3 First order nonlinear equations

\[ \frac{dy}{dx} = f(x, y) \]

where \( f(x, y) \) is not (in general) a linear function of \( y \).

The equation is autonomous if there is no explicit dependence on the independent variable, i.e. \( f(x, y) = f(y) \).

3.1 Separable equations

These are often straightforward to solve.

Suppose we can write

\[ f(x, y) = \frac{g(x)}{h(y)} \]

Then the ode may be written as

\[ h(y) \frac{dy}{dx} = g(x) \]

\[ \Rightarrow \int h(y) \, dy = \int g(x) \, dx \]

\[ \Rightarrow H(y) = G(x) + \text{const} \]

\[ h = \frac{dH}{dy}; \ g = \frac{dG}{dx} \]

\[ \frac{dy}{dx} = \frac{y \sin x}{\sqrt{1 + y^2}} \]

Rearrange:

\[ \frac{1 + y^2}{y} \frac{dy}{dx} = \left( \frac{1}{y} + y \right) \frac{dy}{dx} = \sin x \]

Integrate

\[ \int \frac{1}{y} + y \, dy = \ln y + \frac{1}{2} y^2 = \int \sin x \, dx = -\cos x + \frac{1}{2} A \]

Arbitrary constant of integration \( A \) determined by boundary condition \( y(0) = 1 \)

\[ \Rightarrow \quad 0 + \frac{1}{2} = -1 + \frac{1}{2} A \Rightarrow A = 3 \]

\[ \Rightarrow \quad \ln y + \frac{1}{2} y^2 = \frac{3}{2} - \cos x \]

\[ \ln y = \frac{1}{2} \ln y^2 \Rightarrow \quad \ln y^2 + y^2 = 3 - 2 \cos x \quad (\ast) \]

End of Lecture 7

Cannot give explicit solution for \( y \), but clearly periodic with period \( 2\pi \).

For some quantity \( u \), the value of \( u + \ln u \) is a monotonically increasing function of \( u \).

The minimum value of the right-hand-side of \((\ast)\) is \( 3 - 2 = 1 \), at \( x = 2n\pi \) and the maximum value is \( 3 + 2 = 5 \) at \( x = 2(n+1)\pi \).

\[ y^2 + \ln y^2 = 1 \Rightarrow y = 1 \] is the minimum value. The maximum value is around \( y = 1.92 \).

Note: if \( y \) is a solution, then \(-y\) is also a solution to this equation.
3.1.1 FAMILIES OF SOLUTIONS

In the above example we considered a specific solution, selecting \( A = 3 \) in order to satisfy the boundary condition \( y(0) = 1 \). This is one of a family of solutions that may be obtained using other values of \( A \).

\[
\ln y^2 + y^2 = A - 2 \cos x \; ; \; \text{red } A=3, \; \text{green } A=4, \; \text{blue } A=5, \; \text{black } A=6; \; \text{magenta } A=0.5, \; \text{cyan } A=0.1, \; \text{dark grey } A=0.01.
\]

[See S3_1_FamilyOfSolutionsForSeparableEquation.nb]

As \( A \to \infty \), \( y \to \infty \) and dominated by \( y^2 \sim A \Rightarrow \) amplitude of oscillation becomes small.

As \( A \to -\infty \), \( y \to 0 \) and dominated by \( \ln y^2 \sim A \Rightarrow \) amplitude of oscillation becomes small.

Isoclines: contours of constant \( dy/dx \).

- Here vertical lines at \( x = n\pi \) (where \( dy/dx \) vanishes) are isoclines
- Can be helpful for sketching solution.

Solution with greatest amplitude for \( \ln y^2 + y^2 = A - 2 \cos x \).

\[
\begin{align*}
\max(y) &= y_t \\
&\quad \Rightarrow y_t^2 + 2 \ln y_t = A + 2 \\
\min(y) &= y_b \\
&\quad \Rightarrow y_b^2 + 2 \ln y_b = A - 2 \\
&\quad \Rightarrow y_t^2 - y_b^2 + 2 \ln(y_t/y_b) = 4
\end{align*}
\]

Let \( Y \equiv \frac{1}{2} (y_t + y_b), \; B \equiv \frac{1}{2} (y_b - y_t) \Rightarrow y_t = Y + B, \; \; y_b = Y - B \)

\[
\Rightarrow \quad 4BY + 2\ln \frac{Y + B}{Y - B} = 4
\]
Look for extremum ⇒ \( dB/dA = 0 \)

\[
B'Y + B + \frac{1}{2} \left( \frac{1+B'}{Y+B} - \frac{1-B'}{Y-B} \right) = 0
\]

\[
⇒ B + \frac{1}{2} \frac{Y-B-(Y+B)}{Y^2-B^2} = B - \frac{B}{Y^2-B^2} = B \left( 1 - \frac{1}{Y^2-B^2} \right) = 0
\]

\[
⇒ Y^2 = B^2 + 1 ⇒ Y = \pm \sqrt{(B^2 + 1)}
\]

Substitute back

\[
B\sqrt{B^2 + 1} + \frac{1}{2} \ln \frac{\sqrt{B^2 + 1} + B}{\sqrt{B^2 + 1} - B} - 1 = 0
\]

Which requires numerical solution, giving \( B = 0.48 \), \( Y = 1.11 \), \( y_1 = 1.59 \), \( y_b = 0.63 \).

### 3.1.2 Flow Map

Consider \( \frac{dy}{dt} = t(1-y^2) \) [Note: separable, but we won’t separate]

Here we wish to map out the solution, without directly solving the ode.

Construct the flow, \( i.e. \) arrows showing the slope.

\( dy/dt = t \) at \( y = 0 \).

For \( |y| \) large, then \( dy/dt \) large and negative.

For \( |y| < 1 \), then \( dy/dt > 0 \).
Arrows are vectors showing direction of solution; the flow. Arrows often referred to as flow vectors. Note that since $f(t,y)$ is single valued then solutions cannot cross.
Can draw on solution easily using arrows. There is an analogy between the flow vectors and velocity vectors in a fluid flow, and between the solutions and streamlines. Note: solutions drawn with uniform spacing for $y(0)$, but only up to $y(0) = 4$.

The green lines are orthogonal to the red ones. These might represent, for example, height contours above a plane, or contours of constant potential (see IB Fluids next year). Note potential contours drawn with uniform spacing for $t$ at $y = -2$. 

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To construct this map of the solution we did not require the equation to be separable. However, since it is, we can determine the analytical solution.

\[
\frac{dy}{dt} = t(1 - y^2) \Rightarrow \frac{1}{1-y^2} \frac{dy}{dt} = \frac{1}{2} \left( \frac{1}{1-y} + \frac{1}{1+y} \right) \frac{dy}{dt} = t
\]

Integrating

\[
\frac{1}{2} \left[ \ln |1+y| - \ln |1-y| \right] = \frac{1}{2} \ln \frac{1+y}{1-y} = \frac{1}{2} \left( t^2 + c \right)
\]

\[
\Rightarrow 
\frac{1+y}{1-y} = \pm A e^{t^2} = \pm e^{t^2+c}
\]

\[
\Rightarrow 
y = \pm A e^{t^2} - 1 = \frac{A e^{t^2} \mp 1}{A e^{t^2} + 1} = \frac{e^{t^2+c} \mp 1}{e^{t^2+c} + 1}
\]

\[
\Rightarrow 
y = \tanh \frac{1}{2}(t^2+c) \quad \text{or} \quad y = \coth \frac{1}{2}(t^2+c)
\]

Note: We could have approximated the behaviour for large |y| without solving the exact equation by noting that \(1 - y^2 \to -y^2\) so

\[
\frac{dy}{dt} \approx -ty^2 \Rightarrow \frac{1}{y^2} \frac{dy}{dt} \approx -t \Rightarrow -\frac{1}{y} \approx -\frac{1}{2} t^2 + \text{const} \Rightarrow y \approx \frac{2}{t^2 - t_0^2} \quad \text{for some arbitrary } t_0.
\]

Exploring the behaviour

\[c > 0 \Rightarrow y(0) > 0 \quad y = \tanh \frac{1}{2}(t^2+c) \text{ has } 0 < y < 1; \ y \to 1 \text{ as } t \to \infty\]

\[c < 0 \Rightarrow y(0) < 0 \quad y = \coth \frac{1}{2}(t^2+c) \text{ has } -1 < y < 1; \ y \to 1 \text{ as } t \to \infty\]

3.1.3 EQUILIBRIUM SOLUTIONS

A steady or equilibrium solution is one for which \(dy/dt = 0 \Rightarrow y = \text{const.}\)

For the previous example, \(dy/dt = 0 \Rightarrow y = \pm 1 \Rightarrow y = \text{const} \) is a solution if \(y = \pm 1\).

The equation converges towards \(y = 1\) and diverges away from \(y = -1\).

\(y = 1\) is said to be a stable equilibrium.

\(y = -1\) is said to be an unstable equilibrium.

3.1.4 ISOCLINES

These are contours of constant slope, i.e. \(f(t,y) = \text{const.}\)

For the previous example, \(f(t,y) = t(1-y^2)\),
Note: All curves asymptote to $t = 0$ and $y = \pm 1$.

This emphasises that approach to stable equilibrium (and departure from unstable equilibrium) is faster at high $t$.

In the earlier example $\frac{dy}{dx} = \frac{y \sin x}{1 + y^2}$, we can set $f = \frac{y \sin x}{1 + y^2} = \frac{\alpha}{2}$, say. The isoclines are therefore given by

$$\Rightarrow y^2 - (2/\alpha) y \sin x + 1 = 0$$

$$\Rightarrow y = \frac{\sin x}{\alpha} \pm \sqrt{\frac{\sin^2 x}{\alpha^2} - 1}$$

for $|\alpha| \geq 1$

For solution to exist, require $\sin^2 x - \alpha^2 > 0$.

When $\sin x = \alpha$, then $y = 1$ and location of isoclines is single-valued so they must be vertical at $y = 0$.

As $dy/dx = 0$ when $\sin x = 0$, then $x = n\pi$ must have vertical isoclines with $\alpha = 0$.

Upper branch of isocline has $y = \frac{\sin x}{\alpha} + \sqrt{\frac{\sin^2 x}{\alpha^2} - 1} \to 2\frac{\sin x}{\alpha}$ as $|\alpha|$ becomes small.
First order nonlinear equations

\[ y = \frac{\sin x}{\alpha} - \sqrt{\frac{\sin^2 x}{\alpha^2} - 1} = \frac{\sin x}{\alpha} \left( 1 - \sqrt{1 - \frac{\alpha^2}{\sin^2 x}} \right) \]

Lower branch of isolcine has

\[ = \frac{\sin x}{\alpha} \left( 1 - 1 + \frac{1}{2} \frac{\alpha^2}{\sin^2 x} - \frac{1}{8} \frac{\alpha^4}{\sin^4 x} + \cdots \right) \]

\[ \rightarrow \frac{1}{2} \frac{\alpha}{\sin x} = \frac{\alpha}{2} \csc x \]

Yellow line: \( \sin x \) Red line: cosec\( x \).

End of Lecture 8

3.2 Stability

The technique we shall explore here is not really necessary for first order odes, but is much more useful for higher order odes.
3.2.1 PERTURBATION

Suppose that \( y = y_0 \) is the equilibrium solution, i.e. \( f(t,y_0) = 0 \).

Consider the behaviour of \( y = y_0 + \varepsilon u \) where \( 0 < \varepsilon \ll 1 \) and \( u = u(t) = O(1) \).

For the example used in §3.1.2 \( \frac{dy}{dt} = t(1 - y^2) \), \( y_0 = \pm 1 \) and consider \( y = \pm 1 + \varepsilon u \).

Substitute into the ode: \( \frac{d}{dt}(y_0 + \varepsilon u) = t\left(1 - (y_0 + \varepsilon u)^2\right) \)

\[ \Rightarrow \varepsilon \frac{du}{dt} = t\left(1 - (y_0^2 + 2\varepsilon y_0 u + \varepsilon^2 u^2)\right) = -t\varepsilon u(2y_0 + \varepsilon u) \]

and linearise (discard terms in \( \varepsilon^2 \) and higher)

\[ \frac{du}{dt} \approx -2ty_0u \]

Solve with \( u(0) = u_0 \)

\[ u \approx u_0 e^{-y_0 t^2} \quad \Rightarrow \text{Solutions converge towards } y = 1 \]

\( y_0 = 1 \) \quad \Rightarrow y = 1 \text{ is stable.}

\[ y_0 = -1 \] \quad \Rightarrow \text{Solutions diverge from } y = -1 \]

\( \Rightarrow y = -1 \text{ is unstable.} \)

As we shall see, a solution is stable if \( \frac{\partial f}{\partial y} < 0 \) and unstable if \( \frac{\partial f}{\partial y} > 0 \), but what if \( \frac{\partial f}{\partial y} = 0 \)?

3.2.2 SEMI-STABLE

Consider \( \frac{dy}{dt} = (1 - y)^2 \equiv f(y) \geq 0 \)

Equilibrium solution requires \( f(y) = 0 \Rightarrow y = 1 \).

Since \( f \) is independent of \( t \), all solutions have the same shape.
Solutions for $y < 1$ converge on equilibrium $y = 1$, but for $y > 1$ solutions diverge.

The equation is separable: 

$$\int (1 - y)^{-2} dy = -\frac{1}{y - 1} = t - t_0 \implies y = 1 - \frac{1}{t - t_0}$$

**Stability**

For $t > t_0$, $y \sim 1 - 1/t \to 1$ from below as $t \to \infty$.

For $t < t_0$, $y \sim 1/(t_0 - t) \to +\infty$ as $t \to t_0$ from below, so $y$ rises from $-\infty$ as $t$ increases from $t_0$.

Stability analysis: let $y = 1 + \varepsilon u$

$$\implies \frac{d}{dt}(1 + \varepsilon u) = \varepsilon \frac{du}{dt} = (1 - 1 - \varepsilon u)^2 = \varepsilon^2 u^2$$

$$\implies \frac{du}{dt} = \varepsilon u^2 \approx 0$$
3.2.3 GENERIC STABILITY PROBLEM

Consider \( \frac{dy}{dt} = f(y, t) \)

1. Find fixed points (equilibrium solutions) by solving \( f(y, t) = 0 \) to find solutions with \( y = \text{const} \quad \forall \ t. \) [It is not sufficient for \( dy/dt \) to vanish for only some period of time.]

2. Expand \( f \) about an equilibrium solution \( y = y_0, \) say. Note: Each equilibrium solution must be investigated separately.
   a. \( y = y_0 + u, \) say, and expand \( f \) in Taylor series about \( y = y_0: \)
   
   \[
   f(y, t) = f(y_0, t) + u \frac{\partial f}{\partial y} \bigg|_{y=y_0} + \frac{1}{2} u^2 \frac{\partial^2 f}{\partial y^2} \bigg|_{y=y_0} + \cdots
   \]

3. Substitute into equation
   a. \( \frac{du}{dt} = u \frac{\partial f}{\partial y} \bigg|_{y=y_0} + \frac{1}{2} u^2 \frac{\partial^2 f}{\partial y^2} \bigg|_{y=y_0} + \cdots \)

Solutions:

- If \( \frac{\partial f}{\partial y} \bigg|_{y=y_0} = \alpha(t) \neq 0, \) then \( \frac{du}{dt} \approx \alpha u \) for sufficiently small \( u. \)

\[
\Rightarrow u \approx u_0 e^{\int \alpha \, dt}
\]

\( \to \infty \) if \( \alpha > 0 \quad \Rightarrow \) unstable

\( \to 0 \) if \( \alpha < 0 \quad \Rightarrow \) stable.

- If \( \alpha(t) \) changes sign as \( t \) varies, then must study with greater care; may be stable or unstable. Fortunately, there are a large number of problems in which \( f \) is independent of \( t, \) so \( \alpha \) is constant.

- If \( \alpha = 0 \quad \forall \ t, \) then \( \frac{du}{dt} \approx \beta u^2 \) with \( \beta(t) = \frac{1}{2} \frac{\partial^2 f}{\partial y^2} \bigg|_{y=y_0} \)

\[
\Rightarrow \int \frac{1}{u^2} \, du = -\frac{1}{u} = \int \beta \, dt
\]

\[
\Rightarrow u = \frac{-1}{\beta \int dt},
\]

and system can be semi-stable.

For \( \beta = \text{const}, \) have \( u = \frac{-1}{\beta(t-t_0)}. \) If \( t > t_0, \) then \( u \to 0 \) as \( t \) increases. If \( t < t_0, \) then \( u \to \text{sign}(\beta)\infty \) as \( t \) increases towards \( t_0 \) from below.
First order nonlinear equations

Stability

[For previous example, \( \frac{dy}{dt} = (1-y)^2 \rightarrow \frac{\partial f}{\partial y} = 2(y-1), \ \frac{\partial^2 f}{\partial y^2} = 2 \), so at \( y = 1 \) equilibrium, \( u = \frac{-1}{2(t-t_0)} \) and semi-stable.]

- If \( \beta = 0 \), need to look at further terms of Taylor series in order to determine stability.

3.2.4 PHASE PORTRAITS

Further insight may be gained by considering the phase portrait of the differential function. The phase portrait is a plot of \( \frac{dy}{dt} \) against \( y \) (effectively \( f(y,t) \) against \( t \)).

Consider

\[
\frac{dy}{dt} = y^2 - 1.
\]

This clearly has equilibrium solutions or fixed points at \( y = \pm 1 \).

Recall that \( \frac{\partial f}{\partial y} \) indicates the stability of the fixed points. When \( \frac{\partial f}{\partial y} < 0 \) the solution is stable as \( \frac{dy}{dt} \) will become negative if a perturbation moves \( y \) above the fixed point, thus causing it to move back towards the fixed point. Conversely, \( \frac{\partial f}{\partial y} > 0 \) leads to \( \frac{dy}{dt} \) increasing as we move away from the fixed point, giving an unstable solution.

A stable fixed point is an attractor, while an unstable fixed point is a repellor.

The semi-stable equation \( \frac{dy}{dt} = (1-y)^2 \) explored in §3.2.2 has the attractor and repellor merging:
End of Lecture 9

### 3.3 Exact equations

Recall that we can convert a linear equation of the form \( y' + qy = f \) into an exact equation of the form \( \frac{d}{dx}(Iy) = If \) using the integrating factor \( I = e^{\int q \, dx} \). Here we shall try to do a similar thing for nonlinear equations.

**Note:** There is no simple guaranteed procedure for this – it is not always possible. It will be introduced by example.

Consider \( 3xy^2 \frac{dy}{dx} + y^3 + 2x = 0 \).

This equation is neither linear nor separable. Can we solve it?

By inspection we can see \( 3xy^2 \frac{dy}{dx} + y^3 + 2x = \frac{d}{dx} \left( xy^3 + x^2 \right) = 0 \), and the equation is **exact**.

Hence the solution is \( \psi(x, y) \equiv xy^3 + x^2 = c \quad (c = \text{const}) \Rightarrow y = \left( \frac{c - x^2}{x} \right)^{\frac{1}{3}}. \)

\( d\psi \) is an **exact differential**.

\( \psi \) is sometimes called a **potential**. It is a conserved quantity.

#### 3.3.1 Finding an exact equation

Consider \( f(x, y) \frac{dy}{dx} + g(x, y) = 0 \), which we wish to write as \( \frac{d\psi}{dx} = 0 \) for some \( \psi(x, y) \).

Recall \( \frac{d\psi}{dx} = \frac{\partial\psi}{\partial y} \frac{dy}{dx} + \frac{\partial\psi}{\partial x} \), so we need \( \frac{\partial\psi}{\partial y} = f(x, y) \) and \( \frac{\partial\psi}{\partial x} = g(x, y) \).

In the previous example,
First order nonlinear equations

\[ \frac{\partial \psi}{\partial y} = 3xy^2 \implies \psi = xy^3 + A(x), \]

\[ \frac{\partial \psi}{\partial x} = y^3 + 2x \implies \psi = xy^3 + x^2 + B(y), \]

where \( A(x) \) and \( B(y) \) are the constants of integration.

Now the two expressions for \( \psi \) must be identical, so \( A(x) = x^2 + c \) and \( B(y) = c \), thus

\[ \psi = xy^3 + x^2 + c. \]

However, not all equations are exact, so this procedure is not always possible.

Since \( \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x} \), then we must have \( \frac{\partial f}{\partial x} = \frac{\partial g}{\partial y} \); can therefore test to see if an exact differential is possible before trying to find it!

\[ 3xy^2 \frac{dy}{dx} + 2y^3 = 0 \]

Note: separable, so it can be written as \( \frac{1}{y} \frac{dy}{dx} = -\frac{2}{3x} \), but we look to make it exact.

\[ f = 3xy^2 \implies \frac{\partial f}{\partial x} = 3y^2 \]

\[ g = 2y^3 \implies \frac{\partial g}{\partial y} = 6y^2 \neq \frac{\partial f}{\partial x} \implies \text{equation is not exact.} \]

**But** the equation can be made exact by multiplying the equation by \( x \):

\[ 3x^2 y^2 \frac{dy}{dx} + 2xy^3 = 0 \rightarrow \frac{\partial f}{\partial x} = 6xy^2 = \frac{\partial g}{\partial y}. \]

Integrating \( f = \partial \psi / \partial y \rightarrow \psi = x^2 y^3 + A(x) \)

Integrating \( g = \partial \psi / \partial x \rightarrow \psi = x^2 y^3 + B(x) \).

Equating \( \implies \psi = x^2 y^3 + \text{const} \), hence \( y = cx^{-2/3} \).

Here we have converted the equation into an exact equation by multiplying it by an integrating factor (here \( x \)).

### 3.3.2 INTEGRATING FACTOR FOR NONLINEAR EQUATIONS

The idea of an integrating factor was introduced in §2.3.2 for linear equations.

The process of finding the integrating factor for a nonlinear equation is more complex.

Multiply the general equation \( f \frac{dy}{dx} + g = 0 \) by the function \( \mu(x,y) \), so

\[ \mu f \frac{dy}{dx} + \mu g = 0. \]

For this new equation to be exact, we must have

\[ \frac{\partial}{\partial x}(\mu f) = \frac{\partial}{\partial y}(\mu g) \] (\(^*\))
and
\[ \frac{\partial \psi}{\partial y} = \mu f \quad \Rightarrow \quad \psi(x, y) = \int \mu f \, dy + A(x) \]
\[ \frac{\partial \psi}{\partial x} = \mu g \quad \Rightarrow \quad \psi(x, y) = \int \mu g \, dx + B(y) \]

Unfortunately, determining \( \mu \) by solving (*) may be as difficult (or impossible) as solving the original differential equation. [Of course, equations on the Examples Sheet may be possible…]

[Finding appropriate \( A(x) \) and \( B(y) \) is typically more straight forward.]

Sometimes the solution of (*) may be simplified by proceeding on the assumption that \( \mu \) is a function of \( x \) or \( y \) alone.

\[ 3xy^2 \frac{dy}{dx} + 2y^3 = 0 \]

[Our previous example]

Multiply by integrating factor, assuming \( \mu = \mu(y) \)

\[ \frac{\partial}{\partial x} (\mu f) = \mu \frac{\partial f}{\partial x} = 3\mu y^2 = \frac{\partial}{\partial y} (\mu g) = \mu \frac{\partial g}{\partial y} + \mu \frac{\partial \mu}{\partial y} = 2y^3 + 6\mu y^2 \]

\[ \Rightarrow \quad \frac{1}{\mu} \frac{d\mu}{dy} = -\frac{3}{2y} \quad \Rightarrow \mu \sim y^{-3/2} \]

So

\[ \frac{\partial \psi}{\partial y} = \mu f = 3xy^{1/2} \quad \Rightarrow \psi = 2xy^{3/2} + A(x) \]

and

\[ \frac{\partial \psi}{\partial x} = \mu g = 2y^{3/2} \quad \Rightarrow \psi = 2xy^{3/2} + B(y) \]

Hence

\[ \psi = 2xy^{3/2} + c \quad \text{and} \quad y = Cy^{-2/3} \]

as before!

Note that the integrating factor is therefore not unique.

Since the solution has \( y \sim x^{-2/3} \), then \( \mu \sim y^{-3/2} \sim x \), the previous integrating factor.

Moreover, one would expect any combination of \( x \) and \( y \) that is equal to the integrating factor \( x \) would also be an integrating factor, e.g. \( \mu = x^{-1}y^{-3} \).

### 3.4 Examples

#### 3.4.1 Chemical Kinetics

Consider the chemical reaction \( X + Y \rightarrow U + V \)

\( \text{e.g. NaOH} + \text{HCl} \rightarrow \text{H}_2\text{O} + \text{NaCl} \)

Suppose the concentration of \( X \) is \( x(t) \), of \( Y \) is \( y(t) \) and \( U \) is \( u(t) \) (the reaction gives equal quantities of \( U \) and \( V \), so \( v(t) = u(t) \)).

If \( x(0) = x_0, y(0) = y_0 \) and \( u(0) = 0 \), then the conservation relations give \( x + u = x_0 \) and \( y + u = y_0 \).

Suppose the reaction rate \( r \) is \( \lambda xy \), where \( \lambda = \lambda(T) \) [Note that if \( 2X + Y \), then \( \lambda x^2y \), etc.]

Then

\[ \frac{du}{dt} = r = \lambda xy = \lambda(x_0 - u)(y_0 - u) \equiv f(u) . \]

Fixed points (equilibria) are where \( f(u) = 0 \Rightarrow u = x_0 \) and \( u = y_0 \) (i.e. all of one of the species is converted).
Need to keep in mind the physical meaning of $x$ and $y$: clearly cannot have a negative concentration!

We shall assume $x_0 < y_0$ (the converse may be treated similarly). The phase portrait may then be drawn:

![Phase Portrait](image)

As $u(0) = 0$, we have the reaction rate $f(u)$ decreases monotonically towards the equilibrium $u = x_0$.

![Equilibrium](image)

Of course, here we can determine the exact solution as equation is separable:

\[
\frac{1}{(x_0 - u)(y_0 - u)} \frac{du}{dt} = \frac{1}{y_0 - x_0} \left[ \frac{1}{x_0 - u} - \frac{1}{y_0 - u} \right] \frac{du}{dt} = \lambda,
\]

\[\Rightarrow \quad \frac{-1}{y_0 - x_0} \left[ \ln(x_0 - u) - \ln(y_0 - u) \right] = \lambda(t - t_0)\]

\[\Rightarrow \quad \frac{y_0 - u}{x_0 - u} = e^{\lambda(t - t_0)(y_0 - x_0)} = Ae^{\lambda(t - t_0)(y_0 - x_0)}\]

\[\Rightarrow \quad u(1 - Ae^{\lambda(t - t_0)(y_0 - x_0)}) = y_0 - x_0 Ae^{\lambda(t - t_0)(y_0 - x_0)}\]
First order nonlinear equations

\[ u = x_0 A e^{\lambda y_0 t} - y_0 e^{\lambda x_0 t} \]

Noting that \( u(0) = 0 \) gives \( A = y_0/x_0 \) so

\[ u = x_0 y_0 e^{\lambda y_0 t} - y_0 e^{\lambda x_0 t} \]

Note: Problems such as this – including the derivation of the ode(s) – are fair game for examiners. Maths is not only about solving the equations, but also about deriving them and drawing conclusions from the solutions. In Part IA the examiners would help you with any derivation required in an exam.

### 3.4.2 POPULATION DYNAMICS

**Constant birth and death rates**

Suppose birth rate is proportional to population \( y \),

\[ \text{birth rate} = \alpha y \]

[For many species, the number of females matter, but not the number of males, so \( \alpha \) might not be constant. Moreover, for some species, e.g. rabbits, overcrowding causes the birth rate to decline.]

Suppose also that the death rate is proportional to the population,

\[ \text{death rate} = \beta y \]

[This may be true for death by ‘natural’ causes and hunting by another species. It will typically not be true for epidemics, combat between members, etc.]

\[ \Rightarrow \frac{dy}{dt} = (\alpha - \beta)y = ry \]

\[ \Rightarrow y = y_0 e^{rt} \]

\[ \Rightarrow \] population grows or decreases exponentially, depending on sign of \( \alpha - \beta \).

**End of Lecture 10**

**Fighting for limited resources**

If \( \alpha > \beta \) then the population will grow too large for the available resources (e.g. food supply).

Scarcity of food may lead to fighting (& death) between individuals who happen across the same food at the same time. The probability of one individual being at a given location at a given time is proportional to the population, \( y \), so the probability of two individuals meeting at a given location is proportional to \( y^2 \), hence

\[ \text{death rate due to fighting} = sy^2 \]

where \( s \) is a constant. Thus

\[ \frac{dy}{dt} = ry - sy^2 = r\left(1 - \frac{y}{Y}\right)y = f(y) \]

with \( Y = r/s \). Note that this should hold even if \( \alpha < \beta \). This is the **logistic differential equation**. The equation is autonomous as it has no explicit \( t \) dependence.
The phase portrait shows critical points (fixed points or equilibria) at $y = 0, Y$.

The $y = 0$ fixed point has $\frac{\partial f}{\partial y} > 0$ so is unstable, whereas at $y = Y$ $\frac{\partial f}{\partial y} < 0$ so the fixed point is stable.
Clearly, \( y = Y \) is a stable equilibrium, whereas \( y = 0 \) is unstable.

Logistics equation is separable and can be solved explicitly:

\[
\frac{1}{1 - \frac{y}{Y}} \frac{dy}{dt} = \left( \frac{1}{y} + \frac{1}{Y - y} \right) \frac{dy}{dt} = r
\]

\[\Rightarrow \ln \left| \frac{y}{Y - y} \right| = rt + c\]

\[\Rightarrow \frac{y}{Y - y} = Ae^{rt}\]

Suppose \( y = y_0 \) at \( t = t_0 \), then \( A = y_0 e^{-rt_0}/(Y-y_0) \) and
First order nonlinear equations

\[ y = \frac{Y_0 e^{r(t-t_0)}}{Y - y_0 + y_0 e^{r(t-t_0)}} \]

which gives \( y \to Y \) as \( t \to \infty \).

But can population really rise from just above zero? With careful management it is possible (e.g. Old Blue – see later), but in general…

**Mating opportunities**

Individuals need to find mates to procreate. This can become difficult when population densities are very low. The probability of finding a mate is similar to the probability of fighting over food and is again proportional to \( y^2 \). Of course, fighting or procreating depends on who meets!

For low population densities (where fighting is not an issue) the birth rate is \( \lambda y^2 \), while the death rate remains \( \beta y \), so

\[
\frac{dy}{dt} = -\beta y + \lambda y^2 = -\beta \left(1 - \frac{y}{X}\right) y = f(y)
\]

This equation has the same structure as previously, with critical points at \( y = 0 \) and \( y = X \).

Now, \( y = X \) is **unstable**, so if population falls below \( y = X \), then it will fall to zero and extinction!
The North American **passenger pigeon** is one of the most famous recent examples of extinction.

- In mid 19\textsuperscript{th} century, \( y \sim 10^9 \), the most numerous bird on earth.
- Widely hunted for sport and food in late 19\textsuperscript{th} century; as many as 30,000 shot at a time!
- By 1896 only 250,000 remaining
- Extinct by 1900.

Extinction was ultimately caused not by hunters killing the remaining birds, but by their inability to find mates.

Their extinction was one of the early triggers for conservation.

[Lots of information on the web.]

**Coming back from the brink**

Introduction of mammals (cats, rats, ferrets, opossum, \textit{etc.}) to New Zealand has devastated the populations of many species of birds. Some have become extinct, but others have brushed with the edge of extinction only to recover through careful management, with human intervention (“divine intervention”?) effectively bypassing the problem of finding a mate.
The New Zealand black robin is an extreme example: in 1980 there was only a single breading female and four males…. Now there are around 250 and the population is growing.

**Logistic growth with a threshold**

Populations above the critical threshold \( y = X \) cannot really grow without limit, so some new limiting factor must come into play when \( y \) is large enough.

Could model this as

\[
\frac{dy}{dt} = -r \left( 1 - \frac{y}{Y} \right) \left( 1 - \frac{y}{Z} \right) y = f
\]

For \( Y < Z \) the phase portrait is

Clearly equilibria are \( y = 0, Y, Z \), with two stable and one unstable fixed points.
Stable equilibria at $y = 0$, $Z$, and unstable at $y = Y$.

### 3.5 Comparison with discrete equations

Consider the logistics equation

$$\frac{dy}{dt} = r \left(1 - \frac{y}{Y}\right)y$$

$r > 0$; $Y > 0$.

$\left[y = Y \text{ stable, } y = 0 \text{ unstable}\right]$

The Euler finite difference approximation (see §2.2.3) to this may be written as

$$y_{n+1} - y_n = r \ dt \ (1 - y_n/Y) \ y_n$$

$$y_{n+1} = y_n + \rho \left(1 - \frac{y_n}{Y}\right) y_n$$

$$\Rightarrow$$

$$= \left(1 + \rho\right) \left(1 - \frac{\rho \ y_n}{1 + \rho \ Y}\right) y_n$$

$$= \lambda \left(1 - \frac{y_n}{\xi}\right) y_n$$

where $\rho = r \ dt$, $\lambda = (1 + \rho)$ and $\xi = Y(1+\rho)/\rho$. Let $u_n = y_n/\xi$

$$\Rightarrow$$

$$u_{n+1} = \lambda \ (1 - u_n) \ u_n$$

This ‘normalised’ nonlinear first order difference equation has only one important parameter, $\lambda$, whereas in the original we had $r$ and $Y$. 

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First order nonlinear equations  

Comparison with discrete equations
3.5.1 EQUILIBRIUM SOLUTIONS

Put \( u_{n+1} = u_n \):

\[
u_n = \lambda \left( 1 - u_n \right) u_n
\]

\[
\Rightarrow (\lambda - 1 - \lambda u_n)u_n = 0
\]

\[
\Rightarrow u_n = 0 \text{ or } u_n = (\lambda - 1)/\lambda \equiv U.
\]

[c.f. \( y = 0, Y \)]

3.5.2 STABILITY

Similar to what we did for the differential equations.

Near \( u_n = 0 \), have \( u_n^2 \ll |u_n| \)

\[
\Rightarrow u_{n+1} \approx \lambda u_n
\]

\[
\Rightarrow u_n \approx \lambda^n u_0
\]

Hence unstable if \( |\lambda| > 1 \). Now since \( \lambda = 1 + \rho \), and \( \rho = r \, dt > 0 \), then \( u_n = 0 \) unstable equilibrium.

Near \( u_n = U = (\lambda - 1)/\lambda \), suppose \( u_n = (\lambda - 1)/\lambda + v_n \)

\[
|v_n| \ll U
\]

\[
\frac{\lambda - 1}{\lambda} + v_{n+1} = \lambda \left( 1 - \frac{\lambda - 1}{\lambda} - v_n \right) \left( \frac{\lambda - 1}{\lambda} + v_n \right) = \lambda \left( 1 - v_n \right) \left( \frac{\lambda - 1}{\lambda} + v_n \right)
\]

\[
\Rightarrow \quad \frac{\lambda - 1}{\lambda} + \frac{\lambda - 1}{\lambda} + (1 - (\lambda - 1))v_n - \lambda v_n^2 = \frac{\lambda - 1}{\lambda} + (2 - \lambda) v_n - \lambda v_n^2
\]

\[
\Rightarrow \quad v_{n+1} \approx (2 - \lambda) v_n
\]

\[
\Rightarrow \quad v_n \approx (2 - \lambda)^n v_0
\]

Stable if \( |2 - \lambda| < 1 \)

\[
\Rightarrow \quad -1 < 2 - \lambda < 1
\]

\[
\Rightarrow \quad 1 < \lambda < 3
\]

If we are using the difference equation as an approximation to the logistic differential equation, then \( \rho = r \, dt \) will be small, so \( \lambda < 3 \) as required for stability. However, for real populations, the population may be better approximated by the difference equation than the differential equation. For example, most births occur at almost the same time in the breeding season, so \( \rho \) may represent a whole year, or even a whole generation, thus the meaning of \( \lambda \) may be different and may not be close to 1.

Following the same arguments as for the stability of differential equations (§3.2.3), we can relate the stability to the gradient in the difference formula.

Let \( u = U + v \)

\( u = U \) is equilibrium solution

Then

\[
u_{n+1} = U + v_{n+1} = f(u_n) = f(U + v_n) \approx f(U) + v_n f'(U)
\]
so we require
\[ \left| \frac{v_{n+1}}{v_n} \right| \approx \left| f'(U) \right| < 1 \]
for stability \((v_n \to 0 \text{ as } n \to \infty)\).

For the current equation, we therefore require
\[
\frac{\partial f}{\partial u} = \frac{d}{du} \left( \lambda (1-u)u \right) = \lambda (1-2u)
\]
is less than unity when \(u = U = (\lambda - 1)/\lambda \) (the equilibrium solution). Substituting
\[
\left| \lambda \left( 1 - 2 + \frac{2}{\lambda} \right) \right| = |\lambda - 2| < 1,
\]
which gives the same result as before.

3.5.3 Behaviour close to limit of stability
How does \(u_n\) approach \(U\) as \(n \to \infty\) for different values of \(\lambda\)?
The $\lambda = 1.5$ and $\lambda = 2.8$ solutions are stable and approach the equilibrium solution $u = U$. The convergence can be monotonic, or oscillatory. Why?

**End of Lecture 11**

### 3.5.4 Graphical Approach to Difference Equation

Plotting $u_{n+1}$ against $u_n$ provides a straightforward and instructive method of graphically solving the difference equation. If $u_{n+1} = f(u_n)$ then begin by plotting $f(u_n)$ and the line $u_{n+1} = u_n$.

For $\lambda = 0.8$: 

---

---
Clearly $u_n \to 0$ as $n \to \infty$ regardless of starting point.

For $\lambda = 1.5$:

Convergence is monotonic close to equilibrium solution.

For $\lambda = 2.8$:
Initial monotonic convergence changes to oscillatory convergence.

For $\lambda = 3.2$:

Initial convergence changes to a steady oscillation with period two. The appearance of this new oscillatory solution is called a bifurcation. In this case the period two oscillation is stable; the original solution still exists but it is now unstable. The phenomenon is called period-doubling.

At $\lambda = 3.449$ there is a second period-doubling bifurcation in which the period two solution becomes unstable and a stable period four oscillation appears.
For $\lambda$ slightly larger get another period doubling to an oscillation with period eight, then to sixteen, etc., with values of $\lambda$ converging to $\lambda_c = 3.5699…$

3.5.5 LOGISTIC MAP

Valuable insight into the behaviour of the difference equation may be gained by plotting its asymptotic solutions (as $n \to \infty$) as a function of the control parameter(s).

For our logistic difference equation $u_{n+1} = \lambda (1 - u_n) u_n$, this means plotting $u_n$ against $\lambda$ for large values of $n$. 
When $\lambda < 1$, the difference equation converges on $u = 0$. In the range $0 < \lambda < 3$ a steady solution is obtained with $u = (\lambda - 1)/\lambda$. As noted before, we have a bifurcation at $\lambda = 3$ to a period two oscillatory solution (hence two values of $u$ are observed). From $\lambda \approx 0.3499$, four solutions are observed following a period-doubling bifurcation. Increasing $\lambda$ further results in additional period-doubling bifurcations until $\lambda = \lambda_c = 3.5699$. Beyond this point all periodic solutions are unstable and we have deterministic chaos (the nearly solid black bands in the figure). Typically two solutions that may be close together at one $n$ diverge rapidly as $n$ increases.
For $\lambda > \lambda_c$ there are intervals of values of $\lambda$ in which stable periodic orbits exist. These attract almost all initial conditions and so show up as clear bands in the diagram. If we look hard enough it turns out that there is an interval containing a stable periodic orbit of every integer period.
For $\lambda > \lambda_c$ there are intervals of values of $\lambda$ in which stable periodic orbits exist. These attract almost all initial conditions and so show up as clear bands in the diagram. If we look hard enough it turns out that there is an interval containing a stable periodic orbit of every integer period.

For $\lambda > 4$, typical initial conditions are mapped out of the interval $[0,1]$ and of to $-\infty$.

3.5.7 LOCATION OF BIFURCATIONS

Recall that when we were looking for equilibrium solutions we set $u_{n+1} = u_n$ and solving

$$u_n = \lambda (1 - u_n) u_n$$

For the period two orbit, we can do a similar thing, but set $u_{n+2} = u_n$. Now

$$u_n = u_{n+2} = \lambda (1 - u_{n+1}) \ u_{n+1} = \lambda (1 - \lambda (1 - u_n)) \ \lambda (1 - u_n) \ u_n$$

which yields a quartic in $u_n$. However, we already know two of the roots: since $u_{n+1} = 0$, $U$ satisfy both $u_{n+2} = u_{n+1}$ and $u_{n+1} = u_n$, then they are also solutions to (*). Factorising,

$$u_n \ (1 - \lambda + \lambda u_n)[1 + \lambda - \lambda(1+\lambda)u_n + \lambda^2 u_n^2] = 0.$$
The term in square brackets provides additional roots

\[ u_n = \frac{1 + \lambda \pm \sqrt{\lambda^2 - 2\lambda - 3}}{2\lambda} \]  

(\text{**})

provided \( \lambda^2 - 2\lambda - 3 > 0 \Rightarrow \lambda < -1 \) or \( \lambda > 3 \).

The new roots exist when \( \lambda \geq 3 \) and correspond to the new period two orbit. From a perturbation analysis we can show that they are stable for \( \lambda \) close to 3; in this case the period-doubling bifurcation is termed \textit{supercritical}.

Note that at \( \lambda = 3 \), the new root is a double root with \( u_n = 2/3 \), whereas the equilibrium root also has \( u_n = 2/3 \).

In §3.5.2 we saw that \( |\partial f/\partial u| < 1 \) for stability of the equilibrium solution. We can use the same ideas here for the oscillatory solution, where we have \( u_n = f(f(u_n)) \). The stability boundary therefore requires

\[ \frac{\partial f(f(u_n))}{\partial u} = \lambda^2(1-2u)(1-2\lambda u + 2\lambda u^2) = \pm 1 \]

Substituting in either of the oscillatory solutions (\text{**}) and simplifying gives

\[ 4 + 2\lambda - \lambda^2 = \pm 1, \]

with roots \( \lambda = 1 - \sqrt{6}, \lambda = -1, \lambda = 3 \) and \( \lambda = 1 + \sqrt{6} \). Only the last two of these are relevant, and show that the period two solution is stable for \( 3 < \lambda < 3.449 \).

We may repeat this analysis to find where the period four values of \( u_n \) become possible by setting \( u_{n+4} = u_n \). This leads to a sixteenth order polynomial for \( u_n \), sharing the four roots of the period two orbits.

Plotting the difference equation, looking for potential orbits up to period eight, shows how the structure becomes more complex as \( \lambda \) increases, with each successive bifurcation adding a new periodic solution branch on top of a previously stable branch of periodic points that is now unstable.
First order nonlinear equations

Comparison with discrete equations

\[ u(n) \]

\[ u(n+1) \]

\[ u(n+2) \]

\[ u(n+3) \]

\[ u(n+4) \]

\[ u(n+5) \]

\[ u(n+6) \]

\[ u(n+7) \]

\[ u(n+8) \]

\[ \lambda = 2.90 \]

\[ \lambda = 3.10 \]

\[ \lambda = 3.50 \]

\[ \lambda = 3.57 \]