 32$^3$, one with a refinement factor of 4 and the other with two levels with refinement factor 2. The grids were configured to cover 0%, 5%, 25%, 56% and 75% of the domain. The times are shown in figure 4.6, normalised by the length of time taken at the control resolution.

It is clear that the AMR can produce substantial benefits in computational expense. It appears than a single level of refinement factor four is the cheapest configuration. With a 40% coverage, the calculation takes less than half the time to complete. Above 25% coverage, there is little difference between using two levels of refinement, or a single level with a base grid with twice the resolution. Using a refinement factor of 2, with one or two levels, the calculation takes the same length of time as the control when there is approximately two-thirds of the domain covered. For a refinement factor of 4, the calculations take the same length of time when the coverage is approximately three-quarters of the domain.

![Figure 4.6: Relative timings versus domain coverage.](image)
Figure 4.7: Comparison of the simulations using different gridding arrangements. (a) Evolution of the integral quantities, $\tilde{u}_2^2$, $\tilde{b}^3$ and $10\tilde{u}'_z$, as plotted in figure 4.2(a). (b) Evolution of the error in the integral velocity scale. (c) Evolution of the error in the integral length scale. (d) Evolution of the relative expense of each calculation.

Accuracy

Simulations were run with the same effective resolution, $128^3$. An unrefined case was run as a control. The AMR grids were dynamically arranged to cover regions of tracer and magnitude of vorticity. Specifically, thresholds were specified and any cell with tracer or vorticity above the respective threshold was tagged for refinement. A buffer region can be specified to tag the surrounding cells. Two simulations were run with a base grid of $64^3$ with a single level of refinement, factor 2, one with a single cell buffer, and a second with a four cell buffer. Four simulations were run with a base grid of $32^3$. One used two levels of refinement, each of factor 2, with a single cell buffer. Three simulations were run with a single level with refinement factor 4, which had one, two and four cell buffers.

The evolution of the integral scales is shown in figure 4.7(a). Until time $t = 20$ there is very little difference between the simulations. Between times $t = 20$ and 40 the discrepancies grow, but remain small. A more quantitative comparison is made in figures 4.7(b) and 4.7(c). These figures show the relative error in the
integral velocity and width scales, respectively. Because the time steps are out of synchronisation, each quantity in the control case has been estimated at the corresponding time in the test case. A simple relative error was then calculated. For example, let the test time be \( t = t_0 \), and the two times in the control case be \( t = t_i < t_0 \) and \( t = t_{i+1} > t_0 \). To estimate the length scale in the control case at \( t = t_0 \), \( \tilde{b}_0 \) say, a simple linear interpolation can be expressed as follows

\[
\tilde{b}_0 = \tilde{b}_i + \left( \frac{\tilde{b}_{i+1} - \tilde{b}_i}{t_{i+1} - t_i} \right) (t_0 - t_i),
\]

where \( \tilde{b}_i \) and \( \tilde{b}_{i+1} \) are the widths in the test case at times \( t_i \) and \( t_{i+1} \). Then relative error is then simply defined as

\[
\epsilon(t_0) = \text{abs}\left( \frac{b(t_0) - \tilde{b}_0}{\tilde{b}_0} \right).
\]

In each case, the error before \( t = 15 \) is very small. The error grows to a peak at approximately \( t = 28 \), whereafter it remains around the same size. In the worst case, the \( 32^3 \) simulation with one level of refinement factor 4 and one cell buffer peaks with just over 2% relative error in the integral velocity scale and just over 1% relative error in the integral length scale. This error is greatly reduced using a larger buffer or two levels of factor 2 refinement, but the best improvement is observed by using a finer base grid. The \( 64^3 \) simulation with one cell buffer has relative errors less than 0.5% and 0.2% in the integral velocity and length scales, respectively, and again improves with a larger buffer.

Figure 4.7(d) shows the relative expense of each calculation. This has been calculated by multiplying the percentage coverage by a scaling depending on whether the refinement factor used was 2 or 4. These scaling factors were chosen to be \( \frac{3}{2} \) and \( \frac{4}{3} \), respectively, as this approximates the relations observed in figure 4.6. Note that this expense is the instantaneous expense and so the cumulative expense is the integral of this function. It is clear that for this flow, the growth in the shear layer quickly fills the computational domain, and when a large buffer is used the expense soon becomes greater than the unrefined case. However, when a small buffer is used the relative expense only becomes comparable after \( t = 25 \), by which stage the benefit has been great. After this time, the relative expense is only slightly greater than the unrefined case and the one cell buffer simulations were cheaper than the unrefined case overall.

These plots demonstrate that the use of AMR has to be carefully controlled, and can actually be counterproductive - not only can a simulation take longer, but the answer is less accurate than the unrefined case. It is clear that a balance has to be found between desired accuracy and computational expense. Taking into account the accuracy and expense that was demonstrated above, the optimal configuration tested is \( 64^3 \) simulation with one level of refinement of factor 2 with a single cell buffer. It should be borne in mind that in this simulation a significant proportion of the computational domain is classed as interesting and so is not that well suited for AMR. However, the benefit of AMR would be invaluable if the effects of the boundary conditions were to be reduced. If the transverse extent of the domain was dou-
bled, the unrefined simulation would take four times as long to execute. Using the optimal configuration above, which had a relative cost of approximately 83%, a simulation could be run with a base grid of $128 \times 128 \times 64$ with a single level of refinement, factor 2. The extra cost of this calculation is equivalent to running three simulations at $64^3$, which adds three-sixteenths to the original unrefined simulation time. Therefore, the total relative cost of this simulation is 1.0125, which, compared with a factor 4 increase if the calculation were unrefined, is a massive saving.
4.7 Discussion and Conclusions

A number of simulations have been run to investigate how the code handles a temporally evolving circular shear layer, and the response to heating. The motivation for this investigation is twofold. First, it is a validation exercise to compare how the code agrees with a similar study in the literature, Basu and Narasimha [8], with a pseudo-spectral DNS code. Second, to investigate the application of the MILES approach to a free shear flow.

The study of BN2 has been followed as closely as possible, which proved to be somewhat problematic due to the contradictory information in the paper. However, this study was not substantially different, but it was not possible to make direct quantitative comparisons. There is sufficient theoretical background with which to complement the comparisons.

The general evolution of the flow was well matched. The initial shear layer was seen to roll up into circular vortex rings. These vortices underwent a pairing, which was followed quickly by a transition to three-dimensionality. There then appeared to be a period of self-similar evolution, until the effects of the boundaries began to play a role. The characteristic velocity and length scales appeared to evolve close to the predicted power-laws. The radial profiles of the velocity field collapsed onto a self-similar profile, where the same evolution near the axis as BN2 was observed. The second-order correlations also presented a self-similar collapse, albeit with slightly more scatter, which is unsurprising for second-order correlations. An estimate of the eddy viscosity was made, which presented a reasonable collapse to self-similarity, but with significant scatter. A clear indication from this estimate is that a constant eddy viscosity is not appropriate, and this was reflected by the fact that the velocity field had a non-Gaussian profile, which a constant eddy viscosity analysis suggests. A second important conclusion that can be drawn from the eddy viscosity estimate is that the effect of the true viscosity is much smaller than the effect of the turbulence on the large scale evolution, i.e. the velocity decay and spread rate, certainly at this Reynolds number. This result was explored further in section 4.5, and provides definite support for the use of the MILES approach for this kind of problem.

The response to heating was investigated in the same way as BN2, and the general response was found to be in good qualitative agreement. The velocity decay was arrested and even reversed at the highest heating rates. The spread rate was observed to increase and decrease depending critically on the profile of the heat injection. The response of the second-order correlations was consistent with BN2, and again the heating profile was found to have a significant role in the response. In fact, the only result substantially different to those presented by BN2 was that the global response (i.e. spread rate and velocity decay) depended solely on the heating rate. This disagreement is likely to be due to the disparities in the configuration of the simulations.
Chapter 5

Spatially Evolving Jets
5.1 Introduction

This chapter is concerned with simulating spatially evolving turbulent jets with and without off-source heating. In chapters 3 and 4, it was that the numerical algorithm can reproduce turbulent statistics comparable with those in the literature. The comparisons made in chapter 4 with a temporally evolving circular shear layer study with enhanced buoyancy presented reasonable agreement with the literature. The code is capable of capturing the large-scale advection accurately and the implicit dissipation at small scales emulates the effects physical viscosity. In section 2.6, high Reynolds number jets were discussed. Consideration of the averaged equations of motion under the boundary layer approximation suggested that for high Reynolds number jets turbulent effects dominate viscous effects. This suggests that the specific details of the implicit diffusion inherent in this code will not be a dominant factor in resolving the flow. The conclusion is that this kind of approach is a plausible method for an investigation of this type, and is an issue to be addressed. The careful use of AMR has been shown to maintain a high level of accuracy whilst significantly reducing the computational expense. This will be crucial in capturing the broad range of time and length scales in turbulent jets.

There are many parameters that can affect the flow, both in terms of the physical quantities, and in the details of the simulation, such as the arrangement of the AMR grids. These parameters are discussed in section 5.2 to determine a suitable configuration for the final investigation. Sections 5.3-5.5 consider control experiments to establish the behaviour in the chosen configuration, specifically a pure jet, a pure plume and the mixed case of a buoyant jet. The crux investigation of the response of a turbulent jet to volumetric heating will be presented in the next chapter.

5.2 Configuration

The parameters that require specification for simulating turbulent jets and plumes are the size of the nozzle (point sources will not be considered), the momentum flux, the buoyancy flux, the streamwise and lateral extent of the domain, any perturbation to the inflow, and the resolution and configuration of the AMR. Viscous effects will be neglected throughout this study.

First consider the case of a pure jet, for simplicity. Neglecting any inflow perturbation, the flow is characterised by the nozzle diameter and the inflow speed. Since the equations that are modelled are the incompressible Euler equations, non-dimensionalising the equations removes both of these characteristic scales, and so there is no parameter space to explore. Computationally speaking, the length scale is determined by the number of cells across the nozzle. Since the nozzle velocity is the greatest velocity in the flow, this determines the time step (once the cell width and CFL number have been specified). Since the flow is statistically steady, the time step is only relevant to the length of the simulation, i.e. how many
time steps are required. In conclusion, the only relevant measures of the experiment are the number of cells across the nozzle diameter and the number of time steps the simulation is run for. The domain size is then measured in terms of the nozzle diameter.

With the inclusion of buoyancy, the Morton length scale (2.156) and the laziness parameter (2.155) become important. These determine the streamwise evolution of the plume, and therefore give a measure of the domain required for jet-plume transition.

When volumetric heating is introduced, the streamwise position of the start and finish of the HIZ, as well as the heating rate will become important. The single dimensionless number defined in section 2.6 may give a general description of the global evolution, but the details of the HIZ itself must depend on all three variables. For this reason, the computational domain has been chosen to be close to the laboratory dimensions, and all quantities will be given in SI units.

In each case, there is a region near the source where the flow establishes itself. This region is dominated by shear instabilities and coherent vortices that depend strongly on the details of the inflow conditions. Numerically, without a perturbation, unphysical flow fields can result, an example of which is shown in figure 5.1. This results from resolving a circular nozzle on a Cartesian grid, and is a highly undesirable outcome. A perturbation can be introduced to ensure that flow establishment happens quickly and that the ‘clipping’ effect above is not dominant. If the perturbation has a frequency close to the natural frequency of the jet, large circular vortices, as in the circular shear layer from the previous chapter, can be generated, which then have to undergo secondary instabilities before the flow transitions to a developed state. Determining the exact inflow conditions from a laboratory experiment is an extremely difficult
exercise, and one that is not likely to modify the far field results significantly. Therefore, an artificial perturbation was used. This was achieved by using a velocity field defined over a high aspect ratio box that is square in cross-section and triply-periodic. The velocity field was initiated in the same way as the initial conditions used for the homogeneous turbulence, as described in section 3.1.1. Denoting this velocity field as $u'_I(x, y, z)$, the inflow velocity was specified by

$$u_I(x, y, t) = \bar{u}_I(r) (e_z + \alpha_0 u'_I (x, y, -\bar{u}_0 t)), \quad (5.1)$$

where $\bar{u}_I(r)$ is the mean inflow velocity, where mean is defined by (2.118), $r = \sqrt{x^2 + y^2}$ is the radius, $e_z$ is the unit vector in the streamwise direction, and $\alpha_0$ is the amplitude of the perturbation. In addition, a hyperbolic tangent profile was also used

$$\bar{u}_I(r) = \frac{1}{2} \left( 1 - \tanh \left( \frac{2(r - r_0)}{\delta_0} \right) \right), \quad (5.2)$$

where $r_0$ is the nozzle radius and $\delta_0$ is the shear thickness. These choices were made so that the perturbation was incompressible to prevent numerical difficulties, ‘clipping’ effects were reduced, and the flow broke down quickly and without forming the large circular vortices that delay transition to self-similarity. The amplitude of the perturbation was chosen so that the RMS fluctuation was 2% of the inflow speed. This was found to be sufficient to dominate the clipping effects, but still smaller than the fluctuations expected in a jet flow, which are around 25%, see section 2.6.

For the purposes of establishing an adequate resolution, consider first the laboratory experiment of ASP, where the tank was 600 diameters tall and 250 diameters in each horizontal direction. Suppose the domain is to be discretised using a uniform Cartesian grid, without AMR. Let there be $N_d$ cells across the nozzle diameter, so the domain has a total of $3.75 \times 10^7 N_d^3$ cells. If it is assumed that $N_d = 32$ is a sufficient number of cells, which is likely to be the bare minimum, this gives a resolution of $8000 \times 8000 \times 19200$. A calculation of this magnitude would be around 16 times the number of cells of Kaneda’s 40963 huge homogeneous isotropic turbulence calculation, which is one of the largest calculations to date, and would require many more time steps for statistical convergence. Even with AMR, this is beyond the computational resources available anywhere in the world today. This means that a study of spatially evolving jets has to concentrate on a smaller subsection of the domain. There are essentially two choices, concentrate on the transitional region close to the nozzle, or on the far field and look at the self-similar region.

For the problem in question, the self-similar region in the far field is important. Therefore sacrifices must be made near the nozzle, specifically, there can only be a limited number of cells across the nozzle. However, there must be a sufficient number of cells so that, although the details of the transition to self-similarity are not captured fully, there is a horizontal plane across which the mass and momentum fluxes are such that the flow above that plane is statistically comparable with a developed jet. This is an extremely loose assessment, and it is very difficult to determine exactly what is sufficient. Ultimately,
restrictions on available computing power determine what can be attained. The jet nozzle resolution provides the lower bound for resolutions suitable for simulating jet flows. The upper bound, i.e. the size of the domain, comes from the length of time needed to run the simulation to get time-averaged data. In this sense, the pure jet is the most difficult case because the time scale grows with the square of streamwise distance. In the plume, this growth is only a four-thirds power.

Because a restricted section of the domain was resolved, transmissive lateral boundary conditions were used. This involves matching the velocity field in the ghost cells to those within the domain, see section 2.2. This then allows fluid to be drawn into the domain as required, making the boundaries appear almost transparent. This helps to avoid recirculation effects that would occur if solid walls were used. The upper boundary was also set to be transmissive to allow fluid to pass out of the domain. It was found that these transmissive boundaries were not perfectly transparent. Figure 5.2 shows a schematic of a cross-section of a vortex ring at an outflow boundary. It demonstrates the difficulty in capturing the coherence of a structure like a vortex ring. Since the ghost cells just match the cells within the domain, the downward part of such a structure gets misinterpreted by the boundary and leads to a strong spurious inflow, rather than the structure being simply advected out of the domain. For the upper boundary, this kind of effect was reduced in two ways. Firstly, where inflow was detected, the vertical velocity in the ghost cell was set to zero instead. Secondly, the boundary was moved away from the region of interest using AMR, to reduce the impact of the boundary without significantly increasing computational expense. The AMR was also used to move the lateral boundaries further away. Since there should be no coherent structures leaving these side boundaries, and as inflow is required here, the ghost cells were not adjusted to prevent the spurious injection of fluid seen at the top boundary. The inflow at the lower boundary was set to the inflow within the jet nozzle, and zero outside it.

By using two levels of AMR the domain was separated into three different regions. Firstly, a high resolution region to capture the nozzle. Secondly, a region of interest where the flow was self-similar, and finally, a low resolution region around the first two that acts as a buffer region to minimise the effects of the lateral and upper boundary conditions. It was found that the behaviour of the jet was affected by a

![Figure 5.2: Schematic to show the effect of the transmissive boundary conditions on an upwardly propagating vortex ring. The left-hand plot shows the true velocity field, and the right-hand plot shows the effect due to the boundary condition. Because the ghost cells are matched by the uppermost cell in the domain, an unwanted inflow is generated at the outer edge of the ring.](image-url)
coarse-fine interface. This is not unexpected since it has the effect of halving the implicit Kolmogorov length scale, see section 3.5. For this reason, the extent of the high resolution region was restricted so that this coarse-fine interface did not interfere with the region of interest, which also conveniently reduces computational expense. A more detailed account of the arrangements of the AMR grids will be given in the next section.

The calculations presented in this chapter and the next were conducted at AWE Aldermaston. Sixty-four processors (each clocked at 375MHz) were allocated for twelve days. Each calculation was run on sixteen processors, allowing four jobs to be run concurrently. With this computational power available the base grid resolution was $96 \times 96 \times 128$. The physical domain size was specified as $0.768m \times 0.768m \times 1.024m$. One level of AMR, with a refinement factor of two, was used for the self-similar region. This gives an effective resolution of $192 \times 192 \times 256$, which means that length of each side of a cell is 4mm. Another level with refinement factor two was used around the nozzle, and the nozzle was resolved using eight cells across its diameter at this resolution. This makes the entire domain size $48 \times 48 \times 64$ nozzle diameters. The highest resolution was restricted to eight diameters in the streamwise direction, and the self-similar region to forty-five diameters. It should be noted that if this configuration was rescaled to the experiment of ASP such that the diameters were the same size, then these simulations would only occupy a $96mm \times 96mm \times 128mm$ volume at the bottom of the tank, which is approximately 0.4% of the volume of the experimental tank.

Since the AMR is only required to capture the region where there is jet fluid, and not individual coherent structures, identifying the regions that require covering is straightforward. Thresholds for tracer concentration and vorticity were specified such that a cell with either value above the threshold is tagged for refinement. This allows the AMR to adapt to follow the jet. In practice, this will make little difference to the late time arrangement of the AMR as the flow is statistically steady. It does however, allow the initial period of flow establishment to be resolved much more quickly. For the pure jet the domain coverage of the two levels was 16% and 0.1%, which was even lower in the other simulations, for reasons that will become apparent.

The results of three simulations are presented in this chapter. First a pure jet, i.e. only an inflow velocity was specified, and no buoyancy, and then two simulations with both velocity and buoyancy specified. The idea was to capture a pure plume and a buoyant jet (forced plume). Morton’s balance (laziness) parameter depends critically on the definitions of the width ratio ($\lambda$) and the entrainment coefficient ($\alpha_e$), and moreover assumes that the plume is in a fully developed self-similar state. The constants, depending on the interpretation, can lie in the ranges $\lambda = 1.0-1.2$ and $0.53-1.2$, respectively, which can give rise to a laziness values differing by a multiplicative factor of over 3. This proved to be problematic when determining perfect inflow conditions to achieve a balanced plume. In both buoyant cases, the inflow temperature was set to 1K above ambient (corresponding to a reduced gravity, see (2.7),
of $g' = 2.1 \times 10^{-3}$). It also turned out that because the plume is not fully developed at the nozzle, there was an initial acceleration, even for plumes with $\Gamma < 1$ that should be forced plumes. The pure plume simulation presented had an inflow velocity of $0.018\text{ms}^{-1}$. Taking $\lambda = 1$ (ignoring any difference in velocity and scalar width) and $\alpha_x = 0.083$ (appropriate for a plume), gives $\Gamma \approx 0.78$, which suggests that the plume should be slightly forced, but turns out to be slightly lazy. However, the adjustment length is observed to be small and the resulting flow is plume-like, as will become apparent. The forced plume had an inflow velocity of $0.03\text{ms}^{-1}$, which gives $\Gamma \approx 0.28$ and an adjustment length $l_M \approx 23\text{cm}$, see (2.156).

When the simulations were originally run at AWE, the configuration of the temperature equation was such that the non-conservative update was used. This update is required for the low Mach number combustion, which is one of the primary interests of the code’s authors. This is implemented by subtracting $T \nabla \cdot \mathbf{u}$ from the advective flux. Incompressibility implies that this term is zero (in this case, it is non-zero in the low Mach number combustion case), but the approximation to this term that is used in the code is only approximately zero. Numerically, the approximation vanishes in the limit of infinite resolution, so is not incorrect, but simply an approximation. However, this approximation led to a lack of conservation in buoyancy, which is undesirable. Several simulations have been subsequently run to investigate the effect of this difference, specifically the pure plume and the heated jet with the highest heating rate. It was found that although the streamwise evolution of the buoyancy is different, the normalisation process means that the resulting structure and, more importantly, the phenomenology of the flow were not significantly affected. In this chapter, the pure plume with the conservative update and the buoyant jet with the non-conservative update are presented. A comparison of the two cases is given in section 5.6.

The heated jet cases in the next chapter were run at AWE with the non-conservative update. A test case was run using the conservative update, which demonstrated that the differences were small and the same phenomenology was observed.
This section presents the results of the pure jet simulation. The inflow velocity was $0.128 \text{ms}^{-1}$, and the nozzle diameter was $0.016 \text{m}$. If a value of viscosity for water is assumed, which is not likely to be attained in this simulation, then the Reynolds number for this jet is approximately 2000, which should remain constant with height, see section 2.6. Estimating the effective Reynolds number will be discussed in section 5.7.

At the beginning of the simulation, a vortex ring forms. This vortex ring propagates upwards much more quickly than the jet fluid behind it, and leaves the domain at the top. This phenomenon was also reported by ABP in their simulations, and has also been observed in laboratory jets. Since the time scale of the jet grows with the square of streamwise distance, a significant length of time is required for the jet to attain similarity. Therefore, the vortex ring is not thought to have any effect on the results.

Averaging, as defined by (2.118), was performed from $t = 108$ after 2500 iterations, to $t = 308$ after 7100 iterations. Data from every 10 time steps was used, which gives a total of 460 time points over a simulation time of 200 seconds. An estimate of the number of eddy turnover times over which averaging has been taken can be made. The eddy turnover time was estimated by dividing a characteristic velocity scale (e.g. centreline velocity) by a characteristic length scale (e.g. the jet diameter). The integrating time period can then be divided by this estimate to estimate the number of eddy turnover times. For this jet, the result is approximately 14 eddy turnover times at the top of the region of interest. Since the timescale decreases with the square of the streamwise distance, this estimate is the minimum for the region of interest. Whether 14 eddy turnover times is sufficient for the statistics to converge will be discussed when appropriate.

### 5.3.1 Results

**Visualisations**

Figure 5.3 shows two vertical slices through the jet centreline depicting instantaneous tracer concentration (at the same time point) once similarity has been attained, the left-hand plot is the $x$-$z$ plane and the right-hand plot is the $y$-$z$ plane, which also shows the AMR grids, the highest resolution grids near the nozzle are in white and the intermediate grids are shown in blue. Figure 5.4 shows horizontal slices through planes $z = 0.32$ and $z = 0.712$, just inside the top and bottom of the region of interest. The slices compare qualitatively with similar plots in the literature, the general shapes of eddy structures are clear. Undiluted fluid is seen throughout the jet, even at the axis, which from the point of view of Dimotakis [27], suggests a reasonably low Reynolds, i.e. an order of magnitude below the $10^4$ figure proposed to be indicative of a mixing transition.
Figure 5.3: Pure Jet: Instantaneous tracer concentration through two vertical planes through the axis, high concentration is white, low concentration is black. The left-hand side is the $x$-$z$ plane. The right-hand side is the $y$-$z$ plane and also shows the AMR grids, the highest resolution grids are white and the intermediate resolution is in blue.

Figure 5.4: Pure Jet: Instantaneous tracer concentration through horizontal planes at $z = 0.32$, left, and $z = 0.712$, right.
Figure 5.5: Pure Jet: Three-dimensional rendering of the AMR boxes. Highest resolution in red (0.1%) and intermediate resolution in purple (16%).

Figure 5.6: Pure Jet: Three-dimensional rendering of an instantaneous tracer concentration field. The jet looks roughly conical and the fluid that is not passing out of the top boundary can be seen clearly.
The AMR grids are shown in a three-dimensional plot in figure 5.5. The benefit is clear here, the domain coverage for the two levels is 16% and 0.1%. To estimate the saving that the AMR provides, consider first an unrefined simulation at the effective resolution of the intermediate grids (the region of interest), which is $192 \times 192 \times 256$. According to figure 4.6, a coverage of 16% gives a saving of around 70-75%. The second AMR level near the nozzle naturally leads to additional expense. A simulation at the effective resolution of the region of interest (the intermediate AMR level) is likely to be around three times more expensive than this simulation, and would not have the benefit of the second level near the nozzle.

A three-dimensional rendering of the tracer concentration is shown in figure 5.6. The generic conical structure is apparent here. This plot also shows the first sign that the upper boundary is not completely transmissive and leads to an accumulation of fluid near to the top of the domain. The extent to which this affects the results will be discussed shortly.

**Large Scale Evolution**

Figure 5.7 shows contours of time-averaged streamwise velocity and tracer concentration through horizontal slices at three heights within the region of interest. Each slice has been normalised by the maximum value. The increasing width with increasing height is clear, but it is also noticeable that the tracer width appears to be greater than the velocity width. At the lower two heights there appears to be satisfactory level of axisymmetry. This has deteriorated by the greatest height. This highlights the difficulty of simulating such a flow, since the timescale grows with the square of streamwise distance, convergence becomes more and more difficult. This should be borne in mind when considering the statistics. Note that the computational domain extends to ±0.384, but only shown to ±0.2, and so the box geometry is not expected to influence the results.

Figure 5.8 shows a streamline plot of the average velocity field (blue), and tracer contours have been superimposed (green). The solid black horizontal lines show the coarse-fine interfaces, and the dashed black horizontal lines show the region of similarity (to be defined and justified shortly). Near the source, the streamlines begin to turn upwards well before reaching the jet. This can be due to two effects. Firstly, the jet does not break down very quickly, and so the virtual origin is above the actual nozzle. This suggests a higher intensity perturbation could be required. Secondly, if the lateral boundary conditions do not provide sufficient inflow of radial velocity then the cumulative effect along the streamline will result in the observed behaviour. There is a subregion within the region of interest where the streamlines remain almost horizontal to the jet edge, but near the top of the region of interest and above it the streamlines actually descend before reaching the jet. A region of recirculation is also identified near the top of the domain. This all suggests that the boundary conditions are not representative of ideal jet flow, but are not wholly unacceptable. The benefit of the AMR is highlighted here. The upper boundary condition is far from the region of interest, and so the effect of the recirculation that appears there is reduced, without
Figure 5.7: Pure Jet: Contours of streamwise velocity (left) and tracer (right) at three different heights. Axisymmetry deteriorates as height increases. The tracer contours appear to be wider than the velocity contours.
Figure 5.8: Pure Jet: Streamlines. The black lines denote the coarse-fine interfaces of the AMR gridding. The streamlines near the source can be seen to turn upwards well before reaching the jet. The recirculation near the top of the domain is very clear.
a significant increase in computational expense.

Figure 5.9(a) shows the inverses of average streamwise velocity and tracer concentration against streamwise position normalised to take the same value at the origin, i.e. $\bar{u}_0$ and $\bar{c}_0$. The region of interest is bounded by the vertical black lines. The expected linear growth of the two quantities is highlighted by a linear fit to the data in the region between $z = 0.2$ and $z = 0.64$. This region is henceforth referred to as the ‘similarity region’, a term that will be justified shortly. It is again noticeable that towards the top of the region of interest deviation from linearity is significant, although a small amount of deviation is present throughout. The linear fits shown in figure 5.9(a) lead to the values $k_{jw} = 4.2$ and $k_{jc} = 3.2$, in the notation of Wang and Law [91], see (2.200) for the definitions of each $k$. These are lower than the values reported by the various authors quoted there, which are $k_{jw} = 5.8-6.7$ and $k_{jc} = 4.96-5.3$. This implies that the decay is more rapid than observed experimentally.

Figure 5.9(b) shows the widths, defined as the radial point where the quantity is $e^{-1}$ of the centreline value, of streamwise velocity and tracer concentration against streamwise position. The straight-sidedness of the jet is highlighted by the linear fits to the data over the range of similarity. It is very clear that the tracer width is significantly wider than the velocity width. The gradients of the velocity and tracer widths are $\frac{db_u}{dz} = b'_u = 0.164$, and $\frac{db_c}{dz} = b'_c = 0.202$, respectively, which are slightly higher than the values in [91], which are 0.104-0.115 and 0.126-0.136. This gives a ratio of $\frac{b'_c}{b'_u} = 1.23$ for this simulation. Most of the values in [91] are approximately 1.18, but one is 1.32. This suggests that the spread rate is slightly higher than experimental values.

Figures 5.9(c,d,e) show the streamwise evolution of mass, momentum and tracer fluxes, respectively. These fluxes have been calculated by integrating the relevant profiles as prescribed by definitions (2.140) to (2.143), but accounting for the turbulent fluxes as in (2.203). A linear fit to the mass flux data in the region of similarity has been highlighted in cyan. The higher level of mass flux below the similarity region and the lower level above it can be accounted for by looking at the streamline plot, where there is respectively upward and downward flow outside the jet. There is some variation in the total momentum and tracer fluxes from the theoretically constant values expected. In each case, the mean over the similarity region has been evaluated and used to normalise the plots. This allows a comparison of the relative flux due to the mean and the turbulence to be made. The turbulent momentum and tracer fluxes account for approximately 15-20% and 10-15% of the total flux, respectively. In both cases there appears to be a slight transfer from the turbulent flux to the mean. Although the total fluxes are close to being constant throughout the region of interest, there is some variation, and there is greater variation outside the region of interest. This is likely to be due to the lateral boundary conditions.

Using the mass and momentum fluxes, an estimate of the entrainment coefficient can be made

$$\alpha_e = \frac{1}{\sqrt{8\pi M}} \frac{dQ}{dz}$$

(5.3)
Figure 5.9: Pure Jet: (a) Inverse of centreline velocity and tracer concentration as a function of streamwise position. The $z^{-1}$ dependence is highlighted by the linear fit between $z = 0.2$ and $z = 0.64$. The black lines denote the coarse-fine interfaces of the AMR gridding. (b) Velocity and tracer widths as functions of streamwise position. Linear fits highlight the straight-sided growth and the wider tracer profile. (c) Mass flux as a function of streamwise position. The linearity is highlighted by the fit in cyan. (d) Momentum flux as a function of streamwise position, normalised by the mean total flux over the similarity region to allow easy comparison of the relative turbulent flux. (e) Tracer flux as a function of streamwise position, normalised in the same way as the momentum flux. (f) Entrainment coefficient as a function of streamwise position, evaluated according to (5.3). The standard jet and plume values are denoted by the blue and red horizontal lines, respectively.
and is shown in figure 5.9(f). This estimate has been used in preference to using the jet width, as the latter presents a large amount of scatter, which is reduced using integrals across the jet width. Using the linear fit to the mass flux and 80% of the mean of the total momentum flux estimated above, the entrainment coefficient is approximately 0.07. The values proposed by Fischer [29] for jets (0.0535) and plumes (0.0833) are shown by the horizontal blue and red lines, respectively. The entrainment coefficient from the simulation is between the two, and so is higher than expected. The reasons for this will be discussed shortly. Some large scale variation is present and is most likely due to the effect of the lateral boundary conditions and the gradual increase in the mean momentum flux observed in figure 5.9(d). The small scale scatter present is a numerical error from evaluating the gradient of the mass flux. The base of each AMR grid is aligned to a multiple of eight cells, and so every eight cells the AMR grids cover a slightly wider area. This leads to a small increase in the integrated mass flux, which shows up in the numerical approximation to the gradient.

The data presented in figure 5.9 demonstrate that there is a region within the domain throughout which the simulations is statistically comparable with a turbulent jet. The velocity and scalar decay are close to the expected $z^{-1}$, although the decay seems to be slightly quicker than observed experimentally. The velocity and scalar widths both spread linearly, but again slightly more quickly than observed experimentally. In the similarity region, the fluxes of mass, momentum and tracer appear to be following the expected evolution. Outside this region, the fluxes do not present the expected behaviour, but this is likely to be due to the boundary conditions, and is not thought to be too detrimental to the similarity region. The entrainment coefficient appears to be slightly larger than the values reported in the literature, and the reasons for this will be discussed shortly.

### Averaged Profiles

Figure 5.10(a) shows the radial profiles of average streamwise velocity through each height over similarity region. At low heights the profiles are tall and narrow, and as the streamwise position increases the profiles reduce in height and spreads radially. Figure 5.10(b) shows the same profiles, but normalised by dividing by the centreline velocity and dividing the radial position by the width. The self-similar collapse is very clear, and justifies the term ‘similarity region’. A Gaussian profile is superimposed in cyan, which matches the data very well.

Similar treatment of the average radial velocity is shown in figures 5.10(c,d). The profile shown in cyan is that inferred by a Gaussian streamwise velocity profile, the spread rate ($b'_u$) derived above, and (2.186). There is close agreement again, but the inferred profile has a slightly greater magnitude at large radial positions than the numerical data. This fits with the suggestion that the lateral boundaries do not provide sufficient inflow. The azimuthal velocities (not shown) present some scatter but are close to zero, as expected.
Figure 5.10: Pure Jet: Radial profiles of the streamwise velocity (top), radial velocity (middle), and tracer (bottom). Left-hand plots are not normalised. Right-hand plots are normalised. Every fourth height in the similarity region is plotted. The profiles superimposed are as follows: (b) the cyan profile is Gaussian; (d) the profile in cyan is that inferred by a Gaussian streamwise velocity, the spread rate calculated from the gradient of the velocity width, and (2.186); (f) the yellow line is Gaussian.
Profiles of average tracer concentration are shown in figure 5.10(e), and normalised in figure 5.10(f), where the Gaussian profile, shown in yellow, is a very close match to the data.

The self-similar collapse in these profiles is clear, and justifies the term “similarity region”. This backs up the suggestion that this region, albeit somewhat smaller than the domain, is statistically comparable with a turbulent jet. Although the nozzle region is not captured with as high a resolution as desirable, the resolution is sufficient to produce a region that is similar to a turbulent jet.

**Turbulence Structure**

Figure 5.11(a) shows radial profiles of the streamwise velocity fluctuations (or turbulent intensity), normalised by the centreline streamwise velocity and the velocity width. There is a reasonable self-similar collapse, although a lack of convergence is evident here that will be true in all of the second-order quantities. Since the data has been averaged azimuthally, there are fewer data points near the axis and may partly explain the scatter in the data there. At low heights the normalised peak is greater than at greater streamwise positions. The centreline values are approximately 0.27-0.33, which are slightly higher than the values in the literature 0.24-0.27.

Figure 5.11(b) shows the radial profiles of the radial (blue) and azimuthal (yellow) velocity fluctuations, normalised in the same way. These show an extended region of growth at low heights, to a region with reasonable self-similar collapse, followed by a region with more scatter, similar to the streamwise plot above. The centreline values are again slightly higher than the values in the literature, 0.25 compared with 0.17-0.22.

Figure 5.11(c) shows the radial profiles of the normalised Reynolds stress. Although not clear from the plot, there is a region of growth, a self-similar collapse, and a region where some scatter is present. The peak value, like the turbulent fluctuations, is higher than the values in the literature, but occurs around the same radial position. The cyan line is the profile inferred from a Gaussian streamwise velocity, the measured spread rate and (2.198). The data match reasonably well, but there is an element of self-consistency - the Reynolds stress strongly affects the entrainment rate and spread rate, and the spread rate has been used to derive the fit to the data.

The viscous stress profile for a Reynolds number of 1000 has also been plotted in figure 5.11(c). This demonstrates that the Reynolds stress dominates the flow, and so the details of the viscous dissipation should not be important.

Figure 5.11(d) shows the radial profiles of the normalised tracer fluctuations. Again there is a region of initial growth at low streamwise positions, followed by self-similar region where a slight off-centre maximum develops. The centreline values vary between about 0.25 and 0.35, which is once again higher than experimental observations of approximately 0.22.
Figure 5.11: Pure Jet: (a-d) Normalised radial profiles of second-order correlations: (a) $\frac{\sqrt{\overline{u^2}^2}}{\overline{u_0}}$; (b) $\frac{\sqrt{\overline{u^2}^2}}{\overline{u_0}}$ (blue), $\frac{\sqrt{\overline{u^2}^2}}{\overline{u_0}}$ (yellow); (c) $\frac{\overline{\tau_{rz}^2}}{\overline{u_0}^2}$; The cyan profile is inferred from a Gaussian streamwise velocity profile, using the gradient of velocity width and (2.198). (d) $\overline{c_0^2}$. (e-f) Normalised radial profiles of gradients of second-order correlations: (e) streamwise; (f) radial. A more complete description is given in the text.
Figure 5.12: Pure Jet: Normalised radial profiles of an approximation to the eddy-viscosity coefficients: (a) $\nu^u_r$; (b) $\nu^c_r$. See (2.188).

These second-order correlations present a reasonable level of self-similar collapse, further justifying the term "similarity region". There is more scatter here than in the first-order averages, but that is natural for higher-order statistics. The more important point to note is that the magnitude of the correlations is higher than the values reported in the literature. This is likely to be a consequence of using the MILES approach. Although the correct phenomenological behaviour is recovered, the specific details of the turbulence are different to what is expected. It is also likely that these increased levels of turbulence are responsible for the increased spread rate and decay rate of the velocity field. The spread rate is essentially determined by the Reynolds stress, which is greater than expected. Therefore, the turbulence spreads the jet more quickly than expected, which results in a greater velocity decay so that the momentum is conserved.

To consider the validity of the boundary layer approximation, the derivatives of the second-order correlations are now considered, specifically $\frac{\partial}{\partial z} \left( q'_i q'_j \right)$ and $\frac{1}{r} \frac{\partial}{\partial r} \left( r q'_i q'_j \right)$. Figure 5.11(e) shows radial profiles of a numerical approximation to the streamwise gradient of the Reynolds stress, normalised by centreline streamwise velocity and velocity width. Five second-order correlations have been considered, $u'_z \bar{u}'_z$, $u'_r \bar{u}'_r$, $u'_\theta \bar{u}'_\theta$, $u'_r \bar{u}'_z$, and $u'_r \bar{u}'_c$. Each has been differentiated numerically using a second-order centred finite-difference approximation. There is significant scatter in the data, which is unsurprising, given that the quantity is a numerical derivative of a second-order correlation. However, the values remain small, particularly when compared with the radial gradients, as shown in figure 5.11(f). This lends support to the assumption of neglecting the streamwise gradients in the boundary layer approximation.

Figure 5.12(a) shows radial profiles of an estimate of the eddy-viscosity coefficient, defined by taking the ratio of the Reynolds stress to the radial (numerical) derivative of the average streamwise velocity, see (2.188). The scatter in the data is significant, particularly for large radial positions and close to the axis. However, since this is the ratio of a second-order correlation to a numerical derivative, both of which are tending to zero at large radial values and at the axis, it is remarkable that any kind of
collapse is recovered at all. Similar to estimates made for the temporally evolving circular shear layer in section 4.3, since the Reynolds stress and the streamwise velocity present self-similar collapse, then it not surprising that a profile derived from these two quantities is also self-similar. However, the agreement with the theoretical profile (shown in cyan) is important, and is strong support for using the Boussinesq eddy-viscosity assumption for modelling this kind of flow, but it does imply that a radially dependent coefficient is required.

A similar picture emerges when estimating the eddy-viscosity coefficient for the velocity-tracer correlation, as shown in figure 5.12(b). Again, considering the volatility of the quantity being plotted, the self-similar collapse is very good, and the Gaussian-inferred profile (superimposed in yellow) matches reasonably well.

**Energy spectra**

Figure 5.13 shows plots of average wavenumber spectra of velocities and tracer concentration at various horizontal slices, normalised by total energy for each spectrum. These are derived following Catrakis and Dimotakis [21], whereby the wavenumber spectrum has been evaluated on horizontal planes at each point in time, and the ensemble average of these spectra has been evaluated to obtain the average spectrum at each height. The semilogarithmic plots (right) show clear dissipation ranges at the small scales, highlighted by the straight black lines. Similar to chapter 3, assuming the exponential decay from Saddoughi and Veeravalli [74], \( \exp(-5.2\kappa\eta) \), the approximations shown have effective Kolmogorov length scales of around \( 4d_r \), \( 3d_r \), and \( 2.5d_r \) for streamwise velocity, radial velocity and tracer respectively (note \( d_r = 0.004 \), the same as \( d_z \) etc.). The differences in the effective Kolmogorov length scales, not only from each other but also from the three cell widths established in homogeneous isotropic turbulence in chapter 3, are likely to be due to the anisotropy of the flow, or may be due to normalising by the local kinetic energy rather than the dissipation rate. The logarithmic plots (left) show the differences at the large scales. The streamwise velocity shows a growth at the large scales with increasing streamwise position. Each plot has a large scale peak, which moves to larger scales with height, and is accompanied by a power-law decay that extends to the small scales. The decay of this power-law appears to be shallower than the minus five-thirds shown by the black line, similar to those found in chapter 3. The transverse velocities show no sign of a power-law decay whatsoever, and are in fact closer to the dissipative exponential decay. The tracer spectra are very similar to the streamwise velocity with a peak at small wavenumbers, followed by a power-law decay shallower than minus five-thirds, before the dissipation range at small scales. The curved black line in each plot is the exponential decay corresponding to the straight line shown in the semilogarithmic plots. Only the logarithmic plot of scalar concentration (\( e \)) is present in Catrakis and Dimotakis [21], but the two plots are consistent.
Figure 5.13: Pure Jet: Energy wavenumber spectra of flow variables normalised by total energy and effective Kolmogorov length scale: (a,b) streamwise velocity; (c,d) transverse velocities; (e,f) tracer concentration. Plots (a,c,e) are logarithmic, where the straight black line has gradient minus five-thirds, and the curved line is exponential. Plots (b,d,f) are semilogarithmic, straight lines show exponential decay.
The results presented above compare very well with experimental observations of a pure jet. Above a region where the flow establishes itself, there appears to be a region of similarity. In this region the jet has straight sides, i.e. linear growth is observed in both the velocity and tracer widths. The tracer has a wider profile than the velocity with a constant ratio close to that reported in the literature. The velocity decay and tracer dilution rates are close to the theoretical power-law, and coefficients slightly above those in the literature. The streamlines suggest that the boundary conditions are not representative of ideal jet flow, but are not wholly unacceptable. The velocity and tracer concentration profiles collapse when normalised by the centreline values and widths. The streamwise velocity and tracer concentration have profiles that are near Gaussian in nature, and therefore similar to profiles found in experimental observations. Similarly, each of the radial velocity profiles presents a self-similar collapse and has form comparable with experimental observations and theoretical prediction based on a Gaussian streamwise velocity profile. Self-similar collapse is also observed in the second-order correlations, although the intensity is higher than observed experimentally. This may account for the level of entrainment observed, which is also higher than expected. Comparison of radial and streamwise gradients of the second-order correlations shows that the streamwise gradients are small when compared with the radial gradients, which lends support to the boundary layer approximation. Approximations of an eddy-viscosity coefficient suggest that Boussinesq’s equation can be used, provided a radially dependent coefficient is used, and agree with the theoretical profile inferred from a Gaussian streamwise velocity.

Overall, it seems fair to conclude that the configuration of AMR grids and boundary conditions seems to be adequate, and this numerical code with the inherent implicit turbulence model can reproduce the large scale phenomena found in a jet, and turbulent statistics that are close to experimental observations, although there is disagreement in the specific intensity of the turbulence.
This section reports the results from the pure plume simulation. The configuration of the domain is exactly the same as for the jet in the previous section. Fluid was injected with a temperature 1K above the ambient, with a velocity of 0.018ms$^{-1}$. The velocity was perturbed in exactly the same way as the pure jet, the temperature was not perturbed.

Averaging was taken between iterations 2500 and 6380, corresponding to times 507s and 1300s, giving 388 time points spanning 793s. The number of eddy turnover times was approximately 31, many more than in the jet case.

5.4.1 Results

The visual plots of the jet that were presented in the last section, figures 5.3-5.6, are qualitatively similar to those that result in this plume case and are not shown. The only significant differences are that a lower threshold is required for the plume to be visible, and the plume occupies a smaller volume - the AMR grids at the intermediate level occupy approximately 12% of the domain, compared with 16% in the jet case. This suggests a higher rate of dilution than the jet, and a narrower spreading angle, both of which are expected results, see section 2.6.

Large Scale Evolution

Figure 5.14 shows the contours of average streamwise velocity and tracer at the greatest height presented in the corresponding jet plot (figure 5.7). Note that in this simulation the tracer and temperature fields
Figure 5.15: Pure Plume: Streamlines. The recirculation near the top of the domain is much less pronounced here than in the jet due to the inherent buoyancy of a plume.
are identical. It would appear that the axisymmetry is better preserved than in the jet case, as would be expected since more eddy turnover times have occurred. In the jet case, the scalar contours were visibly wider than the velocity contours. This not the case here, there is little visible difference between the extent of the velocity and tracer contours.

Figure 5.15 shows the streamlines and tracer contours, in the same way as for the jet (figure 5.8). As with the jet, the streamlines near the source turn upwards well before reaching the jet. This behaviour is reproduced all through the region of interest, albeit to a lesser extent at greater heights. The recirculation at the top of the domain is much smaller here than in the jet. Buoyancy is the major difference. Fluid that does not pass out of the domain due to its momentum alone is still buoyant, and so there is a continuing generation of upward motion and it will eventually leave through the upper boundary. In the jet case, this fluid has its upward motion destroyed by the upper boundary and, without buoyancy, lingers there, and is gradually pushed sideways by similar fluid arriving behind it.

Figure 5.16(a) shows the centreline evolution of the average streamwise velocity, tracer concentration and temperature. These have been scaled so that a theoretical self-similar region appears to be linear, specifically $\bar{u}^{-3}$, $\bar{c}^{-5}$, and $\bar{T}^{-\frac{5}{3}}$ have been plotted. This choice is preferred to a logarithmic plot as it eliminates the distorting effect of a virtual origin. It is particularly clear here that the tracer and temperature are identical. Linear fits to the data are shown between $z = 0.2$ and $z = 0.64$. The scalar fit is in good agreement with the data, but there appears to be some variation in the velocity. It is possible that the averaging at the centreline is inadequate. Above the similarity region, all quantities deviate from this linearity. Since the number of eddy turnover times over which averaging has been performed is greater than that in the jet, it would be expected that this deviation is not observed. However, it may be due to the upper boundary condition. The behaviour in the similarity region is consistent with the evolution of a plume.

Figure 5.16(b) shows the velocity and temperature widths, along with linear fits to the data over the range of similarity. There is no obvious trend that differentiates between any of the widths. No further insight can be inferred from the ratio of the fits to the data - taking the ratio of the gradients of the linear fits to the temperature and velocity widths gives a value of 1.01.

Figures 5.16(c,d,e) show streamwise evolution of mass, momentum, buoyancy and tracer fluxes, respectively. The mass, momentum and tracer fluxes appear to be reasonably close to the expected results. The power-law growth of the mass flux, as determined by a linear fit, is 1.33, which is smaller than the expected $\frac{5}{3}$. Better agreement is found with the momentum flux, which is 1.30, close to the expected $\frac{4}{3}$.

Figure 5.16(f) plots the entrainment coefficient, evaluated as in (5.3). Once again, a higher value than expected is obtained, about 0.095 as opposed to 0.085 as proposed by Fischer [29]. This is consistent with the jet value obtained in the previous section - the entrainment coefficient has been measured to be higher in the plume than in the jet.
Figure 5.16: Pure Plume: (a) Centreline evolution of streamwise velocity, tracer concentration and temperature. These have been scaled to appear linear for theoretical plume evolution, specifically $\bar{u}_0^{-3}$, $\bar{c}_0^{-1}$, and $\bar{T}_0^{-\frac{3}{2}}$. Fits to the data are shown between $z = 0.2$ and $z = 0.56$. Again, the vertical black lines denote the coarse-fine interfaces of the AMR gridding. (b) Velocity and tracer widths as functions of streamwise position. Linear fits highlight the straight-sided growth. (c) Mass flux as a function of streamwise position. The cyan line is a linear fit. (d) Momentum flux as a function of streamwise position. The cyan line is a linear fit. (e) Buoyancy flux as a function of streamwise position. (f) Entrainment coefficient. The standard jet and plume values are denoted by the blue and red horizontal lines, respectively.
Figure 5.17: Pure Plume: Normalised radial profiles of flow variables at every fourth height in the similarity region: (a) streamwise velocity (the cyan line is Gaussian); (b) radial velocity (the cyan line is the profile for a jet inferred from the spread rate, and the cyan-red dashed line is the plume profile inferred from the spread rate and buoyancy); (c) temperature (the magenta line is Gaussian).

The data presented in figure 5.16 demonstrate that in the similarity region, the behaviour is statistically comparable with a plume. The decay of the streamwise velocity and scalar quantities is close to the theoretical power-law, and straight sides are recovered, as expected. As with the jet, the entrainment coefficient is measured to be higher than expected.

**Averaged Profiles**

Figure 5.17(a) shows the radial profiles of averaged streamwise velocity normalised by the centreline value and velocity width. As with the jet, the Gaussian profile, shown in cyan, is a good match. The normalised average radial velocity profiles are shown in figure 5.17(b). The dashed red and cyan line shows the radial velocity profile inferred from a Gaussian streamwise profile and (2.186), using the value for $b'_u$ from above, and $\frac{\langle b_u \bar{u}_r \rangle'}{\bar{u}_r}$ as the average over the same range of heights. The fit to the data is good for $\eta < 2$, but, as with the jet, has a greater magnitude at large radial positions. For comparison, the corresponding profile for a jet with the same spread rate is shown by the solid cyan line, i.e. the radial
velocity profile inferred from a Gaussian streamwise velocity using only the term involving $b'_u$ (evaluated for this plume) and not the term involving $\frac{(b_u - \overline{u}_z)'}{\overline{u}_z}$. Figure 5.17(c) shows the radial profiles of normalised temperature. The data match well with the Gaussian profile.

As with the jet, these quantities present a self-similar collapse. The data support a Gaussian distribution in both the streamwise velocity and scalar quantities. The radial velocity profile is consistent with the theoretical profile derived for a plume assuming a Gaussian streamwise velocity, and not with the jet profile, as expected.

**Turbulence Structure**

Figure 5.18(a) shows the fluctuation profiles for averaged streamwise velocity. The plume fluctuations appear to have axial values comparable with the jet, but also have a hint of an off-centre peak. There is a greater range of axial values in the literature than for jet flows. Notably, Shabbir and George [78] report a centreline value of 0.32, but this is higher than other results. The fluctuation profiles for averaged radial velocity are shown in figure 5.18(b). The axial values appear to be slightly lower in the plume than in the jet. This is consistent with experimental observations, but the values here are still higher than in the literature, which is also consistent with the jet results from the previous section and with the higher than expected value for the entrainment coefficient.

Figure 5.18(c) shows the radial profiles of the averaged Reynolds stress. The theoretical profile, derived from Gaussian velocity field, is shown in cyan, with the corresponding jet profile in yellow. The data fits the theoretical profile very well, but, as with the jet Reynolds stress, there is a degree of self-consistency in this agreement; plume data are used to predict the profile that the same plume should take.

Fluctuation profiles for temperature are shown in figure 5.18(d). This clearly highlights the much greater magnitude of the scalar fluctuations in a plume than in a jet, and are once again higher than experimental values. The axial values are approximately 0.5 compared with 0.35 in the jet simulation, and experimental values of 0.38 to 0.42.

The velocity-temperature correlations are shown in figure 5.18(e), $\tau^R_{rz}$ in red, $\tau^R_{rT}$ in blue. These terms feature in the transport equations for $\overline{u}_z^2$ and $\overline{u}_zT$, (2.122) and (2.123) respectively, and so will influence the profiles in figures 5.18(a,c). In particular, the radial term will contribute to the Reynolds stress $\tau^R_{rz}$, which plays a significant role in determining the spread rate and entrainment.

The numerical approximations to the radial and streamwise derivatives of the average Reynolds stress are similar to the jet profiles, and are not shown. As with the jet results, these give support for the use of the boundary layer approximation.

Figures 5.19(c,d) show the eddy-viscosity coefficient estimates for velocity and temperature, respectively.
Figure 5.18: Pure Plume: Normalised radial profiles of second-order correlations: (a) $\frac{\sqrt{\overline{u_z'^2}}}{\bar{u}_0}$; (b) $\frac{\sqrt{\overline{w_z'^2}}}{\bar{u}_0}$ (blue), $\frac{\sqrt{\overline{w_z'^2}}}{\bar{u}_0}$ (yellow); (c) $\frac{\sqrt{T'}^2}{\bar{T}_0}$; (d) $\frac{\tau_R}{\bar{u}_0}$, the cyan line is the plume Reynolds stress derived from a Gaussian streamwise velocity field and (2.199), and for comparison, the corresponding jet Reynolds stress is shown in yellow; (e) $\frac{\tau_R}{\bar{u}_0\bar{T}_0}$ (red), $\frac{\tau_R}{\bar{u}_0\bar{T}_0}$ (blue).
Figure 5.19: Pure Plume: Normalised radial profiles of an approximation to the eddy-viscosity coefficients: (a) $\nu_T^u$; (b) $\nu_T^T$. See (2.188).

Figure 5.20: Pure Plume: Energy wavenumber spectra of flow variables normalised by total energy. (a) Streamwise velocity. (b) Temperature.

The collapse is once again very good, given the quantity in question. The negative viscosity results from a tiny undershoot in the Reynolds stress that is exaggerated in this quantity. The profiles inferred from a Gaussian streamwise velocity are shown in all three plots. The derivation uses the quantities $b'_u$ and $b''_u$, which have been evaluated as the mean over the similarity region. The jet profile is shown in yellow and the plume profile in cyan. In the temperature plot, the inclusion of the buoyancy effects improves the match with the data. The inclusion of the buoyancy term makes little difference to the velocity profile. In both plots, the eddy viscosity appears to be wider than the theoretical profile, and decays more rapidly. It would appear that approximations that have been made deriving these profiles are beginning to lead to mismatches in the data, and warrants future study.
Energy Spectra

Figure 5.20 shows the energy wavenumber spectra for streamwise velocity and temperature (radial velocity is omitted because there is little, if any, difference between these and the jet spectra). All of the spectra present the same dissipation ranges as in the jet. The streamwise velocity spectra, again, have a peak at the large scales and a power-law decay exponent that is slightly shallower than five-thirds. The scalar spectra appear to show a power-law decay with an exponent closer to five-thirds.

5.4.2 Summary

The results above compare well with experimental observations and theoretical descriptions of a turbulent plume. As with the jet, the arrangement of the AMR grids and boundaries has allowed for a subregion to be obtained within which there is a self-similar collapse of both averaged flow variables and second-order correlations. The results suggest that two main improvements may lead to results more consistent with those expected. Firstly, the velocity perturbation to the inflow seems so be too small, and secondly, the lateral boundaries may be restricting the inflow, which could be reduced by using the AMRto move them even further away.

The introduction of buoyancy has resulted in the expected change of behaviour, notable agreement was found in the profiles of radial velocity and Reynolds stress with the theoretical profiles inferred from a Gaussian streamwise velocity. The eddy viscosity estimates suggest that the model used to derive the theoretical profiles requires the inclusion of further terms to predict the functional form correctly.
5.5 Buoyant Jet

In this section results are presented for the mixed case of a buoyant jet. The inflow velocity and temperature were specified as $0.03 \text{ms}^{-1}$ and 1K, which gives a value for Morton’s balance parameter as $\Gamma = 0.28$. Averaging was taken over 290 time points spanning 493s. The number of eddy turnover times was approximately 28.

5.5.1 Results

The results from this simulation were found to be very similar to those obtained in the previous two sections for the jet and plume. The aim of this simulation was to capture the jet-plume transition. Therefore, only the data relevant to this transition are presented.

The visualisations, streamlines, and width evolution are similar to the plume plots, and are not shown.

Figure 5.21(a) shows a log-log plot of centreline averaged streamwise velocity decay. The dashed cyan line is proportional to $z^{-1}$, and the solid cyan line to $z^{-\frac{5}{3}}$. For pure jet-like and plume-like evolution, the decay would follow the dashed and solid cyan lines, respectively. The decay presented here clearly shows a transition from jet-like behaviour to plume-like behaviour, although the transition appears to occur at the lower end of the region of interest.

Figure 5.21(b) shows the centreline evolution of averaged tracer concentration and temperature. In this simulation, these two quantities are not identical because the temperature was evolved using the non-conservative update, as described in section 5.2. It is clear that this leads to a slightly greater decay rate in the centreline temperature than in the tracer concentration. Jet-like and plume-like decay are shown by dashed and solid cyan lines, respectively. The decay appears to be closest to the plume-like decay, but is not convincing, and is certainly not jet-like, even at low heights.

Figures 5.21(c,d) show the fluxes of mass and momentum. The mass flux appears to be growing more slowly than the jet-like linear dashed cyan plot, but the growth increases to greater than this, but does not reach the plume-like growth highlighted by the solid cyan line. The momentum flux also grows more slowly than the plume rate (a linear fit has growth of approximately 1.47 instead of the expected five-thirds), and is certainly not constant as would be the case in a jet.

The entrainment coefficient has been evaluated according to (5.3), as usual. The evolution is close to that observed for the plume, and is not shown. A value of approximately 0.09 is observed. This is below that observed in the plume (0.095), but still higher than the value in [29] (0.0833). The value is approximately constant throughout the domain of interest, and does not show a growth from a jet-like value to a plume-like value, as found in Wang and Law [91].
Figure 5.21: Buoyant Jet: (a) Logarithmic plot of centreline velocity decay. The cyan lines are the behaviour expected in a jet (dashed) and plume (solid), respectively. (b) Centreline decay of tracer concentration and temperature. Cyan lines same as in (a). (c) Mass flux as a function of streamwise position. Cyan lines same as in (a). (d) Momentum flux as a function of streamwise position. Solid cyan line is plume-like, jet-like would be constant. (e) Radial profiles of streamwise velocity at each height in the self-similar region, normalised by the centreline streamwise velocity and the velocity width. The blue lines are the profiles in the plume region, the green lines are the profiles below the plume region. The cyan lines are the jet (dashed) and plume (solid) profiles, respectively, inferred from a Gaussian streamwise velocity.
The normalised radial profiles of streamwise velocities, tracer concentration and temperature all remain close to Gaussian throughout the region of interest. The radial velocity profiles are shown in figure 5.21. Profiles between $z = 0.128$ and $z = 0.256$ are shown by dashed cyan lines, and between $z = 0.256$ and $z = 0.640$ profiles are shown by solid blue lines. In the higher of these two regions, the profiles display a self-similar collapse and are close to the theoretical plume-like profile, and are definitely not jet-like. However, below this region the profiles are between the jet-like profile and the plume-like profile. Again, a jet-plume transition is present, but is at the lower end of the region of interest.

The eddy-viscosity coefficients and energy spectra are very similar to the plume plots and are not shown.

5.5.2 Summary

The results presented in this section give a mixed account of a jet-plume transition. The centreline velocity decay and the mass flux appear to indicate such a transition occurs, but this is not reinforced by the scalar decay rates and momentum flux, which suggest that the evolution is closer to being plume-like.

The inflow velocity is clearly too small for the momentum to dominate the buoyancy. An increase in the inflow velocity will allow a greater jet region to be observed, but there is insufficient streamwise extent to then attain a fully-developed plume state. If a jet-plume transition was to be captured completely a greater streamwise extent is likely to be required.
This section compares the effects of the (non)conservative update. Figure 5.22 shows how the streamwise evolution is affected. In each figure in this section, the left-hand plot is the non-conservative update, and the right-hand plot is the conservative update.

The effect of using the non-conservative update is that there is an excessive decay in temperature, i.e. a drop in buoyancy flux. However, it can be seen in figure 5.22(a) that the centreline temperature still decays with a power law close to the theoretical value of minus five-thirds, the coefficient is simply different. The drop in buoyancy flux is clear in figure 5.22(b). Although there is a difference in the streamwise evolution, the normalisation process reveals that the structure is very close. Figure 5.23 compares the radial profiles of radial velocity, temperature and Reynolds stresses. This is a cross-section of the data, similar agreement is found in similar plots of the other quantities.

These plots, along with a similar comparison made with one of the heated jets in the next chapter, demonstrate that although there are differences in the streamwise evolution, the jet structure is not different. The same phenomenology was present in both heated jet cases.

Figure 5.22: Comparison of streamwise evolution in a plume with the non-conservative update (left) and conservative update (right). (a,b) Inverse centreline evolution. (c,d) Buoyancy flux.
Figure 5.23: Comparison of normalised radial profiles in a plume with the non-conservative update (left) and conservative update (right). (a,b) Radial velocity. (c,d) Temperature. (e,f) Reynolds stress.
5.7 Discussion and Conclusions

An unequivocal conclusion that can be drawn from the investigation that has been conducted is that the three-dimensional simulation of spatially evolving turbulent jets and plumes is an extremely difficult problem. There is a huge range in time and length scales, which are compounded by numerous problems with configuration and boundary conditions that have to be overcome.

Challenges

The ratio of tank to nozzle sizes demands high resolutions, yet the time scale required for statistics to converge denies access to these resolutions. A decision had to be made whether to resolve the transitional region near to the nozzle, or to sacrifice resolution there and concentrate on the far field. Neither choice escapes from restricting the domain of interest, which introduces problems with boundary conditions. Adaptive mesh refinement gave massive benefits for overcoming these issues, however, it was not the solution to all of the problems. Even with a significant buffer region, the lateral and upper boundary conditions were far from perfect, and having made the decision to concentrate on the far field, the inflow left much room for improvement. Nevertheless, the results compare well with the literature, and the basic properties of jets and plumes were recovered.

Results

The pure jet has straight sides, and the streamwise velocity and tracer concentration decay close to the power-law expected. The ratio of the scalar width to velocity width is comparable with the values in the literature, an apparent fundamental feature of turbulent jets that is yet to be explained. Self-similarity is recovered in both the flow variables and the second-order correlations. The streamwise velocity and tracer concentration, like experimental observations, are close to Gaussian, and the radial velocity fits well with the profile inferred therefrom. The mass, momentum and tracer fluxes are close to those expected, but the rate of entrainment appears to be somewhat high, which, it could be argued, provides the strongest argument against using an implicit method for turbulent mixing, one to be discussed in more detail later. Numerical approximations to the streamwise and radial derivatives of second-order correlations provide some support for the well-established boundary layer approximation, but it must not be forgotten that the turbulent transport is significant, particularly in non-canonical situations. An estimate of the eddy-viscosity presents a remarkable collapse to self-similarity, and also agrees well with the profiles inferred from a Gaussian streamwise velocity.

Despite the problems balancing the inflow, the pure plume presents reasonable statistics. Again, it is straight-sided with power-law decays close to the expected values. Once more, the entrainment rate is higher than observed experimentally, but similarity is observed in the velocities, scalar quantities and
second-order correlations. The behaviour relative to the jet corresponds well with the literature. First, the plume rate of entrainment is higher than the jet, as is the Reynolds stress. Also, the streamwise turbulent intensity is approximately the same in each case, the transverse fluctuations are marginally smaller, and finally, the scalar fluctuations are significantly higher in the plume than in the jet. Support is also found here for the boundary layer approximation and Boussinesq's eddy-viscosity equation.

The buoyant jet results present contradictory support for a jet-plume transition. The streamwise velocity decay and the profiles of the radial velocity suggest that a jet-plume transition takes place at the lower end of the region of interest, but the scalar quantities do not present jet-like behaviour. Improvements to the inflow will allow for a greater region where the flow is jet-like, but this is likely to prevent the plume state being attained.

**Assessment of the simulations**

The results suggest that the numerical configuration is not quite capable of recovering ideal jet flow. The inflow was slow to break down, giving rise to a virtual origin well within the domain. Although this is not desirable, the effect is not critical, and can be rectified by an increased amplitude of perturbation. The streamlines appear to turn upwards in the nozzle region before becoming entrained into the jet, but this becomes less pronounced once within the region of interest. The upper boundary proved to be a major problem. The issue of coherent structures crossing the transmissive boundary highlighted in section 5.2 was not completely solved by the adjustment made. Despite restricting the inflow, in the worst case, a recirculation region formed in the unheated jet. The use of AMR reduced the impact on the region of interest and so does not appear to significantly affect the region of interest. It may be beneficial to move this boundary even further from the region of interest, as the increase in computational expense is small. Although the streamwise velocity was close to Gaussian in nearly all of the simulations, the radial velocity was smaller in magnitude than the corresponding profile for large radial positions. This suggests that the inflow from the lateral boundary condition is insufficient. It is not clear how to remedy this problem. Artificially injecting more fluid will undoubtedly lead to undesired effects, and the exact implementation is far from clear. Using the AMR once again may prove to be beneficial, but will have to be tested.

The configuration of AMR used in the experiments, despite the problems highlighted above, can be assessed as adequate. The use of the coarse resolution reduces the effect of the boundary conditions. The high resolution grid near the nozzle essentially doubles the streamwise extent that can be captured. The culmination of the arrangement is a subdomain throughout which free shear flow can evolve.

The resolution used in this study is possibly the minimum that will lead to meaningful results for spatially evolving turbulent jets, but was the maximum that could be attained with the computational resources available. Any lower, and resolution at the nozzle would be lost, or the spatial extent, both streamwise
and laterally, of the self-similar region would be severely restricted.

From a MILES point of view, interesting questions concern the differences with other approaches. If a conventional LES approach were taken, the first question has to be whether a lower resolution than the one used here would be sufficient to resolve the filtered equations. If the answer is no, then the added burden of subgrid models and/or transport equations would demand significantly more computational power. If the answer is yes, then the range of scales that can be captured are likely to suffer. At half the resolution the nozzle would only have four cells across it, which is already questionable at eight. If half the resolution were used and the diameter doubled, the domain would become compromised in spatial extent. All before details of the closure model would have to be addressed, which are not obvious a priori, and may adversely affect the behaviour of the flow. A RANS approach is the primary alternative for this kind of flow, but will have the same trouble with subgrid model.

All approaches have their drawbacks, and ultimately, the only way to be able to claim to have solved the equations under the given conditions is to perform a full DNS calculation, and to be able to show consistent results upon refinement. After all debates about the available methods, the overriding determining factor is the available resources, and the ultimate issue the quality of the results obtained using them. The approach taken here has allowed access to a resolution that appears to be capable of recovering a flow that closely approximates free shear. At this resolution, it is not likely to provide results that can be compared in excruciating detail with laboratory experiments, but has yielded phenomenologically consistent results.
Chapter 6

Jets with Volumetric Heating
6.1 Introduction

In this section, the response due to volumetric heating is considered. The equations that were used include the volumetric heating term that depends linearly on the tracer concentration. The equations of motion that are solved are the inviscid form of (2.12)-(2.16). The same configuration of nozzle size, domain size and AMR were used throughout the heated study as in the unheated jets and plumes presented in the previous chapter.

The heat injection zone (hiz) was placed between the streamwise stations \(z_b = 0.32\) and \(z_t = 0.448\), i.e. 20 and 28 nozzle diameters downstream from the nozzle. The inflow velocity was the same as the unheated jet (\(\bar{u}_z = 0.128 \text{ms}^{-1}\)). By varying the coefficient in the temperature source term, different heating rates were achieved. Using local quantities to define the Richardson number means that it can only be determined in post-processing, by evaluating the integral (2.215). Three different heating rates were used, \(\alpha_c = 0.5, 2\) and \(8\), which were determined to correspond to Richardson numbers \(Ri = 0.2, 0.56\) and \(1.09\) (note the suffix * has been dropped for convenience). These are within the range of the experimental data in the literature. This will be referred to as type I heating. A fourth experiment was conducted where the hiz was allowed to persist to the top of the domain, so heat was injected for all \(z > z_b\). The heating rate was the same as the \(Ri = 0.56\) simulation. This will be referred to as type II heating.

In the laboratory experiments and the simulations of ABP, the heating was only turned on once similarity had been attained. This leads to a transitional stage between an unheated jet and a heated jet. The heated jet fluid catches up with the cold fluid that has gone before it and is entraining fluid that is not ambient, which can posses both momentum and tracer. It also involves running the calculation for an extremely long time to attain similarity a second time, particularly if the aforementioned effects are to be minimised. Therefore, in the following experiments heating was applied from the beginning of the calculation. Of course, when there is no tracer present, no heat is added.

Each case was averaged over around 300 time points spanning times of around 200s. The number of eddy turnover times were between 16 for the lowest heating rate up to 27 for the highest.

6.2 Results

Visualisations

Figure 6.1 shows instantaneous vertical slices through each heated case and through the unheated jet. In each of the heated case, there is a visible thinning of the jet, and this thinning is more pronounced with higher heating rates. This thinning is reflected in the contour plots shown in figure 6.2, \(Ri = 0.56\). In the
Figure 6.1: Heated Jets: Instantaneous plots of tracer concentration through vertical slices through the axis of the different heated jets. Left to right are pure jet, low heating ($R_i = 0.20$), intermediate heating ($R_i = 0.56$), high heating ($R_i = 1.09$), and heated to the top of the domain.
Figure 6.2: Heated Jets: Contours of streamwise velocity (left) and tracer (right) at three different heights (Ri = 0.56).
Figure 6.3: Heated Jets: Streamlines. Top to bottom are: Type ii; Type i, Ri = 1.09; Type i, Ri = 0.56; Type i, Ri = 0.2. In each plot the cold jet streamlines are shown in blue, tracer concentration contours in green. The horizontal black lines denote coarse-fine interfaces, and horizontal red lines denote the hiz.
pre-HIZ the tracer contours still appear to be wider than the velocity contours, but in the post-HIZ not only do the contours appear to be narrower, it is also noticeable that axisymmetry is better preserved that in the cold jet, even at the greatest height compare the post-HIZ plots with figure 5.7. This is due to a quicker time scale brought about by the arrest of the velocity decay due to heating, so the number of eddy turnover times over which averaging has been taken is greater.

**Large Scale Evolution**

Figure 6.3 compares the streamlines in each heated case with those of the pure jet. As with the other buoyant cases, the recirculation at the top of the domain is reduced, and at the higher heating rates is almost non-existent. In each case, the deviation from horizontal happens at larger radial positions with heating, below the HIZ, and is more pronounced at higher heating rates. This suggests that the heating produces a global response, rather than a response that is restricted to the HIZ and above. The contours of tracer concentration, particularly at Ri = 1.09, show the visual thinning upon entering the HIZ and a return to growth above.

The arrest in (centreline) velocity decay is shown in figure 6.4(a) for Ri = 0.56 (type i), where the centreline evolution of scalar concentration and temperature are also shown. The inverse of each is plotted to highlight the jet-like decay of velocity and tracer in the pre-HIZ region. The temperature curve falls from infinity because $T = 0$ in the pre-HIZ. Once within the HIZ the centreline temperature is seen to increase to a maximum at the end of the HIZ, above which it decays. The centreline velocity begins to decay more slowly within the HIZ, and at approximately $z = 0.4$ the decay stops completely and an acceleration follows, this acceleration then slows post-HIZ and the velocity becomes approximately constant. The decay of centreline tracer concentration, however, appears to increase within the HIZ, and continues to do so post-HIZ.

Figure 6.4(b) compares the centreline velocity evolution at the different heating rates, note that the streamwise extent has been restricted. It is clear that the effect on velocity is more pronounced at higher heating rates. At the lowest heating rate, the decay slows more gradually and becomes approximately constant once well into the post-HIZ, whereas at the highest heating rate the decay stops very rapidly and begins to accelerate strongly whilst still within the HIZ. With type II heating, the initial response is close to that at the same heating rate, but continues to accelerate as the heating persists (there is no post-HIZ).

The decay of centreline tracer concentration at each heating rate is shown in figure 6.4(c). Heating appears to slightly increase the rate of dilution of the tracer, increasingly so with Richardson number.

Figure 6.4(d) shows the evolution of the centreline temperature at each heating rate. In each of the type I cases, the temperature reaches a maximum at the end of the HIZ. In the type II case, a maximum is
Figure 6.4: Heated Jets: (a) Inverse of centreline velocity, tracer concentration and temperature as functions of streamwise position (Type I, $R_i = 0.56$). The red vertical lines denote the HIZ. (b) Comparison of centreline velocity at different heating rates. (c) Comparison of centreline tracer concentration at different heating rates. (d) Comparison of centreline temperature at different heating rates.

only reached at $z = 0.57$. In all cases, the maximum is followed by a region of decay.

The evolution of the widths of the velocity, tracer and temperature profiles at $R_i = 0.56$ (type I) are shown in figure 6.5(a). The difference in behaviour between velocity and tracer widths is striking. The tracer width hints at a bulge just before the HIZ, and the thinning visible in the instantaneous plots is reflected by a region of constant width that extends over a significant distance to outside the HIZ. After this constant region, there appears to be a return to straight-sidedness, albeit with a shallower gradient than in the pre-HIZ. In contrast, the velocity width increases over the HIZ, and also appears to show nearly linear growth in the post-HIZ, again at a slower rate than in the pre-HIZ. For reasons to be explained shortly, the temperature width is significantly wider in the HIZ, but soon decreases and becomes narrower than the velocity width and wider than the tracer width. The temperature width is, of course, undefined in the pre-HIZ.

A comparison of the velocity widths at different heating rates is made in figure 6.5(b). In each case, the
width increases at first at a rate higher than in the pre-HIZ, and then at a slower rate. At the lowest heating rate, the response is very slight, and is not significantly different from the unheated case. As the heating rate increases, the response becomes much more significant. At the highest rate, the velocity width becomes almost constant after the half-way point of the HIZ, and then shows a post-HIZ increase to a rate of growth slightly slower than the pre-HIZ. The type II case and type I case at Ri = 0.56 are not significantly different in behaviour.

Significantly contrasting behaviour is present in a comparison of the tracer widths at the different heating rates, figure 6.5(c). As the heating rate increases, the tracer width shows a greater reduction in spread rate, and at the highest heating rate even shows a decrease. Note that the velocity and tracer widths for the type II heating present different behaviour relative to the type I case at the same heating rate. The comparison of temperature widths, shown in figure 6.5(d), displays a similar trend to the tracer width; as the heating rate increases a greater decrease in width is observed.

The above data show that heating brings about an arrest in the velocity decay, and can lead to an
acceleration if the heating is strong enough. Heating also leads to enhanced rates of dilution of tracer concentration. Importantly, there is contrasting behaviour in the evolution of velocity and tracer widths - heating leads to a slight increase in the velocity widths at first, followed by a decrease at greater heights, whereas a significant thinning is present in the tracer width. All of these results are consistent with those reported in BN1 and AP.

The behaviour observed in the centreline velocity and temperature evolution is a natural consequence of heating. The increased rates of tracer dilution and contrasting width behaviour cannot be explained simply in terms of heat injection, but depend on the structure of the flow.

Averaged Profiles

Figure 6.6(a) shows the evolution of streamwise velocity profiles at Ri = 0.56 (Type I) between heights z = 0.2 and z = 0.64. The pre-HIZ (z = 0.2-0.32) is shown in blue, the HIZ (z = 0.32-0.448) in red, and the post-HIZ (z = 0.448-0.64) in magenta. The decay in pre-HIZ velocity can be seen, followed by the increase and widening during the HIZ and afterwards. When normalised by the centreline velocity and the velocity width, the profiles still appear to be close to Gaussian, figure 6.6(b), but there is a region where the velocity is slightly lower than the Gaussian profile at the edge of the jet, in the post-HIZ. However, at the lower Richardson number, 0.2, the normalised profiles show a more pronounced deviation from Gaussian, figure 6.6(c). Near the axis, the profiles in the post-HIZ appear to have a flat-top (as it was termed in ASP), and even a hint of an off-centre maximum, although this is small. At the highest heating rate, and in the type II case, the profiles are comparable with the type I Ri = 0.56 case, and do not resemble a flat-top profile.

A significant change in behaviour is detected in the radial velocity profiles, figure 6.6(d) (Type I, Ri = 0.56). Again, the unnormalised plots show the widening of the profiles. Moreover, the heating first brings about a decrease in the outward velocity near the axis, which eventually becomes inflow across the whole section. This inflow persists in the post-HIZ. The normalised profiles are shown in figure 6.6(e). The thick cyan line highlights the jet-like profile in the pre-HIZ. The cyan-red line shows a theoretical plume-like profile in the post-HIZ. This inflow is extremely important for explaining the behaviour in the widths of velocity and tracer concentration, and is discussed below in conjunction with temperature profiles below.

The tracer concentration profiles, shown in figure 6.7(a) (Type I, Ri = 0.56), are distinctly different from the velocity profiles. The centreline continues to decay with heating and the widening of the velocity profiles is not present here. The normalised profiles, figure 6.7(b), show a slight variation from Gaussian, changing, at the edge of the jet, from below to above the highlighted profile.

The temperature profiles, figure 6.7(c) (Type I, Ri = 0.56), show an interesting feature of the development of the heated jet. There is a very clear off-centre maximum in each profile in the lower-HIZ. This will
Figure 6.6: Heated Jets: (a) Radial profiles of streamwise velocity at each height in the self-similar region (Ri = 0.56). (b) Radial profiles of streamwise velocity at each height in the self-similar region, normalised by the centreline value and the velocity width (Ri = 0.56). (c) Radial profiles of streamwise velocity at each height in the self-similar region, normalised by the centreline value and the velocity width (Ri = 0.2). Note that there is a slight hint of a flat-top Gaussian at this lower heating rate. (d) Radial profiles of radial velocity at each height in the self-similar region (Ri = 0.56). (e) Radial profiles of radial velocity at each height in the self-similar region, normalised by the centreline streamwise velocity and the velocity width (Ri = 0.56).
Figure 6.7: Heated Jets: (a) Radial profiles of tracer concentration at each height in the self-similar region (Ri = 0.56). (b) Radial profiles of tracer concentration at each height in the self-similar region, normalised by the centreline value and the tracer width (Ri = 0.56). (c) Radial profiles of temperature at each height in the self-similar region (Ri = 0.56). (d) Radial profiles of temperature at each height in the self-similar region, normalised by the centreline value and the temperature width (Ri = 0.56).

have affected the measurements of the temperature width, which is based on a purely Gaussian profile. The increase of the centreline value to a maximum at the end of the Hiz is evident, followed by the decay in the post-Hiz. The normalised profiles, figure 6.7(d), show that the off-centre maximum can be 20% in excess of the centreline value, but reduces throughout the Hiz. In the post-Hiz, the profiles are close to Gaussian. This is responsible for the observed behaviour in the velocity width. The off-centre maximum leads to greater generation of streamwise velocity away from the axis, and so the velocity width increases. As the temperature maximum returns to the axis, this effect diminishes and the streamwise velocity is affected more by the radial velocity profile, which is negative across the jet and the velocity width decreases.

The off-centre temperature maximum was also found by ASP, and results from a competition between the tracer concentration (upon which heat injection depends linearly) and the residence time of the fluid in the Hiz. Fluid close to the axis has a high tracer concentration and so experiences strong heating, but
is travelling more quickly than fluid towards the edge of the jet and so spends less time being heated. Conversely, fluid at the edge of the jet has a lower tracer concentration so only experiences gentle heating, but because it is travelling slowly, experiences heating for a longer period of time. The results of these two competing effects is the observed off-centre temperature maximum.

This off-centre temperature maximum, coupled with the change in radial velocity profiles can explain the observed behaviour in the velocity and tracer widths. Heating leads to an acceleration of the fluid in the streamwise direction. Incompressibility demands that this fluid is replaced and so fluid is drawn towards the hiz. This is the global effect observed in the streamline plots in figure 6.3, and explains the change of behaviour in the radial velocity - an inflow is experienced at all radial positions. This inflow sweeps the tracer towards the axis, and is responsible for the thinning of the tracer width. In contrast, the streamwise momentum is being continually generated by the injected heat, and because this heat has an off-centre peak, the velocity width does no decrease in the same way as the tracer width. Again, there are two competing effects. The heating leads to a slight widening of the velocity profiles and the radial inflow leads to a decrease. As the temperature profile approaches a Gaussian distribution, the latter effect becomes dominant and the growth rate of the velocity width eventually decreases.

The above data are once again consistent with the observations of AP. The only small difference is that a flat-top Gaussian was clearly observed in the experimental streamwise velocity data of ASP, which was not observed in these simulations. At the lowest heating rate, a small flat-top was observed, but was far from conclusive. This may a restriction of the limited domain size in the calculations, and may depend on the precise details of the hiz (e.g. position and extent), which cannot be characterised by a single Richardson number.

**Entrainment**

Figure 6.8(a) shows the evolution of mass flux in each case. There is a clear difference in the mass fluxes in the pre-hiz. This is due to the global effect that heating produces, which was first observed comparing the streamlines, figure 6.3. As the heating rate increases a greater mass flux is observed. Before the hiz, the flux is close to linear in each case, but grows more quickly than linear during and after the hiz. It should be noted that a higher mass flux does not necessarily imply a greater rate of entrainment, it could be that the fluid is simply moving quicker.

The evolution of the entrainment coefficient is shown in figure 6.8(f). It is clear that heating brings about a higher value for the entrainment coefficient in general. Again, heating has a global effect and so the entrainment coefficient is seen to be higher in the pre-hiz than in the unheated case. More locally though, as the jet enters the hiz it would appear that the entrainment coefficient drops, particularly at the higher heating rates. There also appears to be a difference in behaviour between the lowest heating rate and the other cases. In the post-hiz, the entrainment coefficient for the lowest heating rate settles
Figure 6.8: Heated Jets: Comparisons of different fluxes at different heating rates. (a) Mass flux. (b) Streamwise evolution of the entrainment coefficient at different heating rates. (c) Total momentum flux. (d) Relative turbulent momentum flux. (e) Total temperature flux. (f) Relative turbulent temperature flux.
Figure 6.9: Heated Jets: (a-d) Comparison of the evolution of streamwise and radial turbulent intensities evaluated in different ways, (Type I, $R_i = 0.56$). (a) Axial value, normalised. (b) Axial value, not normalised. (c) Integrated value, normalised. (d) Integrated value, not normalised. (e) Streamwise evolution of the streamwise turbulent intensity integrated radially at different heating rates. (f) Comparison of factors contributing to the entrainment rate. Positive values are due to enhanced buoyancy. Negative values are due to enhanced turbulent intensity.
down to a value comparable with the highest heating rate, and above the middle case. This suggests that there is a subtle complication to the Richardson number dependence. The type II heating results in the highest value of the entrainment coefficient in the post-HIZ.

The momentum flux presents a similar picture to the mass flux, and is shown in figure 6.8(c). In the pre-HIZ the momentum flux is constant and equal in each case. The flux increases as the jet enters the HIZ, more so with a higher heating rate, and shows linear growth in the post-HIZ for type I heating. A comparison of the relative momentum flux due to the turbulence, i.e. the ratio of $M'$ to $\bar{M}$, is shown in figure 6.8(d). It shows that heating causes relatively more of the momentum flux to be carried by the turbulence, an effect that increases with heating rate. The same response is present in a comparison of the relative turbulent tracer flux, (not shown). The total tracer flux is close to being constant in the region of interest in all cases and is not shown.

The total temperature flux is shown in figure 6.8(e), and shows a simple situation of growth during the HIZ and conservation in the post-HIZ. The relative turbulent flux is shown in figure 6.8(f), and like the tracer, shows an increase in the relative proportion of the flux carried by the turbulence. It also shows that the higher heating rates leads to higher relative levels of temperature flux carried by the turbulence.

Figures 6.9(a-d) show four different presentations of the turbulent intensity as a function of streamwise position. These plots demonstrate how different interpretations of the analysis of J. C. R. Hunt can lead to different conclusions, see section 2.6.4. The top two figures are the centreline evolution, the bottom two are the radial integrals, the left-hand two are normalised, and the right-hand two are not.

Figure 6.9(a) shows the centreline evolution of the normalised second-order correlations. The velocity correlations show a marked drop when the jet enters the HIZ. Note that the transverse velocity correlations drop at a higher streamwise position than the streamwise velocity fluctuation does. The corresponding unnormalised plot, figure 6.9(b), shows an arrest in the decay of the velocity fluctuations and a small growth in the post-HIZ. The streamwise evolution of the integrals of the normalised second-order fluctuations is shown in figure 6.9(c), where a drop is still observed in the velocity fluctuations. A stark difference is present when the integrals are not normalised, figure 6.9(d). The scalar fluctuations decrease, as expected. The velocity fluctuations, on the other hand, increase after entering the HIZ, again the transverse velocity fluctuations respond more slowly than the streamwise velocity fluctuation.

These results highlight the differences between the interpretations of Hunt’s analysis. The comparisons made in the literature (BN1 and AP) refer to the normalised centreline value, figure 6.9(a). This would suggest that the change in turbulent intensity should lead to an increase in entrainment. However, the revised interpretation of the data, figure 6.9(d), suggests that the change in turbulent intensity should indeed lead to a decrease in entrainment.

A comparison of the streamwise turbulent intensities integrated radially is shown in figure 6.9(e), where
Figure 6.10: Heated Jets: (a,b) Normalised profiles of streamwise velocity fluctuations, type i, $R_i = 0.2$ and $R_i = 1.09$, respectively. (c,d) Normalised profiles of radial velocity fluctuations, type i, $R_i = 0.2$ and $R_i = 1.09$, respectively. (e,f) Normalised profiles of Reynolds stress fluctuations, type i, $R_i = 0.2$ and $R_i = 1.09$, respectively.
it can be seen that the effect is more pronounced with heating.

Figure 6.9(f) compares the two effects highlighted in the reworked analysis of Hunt. Specifically, the two terms in (2.210) due to the positive forcing \( \left( \frac{F}{\sqrt{M}} \right) \) and the turbulent intensity \( \left( \frac{1}{\sqrt{M}} \frac{dM'}{dz} \right) \), see section 2.6.4. The heating leads to an increase in the entrainment coefficient, giving rise to the positive curves, whereas the turbulent term shows a definite decrease in entrainment coefficient.

Turbulence Structure

The question remains as to what causes the change in the turbulent intensity, and how it fits with the observations of BN1, and AP.

Throughout this section, profiles of second-order correlations are presented for type i heating at \( \text{Ri} = 0.2 \) and \( 1.09 \), respectively. The type i \( \text{Ri} = 0.56 \) and type ii cases present comparable data, and are not shown. Any differences will be discussed explicitly.

The profiles of normalised streamwise turbulent intensity are shown in figure 6.10(a,b). It is very clear that heating brings about a significant change in the profiles. All cases show similar profiles for the pre-HIZ. A decrease in all of the axial values is observed when heating is applied, which explains the trend observed in figure 6.9(a). Heating leads to a pronounced off-centre maximum, which can exceed the value at the same position in the pre-HIZ profiles. In the lowest Richardson number case, the heating does not appear to be strong enough for the minimum axial value to be attained in the HIZ, as this is only reached in the post-HIZ. In each case, the post-HIZ profile appears to reach a self-similar state that persists to the top of the domain. In the strongest heating case, the off-centre maximum is well in excess of the pre-HIZ value at that radial position, and it appears that in the post-HIZ, it relaxes and brings about a small increase in the axial value, resulting in a slightly different post-HIZ profile to the other heating cases. It should be noted that in the type ii case (not shown), the self-similar state that is reached is very similar to that in the type i cases at the lower two heating rates.

It is this change in self-similar structure that leads to different results in the interpretations of the turbulent intensity data presented in figures 6.9(a-d). The proposal of J. C. R. Hunt deals with entrainment in terms of the integral of the turbulent intensity. Understandably, it is difficult to measure accurately the entire fluctuation profile in a laboratory experiment. Therefore, it seems natural to assume that the turbulence is self-similar and infer the value of the integral from the centreline value, which is much easier to measure. However, the above data demonstrate that self-similarity is lost, and so this procedure is no longer a valid way to evaluate the integral of the velocity fluctuations. There is a significant decrease in the centreline values presented above, but the off-axis values present a significant increase. When these profiles are integrated radially, which includes a radial weighting from the Jacobian, heating leads to an increase in value, and not a decrease as the centreline would imply. When this is compounded with
Figure 6.11: Heated Jets: (a,b) Normalised profiles of tracer fluctuations, type 1, $\text{Ri} = 0.2$ and $\text{Ri} = 1.09$, respectively. (c,d) Normalised profiles of temperature fluctuations, type 1, $\text{Ri} = 0.2$ and $\text{Ri} = 1.09$, respectively.

With the normalisation issues highlighted in section 2.6.4, the link between the rate of entrainment and the turbulent intensity can be misinterpreted. This important point will be discussed further in section 6.3, once the cause of this change in behaviour has been established.

Figures 6.10(c,d) show the radial velocity turbulent intensity. Again, heating brings about a significant change in the shapes of the profiles. In each case the profiles in the pre-HIZ are comparable. The post-HIZ profiles are lower than the pre-HIZ profiles. This is possibly due to the normalisation as the increase in centreline velocity will lead to a decrease in these profiles unless there is the same relative growth. In the lowest heating case there is an initial period of increase before the decay. At the higher heating rate, it is clear that there are several profiles in the HIZ that are at the same magnitude as the pre-HIZ self-similar state. This reiterates the delayed response to the heating observed in the radial fluctuations compared with the streamwise fluctuations, see figures 6.9(a-d). The reason for this is related to the cause for the change in structure of the streamwise fluctuations and will be discussed below.

The Reynolds stress profiles are shown in figures 6.10(e,f). At the lowest heating rate, the heating only
Figure 6.12: Heated Jets: (a,b) Normalised profiles of streamwise velocity-temperature Reynolds stress ($\tau_{zT}^R$), type $i$, $Ri = 0.2$ and $Ri = 1.09$, respectively. (c,d) Normalised profiles of radial velocity-temperature Reynolds stress ($\tau_{rT}^R$), type $i$, $Ri = 0.2$ and $Ri = 1.09$, respectively.

brings about a small change in the profile, the maximum appears to move to a greater radial position and the decay to zero looks to be steeper. At higher heating rates, the difference is more significant. The maximum not only moves to a greater radial position, but it is accompanied by a decrease in magnitude. Again, this is most likely due to the normalisation, the increase in centreline velocity results in a decrease in these profiles.

Figures 6.11(a,b) show the radial profiles of the normalised tracer turbulent intensity. In the pre-HIZ the profiles are the same as the unheated case. The heating causes the intensity to increase across the entire profile, and the response to heating is more rapid with a higher heating rate. In the intermediate heating rate (not shown) the profiles resemble the profiles seen in the pure plume simulation. At the lowest heating rate the growth does not reach the same magnitude, and at the highest heating rate the growth is higher.

The profiles of normalised temperature fluctuations are shown in figure 6.11(c,d). During the HIZ the fluctuations present a significant off-centre maximum, which decreases in the post-HIZ, accompanied by an increase in the axial value. At the lowest heating rate, the off-centre maximum is still present in the
post-hiz profile. This is not the case at the highest Richardson number.

Figures 6.12(a,b) show the normalised profiles of streamwise velocity-temperature Reynolds stress \( \tau_{RzT} \), note that the width is normalised by the velocity width, not the temperature width. Similarly, figures 6.12(c,d) show the radial velocity-temperature Reynolds stress, normalised in the same way, \( \tau_{RzT} \). These two Reynolds stresses are of particular interest, as they appear as source terms in the Reynolds stress transport equations for the streamwise velocity fluctuations and velocity Reynolds stress, respectively, see (2.122) and (2.123). At the jet edge, the correlation appears not to tend to zero in a straightforward manner. This is likely to be due to mismatch of profile widths between the velocity, temperature and tracer. However, two things are particularly evident. First, there is a negative region close to the axis, particularly at the low heating rate. Second, there is a large positive peak away from the axis. It is these two features that are responsible for the change in structure of the velocity fluctuations in figure 6.10(a,b). Heat injection leads to fluctuating temperature and velocity correlations that are negative close to the axis and positive away from the axis. These correlations affect the streamwise velocity fluctuations (and velocity Reynolds stress), which compounded with an increasing velocity, lead to normalised streamwise velocity fluctuations that decreases near the axis and increases away from the axis, i.e. there is a change in the structure, and is no longer self-similar.

The radial and streamwise gradients of the second-order correlations are shown in figure 6.13. As usual, most of the streamwise gradients are close to zero, apart from one or two of the temperature profiles. This suggests that at the lower end of the hiz, the streamwise changes cannot be neglected, as the boundary layer approximation would suggest - changes in the streamwise direction are no longer small compared with radial changes and so the boundary layer approximation is invalid. The radial gradients present the usual behaviour, except in the velocity-temperature correlation, which is distinctly different due to the shape of the profiles presented above.
Figure 6.14: Heated Jets: (a) Radial profiles of normalised approximation to the eddy-viscosity coefficient (Type i, Ri = 0.56). (b) Radial profiles of normalised approximation to the eddy-viscosity coefficient for tracer concentration (Type i, Ri = 0.56). (c) Radial profiles of normalised approximation to the eddy-viscosity coefficient for tracer concentration (Type i, Ri = 0.56).

Figure 6.14 shows the estimate of the eddy-viscosity coefficient for velocity. It would appear that the heating causes a decrease in the magnitude across the whole profile, although this may be due to the normalisation. The coefficient for tracer concentration appears to remain largely unchanged, figure 6.14, although there does appear to be a flattening of the peak. It is possible that the profile changes from jet-like to plume-like. The profiles presented in the previous chapter presented the same kind of difference, but the scatter in the data prevents definitive conclusions being drawn. It is important to note that the radial position is normalised by the velocity width. The data do not collapse if normalised by the scalar width. The coefficient for temperature is shown in figure 6.14.

The spectra of both velocities and scalars are comparable with those presented for the plume in section 5.4, and are not shown. There is no clear indication of any change in structure discernible from these spectra.
Simulations have been run to investigate the response of a turbulent jet to volumetric heating. Three heating rates were considered, and an additional case where heating was applied to the entire upper section of the domain. A complicated three-way coupling is observed between the velocity, tracer and temperature fields. This is compounded by the interaction of the mean flow with the turbulence.

**Large scale flow**

In the pre-hiz, the evolution is jet-like, i.e. there is a decay of velocity and tracer that is close to $z^{-1}$, and the sides are approximately straight. Applying the heating naturally increases the temperature of the jet, which, in this configuration, reaches a maximum at the end of the hiz. For type II heating a maximum is eventually reached, but it takes twice the length of the hiz to be attained, at the heating rate used. The increase in temperature, and corresponding increase in buoyancy, arrests the velocity decay, and can lead to an increase. These results are consistent with BN1 and AP.

Since the fluid is accelerated upwards, continuity demands that the fluid is replaced. The streamlines suggest that some of this fluid comes from beneath the hiz (i.e. heating induces a global change), and the radial velocity profiles present a pronounced change in behaviour - an increased inflow is experienced at all radial positions. This change in radial velocity was also observed by AP. The global effect of heating leads to enhanced entrainment throughout the jet, i.e. the mass flux is everywhere greater than in the unheated jet, which is also the case in the entrainment coefficient. A mass flux greater than the heated jet was also reported by AP. The mass flux reported by BN1 will be discussed below.

The change in the radial velocity sweeps the tracer towards the axis, and with the natural absence of tracer generation, the rate of growth of the tracer width decreases, and can even become negative. This is consistent with both BN1 and AP. The reduction in width is not mirrored by the velocity width, because there is generation of streamwise momentum due to heating. At each height in the lower-hiz, the peak temperature occurs away from the axis. This results in an increased rate of growth of the velocity width.

The enhanced growth in velocity width due to the generation of streamwise momentum by the off-centre temperature maximum is in competition with the increased inward radial velocity. Hence, the velocity width increases initially when the off-centre maximum is greatest, and decreases at greater heights, where the temperature profile has become more Gaussian.

Agrawal and Prasad attribute the thinning of the tracer concentration to the disruption of the large eddies, and not to the change in radial velocity profile, which they stress as previously unobserved. “The disruption of large eddies results in a reduction of the scalar width, and eventually, a flattening of the velocity width as well”. In their numerical study, “the difference arises because [tracer] concentration changes from an active scalar in the hiz to a passive scalar in the post-hiz”. In AP, it is only mentioned
in passing that it had been suggested by personal communication that this “anomalous behaviour may be related to an increased mass flux”. The view adopted here is that the heating accelerates the jet fluid upwards, continuity demands that this fluid is replaced, and the resulting change in radial velocity profile sweeps the tracer inwards. The visual thinning and the change in turbulent structure are indeed correlated, but how the latter implies the former is not clear.

The off-centre temperature maximum occurs due to the competition between tracer concentration and the residence time in the HiZ-axial fluid receives intense heating for a limited time, whereas fluid at the jet edge receives gentle heating for longer. This is consistent with AP.

Generally, the profiles of streamwise velocity and tracer remain close to Gaussian, even under heating. A flat-top Gaussian streamwise velocity distribution was reported by AP. This was not found in these simulations, however, at the lowest Richardson number there is a hint of a flat-top Gaussian developing. These results do not give a clear indication that this is always the case, and suggest that a lower Richardson number is required. Alternatively, a different arrangement of the HiZ may also allow this flattening of the profile to occur – it is not likely that the flow can be completely characterised by a single Richardson number.

**Entrainment**

How does heat injection affect the entrainment rate into a turbulent jet? This is clearly a complicated issue, particularly given the different interpretations of the term entrainment.

In BN1, the mass flux presented is arrested by heating. This result should be interpreted carefully. The mass flux was based on the centreline velocity and profile width, under the assumption of self-similarity, which means \( Q \propto b^2 u \). The profile width chosen was tracer width, probably because it is easier to measure. However, the data from this study, along with AP (and even alluded to in BN1) demonstrate significantly differing behaviour in the velocity and scalar widths. Therefore, the mass flux presented in BN1 is not likely to be the true mass flux. Figure 6.15 compares the use of the two different widths for evaluating the mass flux, using the data from this study, type \( \iota \) Ri = 0.56. The difference is clear, and suggests that the mass flux in the experiments of BN1 is consistent with the present study and with the experimental and numerical data of AP. Therefore, it seems fair to conclude that heating leads to an increase in mass flux.

The data from this study suggests that the increase in mass flux is a global effect. The acceleration of the jet fluid due to heating draws fluid from all directions, not just laterally as a traditional laterally entraining model would suggest. This could be due to a high level of heating in a restricted region - the experiments cover a much greater domain and so the global effect could be much lower.

Despite this global effect, the local entrainment has been investigated. The entrainment coefficient has
been estimated according to (5.3). The global effect is clear here, as the heated coefficient is higher than the unheated jet at all streamwise positions. However, there is a pronounced decrease as the fluid enters the HIZ. This suggests that a constant coefficient, self-similar analysis will not correctly capture the effect of heating on entrainment. This clearly needs further investigation.

Hunt [39] proposed that a decrease in entrainment could result from an increase in turbulent intensity. Before this proposal was investigated, the theory has been readdressed to account for a variable mean momentum flux. This has highlighted how different interpretations can lead to different conclusions, particularly those in the literature. The readdressed theory suggests that a decrease in entrainment can result from a relative increase in turbulent intensity, as opposed to an increase in relative turbulent intensity. Experimentally, the radial integral of second-order fluctuations is extremely difficult to measure accurately. Therefore, the solution is to assume self-similarity, and infer the integral value from the centreline value. BN1 measured a decrease in this value, and so concluded that a decrease in entrainment could not result from the observed change in turbulent intensity. However, there was some evidence in BN1 that the turbulent intensity did not remain self-similar, a result that has been confirmed by the present study. This means that the integral has to be evaluated properly. However, if the integral of the relative turbulent intensity is evaluated, a decrease is still observed upon heating. Therefore, the data are not consistent with the original proposal. Only when the integral is evaluated according to the readdressed theory does the data suggest that the change in turbulent intensity can support a decrease in entrainment.

In stark contrast to the above discussion, AP report an increase in the centreline turbulent intensity in both the experimental and numerical results. However, when discussing Hunt’s theory, arguments are only put forward in terms of the eddy disruption. Hunt associates an increase in turbulent intensity with the jet breaking up into smaller eddies. The presence of large eddies in the lower-HIZ is used by AP to
account for the increased mass flux (due to the positive forcing and no influence from the turbulence), and the disruption of the eddies in the upper-HIZ to account for the decrease in mass flux growth. The understanding of this problem is still not clear, and further work is required.

### Turbulent Structure

The radial profiles of streamwise and radial turbulent intensity show a marked change in shape upon heating. This was also observed by both BN1 and AP. Bhat and Narasimha present radial profiles of normalised turbulent intensity just above the HIZ, at different heating rates. At the highest heating rate, an off-centre maximum is clearly present. All of the values are lower than the unheated jet. In stark contrast to BN1 and the results presented in the last section, Agrawal and Prasad observed a substantially higher (60%) value of normalised axial turbulent intensity through the lower-HIZ compared with the unheated jet. A decrease is observed in the upper-HIZ, which becomes constant in the post-HIZ. In the results presented in the last section, the normalised axial values are comparable with the unheated jet in the pre- and lower-HIZ. The same decrease to a constant is observed, but the reason for the contrasting results is far from clear.

Since the shape of the profile changes, self-similarity is lost, so the true integral has to be evaluated to assess the proposal of Hunt. Again, the values are comparable with the unheated jet in the pre-HIZ, but a decrease in the normalised value is observed in the HIZ. Only when the unnormalised values are considered is an increase due to heating observed. So by reworking the analysis of Hunt and evaluating the true unnormalised integral of the turbulent intensity, the essence of the original proposal is confirmed. By considering the entrainment rate in terms of (2.210), as shown in figure 6.9, the link is demonstrated further.

The change in behaviour of the second-order correlations can be explained mathematically by considering the Reynolds stress transport equations (2.121), (2.122) and (2.123). The terms of importance here are the velocity-temperature correlations, $u'zT'$ and $u'rT'$. The negative region near the axis and positive region away from the axis in these term are responsible for the changes in mixing behaviour. This is consistent with one of the main conclusions of the wavelet analysis of Narasimha, Saxena and Kailas [61], who found that heating led to enhanced mixing activity away from the centreline. The complicated profile that these correlations present highlights the intricate interaction between the velocity, temperature and tracer, and the effect on the turbulent structure.

The buoyancy does not feature in the radial turbulent intensity equation, and so there is a delay in the response of the radial fluctuations compared with the streamwise fluctuations. The response of the radial fluctuations can be accounted for by the return-to-isotropy action of the pressure-strain interaction. This is similar to the generation of turbulent fluctuations. In the radial equation the generation terms are small in comparison with the generation terms in the streamwise equation. In each of the plots of second-order
correlations, there is a time-delay between responses in the streamwise and radial fluctuations. Again, the
delay is explained by the difference in the two equations, and the response is due to the return-to-isotropy
action of other terms.

Summary

The pre-HIZ presents jet-like behaviour. Upon heating there is competition between the tracer concentra-
tion and residence time of the fluid in the HIZ. This competition results in a temperature profile with an
off-centre maximum. The increase in temperature arrests the velocity decay and accelerates it upwards.
Continuity demands that the fluid is replaced. This brings about a change in the profile of the radial
velocity, which becomes directed towards the axis at all radial positions. This inward velocity sweeps the
tracer concentration towards the axis, resulting in a reduced width. Generation of upward momentum by
the injected heat initially widens the velocity profile, due to the off-centre maximum. As the temperature
profile becomes more Gaussian, the inward velocity also leads to a reduction in velocity width. The
positive buoyancy leads to an increased rate of entrainment, which is opposed by an increase in turbulent
intensity. The velocity-temperature correlation acts to increase turbulent levels away from the centreline,
and brings about a pronounced change to the profiles of the second-order velocity autocorrelations. The
increase in velocity reduces the normalised centreline turbulent fluctuations. The unnormalised integral
increases, correlated with a decrease in entrainment rate, and so fits with the reworked analysis of Hunt.

None of the results obtained here are in contrast with the Bhat and Narasimha experiments, and on the
whole agree with those of Agrawal and Prasad. The flat-top Gaussian profile is not clearly recovered, but
may be present at the lowest heating rate. The specific details of the size and location of the HIZ relative
to the jet may provide the explanation for this. The only real significant difference between the results
here and AP is in the behaviour of the normalised centreline turbulent intensity. The velocity-temperature
correlation leads to an off-centre maximum and, together with the normalisation, leads to a decrease of
the centreline value. The reason for the substantial increase measured by Agrawal and Prasad is not
explained by these simulations.
Chapter 7

Conclusions and Future Work
7.1 Conclusions

The aim of this study was to reproduce numerically the laboratory experiments of Bhat and Narasimha [14] (BN1), and Agrawal and Prasad [3], and investigate the effects of heat release on spatially evolving turbulent jets.

In order to do this, a numerical approach was taken that employed an implicit turbulence model. The extent to which such an approach can reproduce turbulent structure and statistics was investigated in chapter 3, in the context of homogeneous isotropic turbulence. The results were encouraging. With large scale flow generation, an inertial range with a power-law decay close to the theoretical minus-five thirds was recovered. Probability density functions revealed a structure consistent with other turbulence studies in the literature. Overall, the fundamental ideas behind Kolmogorov’s 1941 theory were found to hold, with one subtle difference. With the absence of a physical viscosity, the fundamental property of the fluid that controls the small scale dissipation was found to be an effective Kolmogorov length scale, which remained fixed at approximately three cell widths, independent of resolution. It was concluded that the small scale behaviour was not the same as in a real-world viscous fluid, but the approach is suitable for high Reynolds number flows where the viscous effects are not dominant and there is large scale flow generation, as in free shear or buoyancy driven flows.

Further code validation was provided in chapter 4, by a study of a temporally evolving circular shear layer with off-source heating, where comparisons were made with an existing study in the literature, Basu and Narasimha [8]. Although, quantitative comparisons were not always possible, the flow evolution was consistent, and the same phenomenology was recovered.

Before the response to off-source heating was investigated, benchmark cases of spatially evolving turbulent jets and plumes were considered in chapter 5. Once again, the general phenomenology were recovered, but differences in the specific numbers were observed. The straight-sided evolution and power-law decays were present, but the entrainment and turbulence levels were higher than values reported in the literature. Basic modelling techniques were examined, and support for the boundary layer approximation and Boussinesq’s eddy viscosity assumption were found.

The crux investigation of the response of spatially evolving turbulent jets to off-source heating was presented in chapter 6. A higher level of statistical convergence than in previous studies was achieved, and the results were in general agreement with the work of BN1 and AP. The change in radial velocity behaviour observed by AP was captured, along with the off-centre temperature maximum. The contrasting behaviour of scalar and velocity widths was confirmed, which was argued to be a direct response to the radial velocity and temperature distributions, rather than the disruption of the small eddies as proposed by AP. The greatest insight was provided by detailed measurements of the second-order fluctuations. A pronounced change in self-similar structure was recovered, and accounted for by the velocity-temperature
correlations, which appear as source terms in the Reynolds stress transport equations. A reappraisal of the analysis of Hunt [39] suggested that not only the present study, but also the previously discounted measurements of BN1, supported a correlation between the local turbulent intensity and the entrainment of ambient fluid.

In conclusion, the specifics of the turbulence were not recovered in minute detail by this implicit method, but the large scale flow and phenomenology were captured, and insight was provided into a very complicated fluid dynamical problem.

### 7.2 Future Work

There are many outstanding problems in the modelling of spatially evolving turbulent buoyant jets. There are differences between pure jets and pure plumes that can only be accounted for in existing integral models by different constants. The entrainment coefficient, ratio of velocity to scalar widths, and scalar fluctuation intensity are three prime examples. In non-canonical situations, these differences have to be reconciled somehow, Wang and Law’s [91] empirical model is testament to this.

A key problem that needs addressing is the instant recovery of straight sides from the incompressibility when self-similarity is assumed. This may be resolved by allowing the streamwise and radial velocities to scale independently, i.e. \( \bar{u}_z(z, \eta) = \bar{u}_0(z)f(\eta) \) and \( \bar{u}_r(z, \eta) = \bar{v}_0(z)g(\eta) \). This makes intuitive sense for a variable entrainment coefficient.

Moreover, an examination of the underlying turbulence is necessary to explain these effects. In particular, consideration of the Reynolds stress transport equations would be a prudent starting point.

Support for Boussinesq’s eddy viscosity hypothesis was provided by estimates of the coefficient, which was found to agree reasonably well with the spatially dependent profile derived from a Gaussian streamwise velocity. However, discrepancies were present, particularly with the inclusion of buoyancy, and moreover, off-source heating. The applicability of such a simple treatment of the Reynolds stress may generalise to higher-order correlations, and form the basis of a closure strategy (generalised gradient diffusion).

An interesting paper that has received curiously little attention, is by So and Aksoy [81], which addressed the issue of jet-plume transition. The entrainment assumption was avoided, and so an extra condition was required to close the equations. The approach taken was to use a streamwise dependent eddy viscosity, assume functional forms for the self-similar profiles in the flow variables and derive equations for the Reynolds stress. By imposing a boundary condition on this Reynolds stress equation, an auxiliary equation was derived that can be used to close the system. Comparisons with experimental data was promising, but the model needs further examination to understand its potential.

Another interesting approach is taken by Kaminski et. al. [45] (see also Carazzo et. al. [20]), which is based
on the work of Priestley and Ball [70],[69], and Fox [30], using a kinetic energy equation. Characteristic scales are defined for length, velocity and reduced gravity in terms of radial integrals, so that the equations can be recast to appear to be written in the top-hat form used by MTT. This appears to collect changes in self-similarity into a term that was referred to as similarity-drift. Again, these interesting ideas required further attention.

An invaluable tool in the formulation of such models is that availability of quality measurements, not only of the mean flow variables, but also of the second- and third-order correlations. Although far from perfect, the simulations presented in chapter 5 are a solid foundation upon which a database of simulations of spatially evolving turbulent buoyant jets can be built. Such a database could be used to compare turbulence model with. These turbulence models could be used in less expensive two-dimensional RANS simulations, for example, but may also allow advanced analytical models to be formulated. It may even be possible to derive new integral models.

Studies need not be restricted to round jet and and plumes, as planar jets, plumes and mixing layers lend themselves to similar treatment. Moreover, complicated behaviour in non-Boussinesq, time-dependent and reacting flows can be considered.

In conclusion, there are important fundamental questions that can be addressed from simulations conducted in this study and further calculations based upon them. This kind of implicit turbulence model approach can give access to phenomenology and turbulent structure that are unobtainable by experiments or other approaches. These results can be a useful guide in constructing a second-order turbulence model for turbulent jets and plumes, whether it is to be used in less expensive calculations or an ambitious analytical description. Ultimately, this study has demonstrated the potential of numerical simulations to further the understanding of the phenomena and underlying mechanisms of turbulent mixing.
Bibliography


